To appear in Communications in Algebra

PRÜFER CONDITIONS IN THE NAGATA RING AND THE SERRE'S CONJECTURE RING

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Abstract. The Nagata ring $R(X)$ and the Serre's conjecture ring $R\langle X\rangle$ are two localizations of the polynomial ring $R[X]$ at the polynomials of unit content and at the monic polynomials, respectively. In this paper, we contribute to the study of Prüfer conditions in $R(X)$ and $R\langle X\rangle$. In particular, we solve the four open questions posed by Glaz and Schwarz in Section 8 of their survey paper [38] related to the transfer of Prüfer conditions to these two constructions.

1. Introduction

Throughout, R will denote a commutative ring with identity and X an indeterminate over R. The Nagata ring $R(X)$ and the Serre's conjecture ring $R\langle X\rangle$ are localizations of the polynomial ring $R[X]$ at the polynomials of unit content and at the monic polynomials, respectively; that is,

$$R(X) := S^{-1}R[X], \text{ with } S := \left\{f \in R[X] \mid c(f) = R\right\}$$

$$R\langle X\rangle := U^{-1}R[X], \text{ with } U := \left\{f \in R[X] \mid f \text{ is monic}\right\}$$

where $c(f)$ denotes the content of the polynomial f (i.e., the ideal of R generated by the coefficients of f) and S is a multiplicatively closed set by the Dedekind-Mertens lemma [42]. We have $R[X] \subset R\langle X\rangle \subset R(X)$. Further, $R(X)$ is a localization of $R\langle X\rangle$ and both constructions are faithfully flat over the base ring R, and hence share many ideal and ring-theoretic properties with R.

The construction $R(X)$ appeared in Krull's 1942 paper [48] and then was studied by Nagata [54, 55] using the notation $R(X)$. Later, it was investigated by Arnold [9], Gilmer & Mott [32], Gilmer [31], Daniel Anderson [3, 4], Ratliff [59], and then was named after Nagata by Querré in [57]. During the last three decades, numerous works appeared in the literature dealing with various aspects of Nagata rings; see for instance [2, 5, 6, 7, 8, 9, 11, 12, 16, 27, 28, 31, 34, 35, 38, 41, 45]. Note that the construction of Nagata ring is distinct from Grothendieck's Nagata rings (also called universally Japanese rings [40] or pseudo-geometric rings [55]). The construction $R\langle X\rangle$ appeared in Quillen-Suslin's solution to Serre's Conjecture [60] proving that every finitely projective $R[X_1, ..., X_n]$-module is free, when R is

Date: August 3, 2017.
2010 *Mathematics Subject Classification.* 13F20, 13B25, 13C10, 13F30, 13D05.
This work was funded by King Fahd University of Petroleum & Minerals Research Grant # RG1331.
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a principal ideal domain [58]. In 1978, Brewer and Costa extended this result to one-dimensional Bézout domains [20]. In 1980, Querré devoted Section 4 of his paper [57] to the study of divisorial ideals in rings of fractions of $R[X]$ and proved, among other results, that R and $R\langle X\rangle$ have the same Picard group (resp., divisor class group) when R is an integrally closed (resp. Krull) domain. Later in the same year, le Riche [49] published a paper entirely devoted to the investigation of the descent and ascent of ring-theoretic properties from and to $R\langle X\rangle$ and R. In 1997, the construction $R\langle X\rangle$ was named in [23] Serre's conjecture ring. During the last three decades, several papers appeared in the literature dealing with various aspects of Serre's conjecture rings; see for instance [5, 6, 21, 34, 35, 38, 52, 56, 63, 64].

Recall that R is semihereditary if every finitely generated ideal is projective [24]; R is arithmetical if every finitely generated ideal is locally principal [30, 44]; R is Gaussian if $c(fg) = c(f)c(g)\ \forall\ f, g \in R[X]$ [62]; R is Prüfer if every finitely generated regular ideal is projective [22, 39]; R is an fqp-ring if every finitely generated ideal is quasi-projective [1, 26]; R is locally Prüfer if R_p is Prüfer for every prime ideal p of R [18]; R is strongly Prüfer if every finitely generated ideal I of R with $(0 : I) = 0$ is locally principal [5, 51]. The following diagram of implications summarizes the relations between the Prüfer conditions involved in this study:

$$\begin{array}{ccccc}
 & & R \text{ is semihereditary} & & \\
 & & \Downarrow & & \\
 & & \mathrm{w.dim}(R) \leq 1 & & \\
 & & \Downarrow & & \\
 & & R \text{ is arithmetical} & & \\
 & & \Downarrow & & \\
 & & R \text{ is an fqp-ring} & & \\
 & & \Downarrow & & \\
 & & R \text{ is Gaussian} & & \\
 & \swarrow & & \searrow & \\
R \text{ is locally Prüfer} & & & & R \text{ is strongly Prüfer} \\
 & \searrow & & \swarrow & \\
 & & R \text{ is Prüfer} & &
\end{array}$$

Recall that all these notions collapse to the concept of Prüfer domain if R is an integral domain. Also, notice that the above implications are irreversible, in general, as shown by examples provided in [1, 13, 14, 15, 17, 18, 25, 36, 37, 38, 43, 50]. Further, as mentioned by Glaz and Schwarz in [38], the question of how the locally Prüfer and strongly Prüfer conditions relate to each other is still open. Next, we collect, in chronological order, some of the main known contributions on the transfer of *Prüfer conditions* to Nagata and Serre's conjecture rings in the literature.

In 1969, Arnold related the ideal theory of R to that of the Kronecker function ring and the Nagata ring. One of his main results asserts that, in the class of integrally closed domains, $R(X)$ is a Prüfer domain if and only if so is R [9, Theorem 4]. In 1978, Brewer and Costa extended Quillen-Suslin's result (cited above) to Bézout domains with Krull dimension 1. They obtained this extension as a corollary of their main result that, given an integral domain R which is not a field, $R\langle X\rangle$ is Prüfer (resp., Bézout) if and only if so is R and $\dim(R) = 1$ [20, Theorem 1].

In 1980, le Riche published a paper entirely devoted to the study of the ascent and descent of various ring-theoretic conditions between R and the Serre's conjecture ring. One section of this paper dealt with Prüfer domains and related rings. One of the main results states that $R\langle X\rangle$ is a semihereditary ring if and only if R is a semihereditary ring with $\dim(R) \leq 1$ [49, Theorem 3.7].

In 1985, the Andersons and Markanda published a long paper on Nagata rings and Serre's conjecture rings, with one section dedicated to the transfer of the arithmetical and Prüfer properties. They proved that $R(X)$ is an arithmetical ring if and only if so is R; and $R\langle X\rangle$ is an arithmetical ring if and only if R is an arithmetical ring with $\dim(R) \leq 1$[5, Theorem 3.1]. As for the Prüfer notion, they showed that $R(X)$ is a Prüfer ring if and only if R is a strongly Prüfer ring; and $R\langle X\rangle$ is a Prüfer ring if and only if R is a strongly Prüfer ring with $\dim(R) \leq 1$ and R_p is a field whenever $p \subsetneqq q$ are prime ideals of R [5, Theorem 3.2].

In 1989, Glaz in [34] studied necessary and sufficient conditions for Nagata rings and Serre's conjecture rings to inherit coherence, and derived exact relations between the weak dimension of R and that of $R(X)$ and $R\langle X\rangle$ when R a stably coherent ring of finite weak dimension [34, Theorem 2]. As a consequence, she established a transfer result for the semihereditary property to Nagata rings; namely, $R(X)$ is a semihereditary ring if and only if so is R [34, Corollary 3]. Recall that R is stably coherent if R and $R[X_1, ..., X_n]$ are coherent for every $n \geq 1$.

In 2011, Glaz and Schwarz published a comprehensive survey [38] on the study of Prüfer conditions in various settings of commutative rings. Section 8 of this survey was dedicated to Nagata rings and Serre's conjecture rings; and featured several open questions on the remaining conditions: Gaussian ring, locally Prüfer ring, and ring with weak global dimension ≤ 1.

The objective of this paper is to contribute to the study of the ascent and descent of Prüfer conditions to and from R to the Nagata ring $R(X)$ and the Serre's conjecture ring $R\langle X\rangle$. Our main objective is to address some open questions posed by Glaz and Schwarz in Section 8 of their survey paper [38] related to the transfer of various Prüfer conditions to these two constructions. As we have seen above, the transfer of some Prüfer conditions is already known in the literature. Accordingly, we will focus on the remaining notions of fqp-ring, Gaussian ring [38, Open Question 10], locally Prüfer ring [38, Open Question 12], and ring with weak global dimension ≤ 1 [38, Open Question 9]. We will also address Open Question 11 of [38] about the relationship between the locally Prüfer and strongly Prüfer conditions.

Throughout, for a ring R, $\mathrm{w.dim}(R)$ will denote the weak global dimension of R and $\mathrm{Z}(R)$ and $\mathrm{Nil}(R)$ will denote, respectively, the set of zero divisors and nilradical of R. For an ideal I of R, $\mathrm{Ann}(I)$ will denote the annihilator of I.

2. Results

We first recall known results on the transfer of the Prüfer conditions to polynomial rings. In 1973, McCarthy proved that a polynomial ring (with one indeterminate) over a von Neumann regular ring is semihereditary [53, Theorem]. In 1974, Gilmer and Parker [33] extended this result, in the more general context of semigroup rings, to Bézout rings (i.e., every finitely generated ideal is principal). Namely, let R be a ring and Γ a torsion-free cancellative Abelian semigroup with zero. They proved that the semigroup ring $R[\Gamma]$ is Prüfer if and only if $R[\Gamma]$ is Bézout if and only if R is von Neumann regular and (up to isomorphism) either Γ is a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ or $\Gamma = G \cap \mathbb{Q}_o$ where G is a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ and $\mathbb{Q}_o$ denote the additive semigroup of nonnegative rationals (see also [3, Theorem 6] and [19, Theorem 15]). Consequently, in the particular case of $\Gamma = \mathbb{N}$, we obtain:

Remark 2.1. The following assertions are equivalent:

(1) R is von Neumann regular;
(2) $R[X]$ is semihereditary;
(3) $\mathrm{w.dim}(R[X]) \leq 1$;
(4) $R[X]$ is arithmetical;
(5) $R[X]$ is fqp;
(6) $R[X]$ is Gaussian;
(7) $R[X]$ is Prüfer.

The first main result solves [38, Open Question 10] by establishing the transfer of the Gaussian property to Nagata and Serre's conjecture rings. Recall that a ring R is Gaussian if $c(fg) = c(f)c(g)$ for any two polynomials f and g in $R[X]$. Also, from [15, Theorem 2.2], a local ring is Gaussian if and only if, for any two elements a, b in the ring, the following two properties hold:

(G1) $(a,b)^2 = (a^2)$ or (b^2), and
(G2) if $(a,b)^2 = (a^2)$ and $ab = 0$, then $b^2 = 0$.

Note, at this point, that in the proof of the Prüfer case for Serre's conjecture rings [5, Theorem 3.2(2)], the authors wrote at the end of the proof "*It now follows as in (2) of Theorem 3.1 that* R_{P_0} *is a field. This shows that both* $\dim(R) \leq 1$ *and* R_P *is a field if* $P \subsetneq Q$ *are prime ideals of* R." But in the proof of Theorem 3.1, they used the fact "$\dim(R) \leq 1$" to prove that R_P is a field. So, in the next result, we cannot conclude from [5, Theorem 3.2(2)($\Rightarrow$)] that R_p is a field for every non-maximal prime ideal p of R if $R\langle X\rangle$ is Gaussian (hence Prüfer).

Theorem 2.2. *Let R be a ring. Then:*

(1) *$R(X)$ is Gaussian if and only if R is Gaussian.*
(2) *$R\langle X\rangle$ is Gaussian if and only if R is Gaussian and R_p is a field for every non-maximal prime ideal p of R.*

Proof. (1) Recall first that the Gaussian property is local and there is a one-to-one correspondence between the maximal ideals of R and the maximal ideals of $R(X)$ given by $\mathfrak{m} \longleftrightarrow \mathfrak{m}R(X)$. Further, for any prime ideal p of R, we have the natural isomorphisms:

$$R_p(X) = R[X]_{p[X]} = R(X)_{pR(X)}.$$

Therefore, we can reduce to the case where R (or, equiv., $R(X)$) is local. Assume $R(X)$ is Gaussian and let $I := (a,b)R$. Hence, by [5, Theorem 2.2(1)], we have

$$\begin{array}{rcl} I^2 & = & I^2R(X) \cap R \\ & = & \Big((a^2)R(X) \text{ or } (b^2)R(X)\Big) \cap R \\ & = & (a^2) \text{ or } (b^2). \end{array}$$

Whence (G1) holds, and so does (G2) since $R \subset R(X)$; that is, R is Gaussian.

Conversely, assume R is Gaussian and let $f,g \in R[X]$ with $f = a_nX^n + ... + a_0$ and $g = b_mX^m + ... + b_0$. The fact that R satisfies (G1) yields

$$a_ia_j \in (a_i^2, a_j^2) \subseteq (a_1^2, ..., a_n^2)$$

for all $1 \leq i \leq j \leq n$; and hence $\big(c(f)\big)^2 = (a_k^2)$ for some $k \in \{1,...,n\}$. Similarly, we get $\big(c(g)\big)^2 = (b_l^2)$ for some $l \in \{1,...,m\}$. It follows that

$$f^2 = a_k^2 h \text{ and } g^2 = b_l^2 h'$$

where h and h' are units in $R(X)$. Moreover, for any $i := 1,...,n$ and $j := 1,...,m$, we have

$$a_ib_j \in (a_i^2, b_j^2) \subseteq (a_k^2, b_l^2) \subseteq (a_k, b_l)^2 = (a_k^2) \text{ or } (b_l^2).$$

Consequently, we obtain

$$(f,g)^2R(X) = (f^2) \text{ or } (g^2)$$

and so (G1) holds for $R(X)$. Now, assume $fg = 0$. Therefore $c(f)c(g) = c(fg) = 0$ which yields $a_kb_l = 0$. This forces $a_k^2 = 0$ or $b_l^2 = 0$; that is, $f^2 = 0$ or $g^2 = 0$. Thus $R(X)$ satisfies (G2), as desired.

(2) ($\Leftarrow$) In this direction, the proof makes use of localization similarly to the arithmetical case [5, Theorem 3.1(2)]. Indeed, let M be a maximal ideal of $R\langle X\rangle$. Then, either $M = \mathfrak{m}\langle X\rangle := \mathfrak{m}R\langle X\rangle$ for some maximal ideal $\mathfrak{m}$ of R or $M = PR\langle X\rangle$ for some prime ideal P of $R[X]$ which is an upper to a non-maximal prime ideal p of R. In the first case, the ring

$$R\langle X\rangle_M \cong R[X]_{\mathfrak{m}[X]} \cong R_{\mathfrak{m}}(X)$$

is Gaussian by (1). In the second case, the ring

$$R\langle X\rangle_M \cong R[X]_P \cong R_p[X]_{PR_p}$$

is Gaussian (in fact, principal) since R_p is a field.

($\Rightarrow$) Assume that $R\langle X\rangle$ is Gaussian. Then, $R(X)$, being a localization of $R\langle X\rangle$, is Gaussian and so is R by (1). Now, let p_o be a minimal prime of R such that $\dim(R) = \dim(\frac{R}{p_o})$. Then, the ring $\frac{R}{p_o}\langle X\rangle$, being isomorphic to $\frac{R\langle X\rangle}{p_o\langle X\rangle}$, is a Gaussian domain (i.e., Prüfer domain) since the Gaussian property is stable under factor rings. It follows, via [20, Theorem 1], that $\dim(R) = \dim(\frac{R}{p_o}) \leq 1$.

Next, let $p \subsetneq \mathfrak{m}$ be a pair of prime ideals of R. Then, $\dim(R_p) = 0$. We prove that R_p is a domain (and, a fortiori, a field). For this purpose, recall that $R_{\mathfrak{m}}\langle X\rangle$ is a localization of $R\langle X\rangle$ [5, Lemma 2.5] and $R_p \cong (R_{\mathfrak{m}})_{pR_{\mathfrak{m}}}$. Therefore, without loss of generality, we may assume that $(R, \mathfrak{m})$ is a one-dimensional local Gaussian ring. In this setting, the prime ideals of R are totally ordered by inclusion and hence $p = \mathrm{Nil}(R)$ is its unique minimal prime ideal; that is, $\frac{R}{p}$ is a valuation domain [15, p. 183]. Let $a \in \mathfrak{m} \setminus p$. Then $\bar{a}X + \bar{1}$ is an irreducible element of $\frac{R}{p}[X]$. Since $\frac{R}{p}[X]$ is a GCD domain, $\bar{a}X + \bar{1}$ is prime in $\frac{R}{p}[X]$ [10]. It follows that $Q = (p[X], aX + 1)$ is a prime ideal of $R[X]$. Moreover, $R[X]_Q$, being a localization of $R\langle X\rangle$, is a local Gaussian ring. Therefore, its prime ideals are totally ordered by inclusion and, necessarily, we have

$$P[X]_Q \subseteq (aX + 1)_Q.$$

At this point, we conclude that R_p is a domain via the same arguments used at the end of the proof of the arithmetical case [5, Theorem 3.1(2)]. □

Remark 2.3. Observe that if R_p is a field for every non-maximal prime ideal p of R, then necessarily $\dim(R) \leq 1$; and the two conditions coincide trivially in the domain case. So, Theorem 2.2 extends the classic results on Prüfer domains.

As a consequence of Theorem 2.2, the next result solves [38, Open Question 9] by establishing conditions on R to force the weak global dimension of $R(X)$ (resp., $R\langle X\rangle$) to be at most 1.

Corollary 2.4. *Let R be a ring. Then:*

(1) $\mathrm{w.dim}(R(X)) \leq 1$ *if and only if* $\mathrm{w.dim}(R) \leq 1$.
(2) $\mathrm{w.dim}(R\langle X\rangle) \leq 1$ *if and only if* $\mathrm{w.dim}(R) \leq 1$ *and* R_p *is a field for every non-maximal prime ideal p of R.*

Proof. Recall first Glaz's well-known result that $\mathrm{w.dim}(R) \leq 1$ if and only if R is a Gaussian reduced ring [36, Theorem 2.2]. Further, since $\mathrm{Nil}(R(X)) = \mathrm{Nil}(R)(X)$ and $\mathrm{Nil}(R\langle X\rangle) = \mathrm{Nil}(R)\langle X\rangle$, then $R(X)$ (resp., $R\langle X\rangle$) is reduced if and only if so is R. Thus, this fact combined with Theorem 2.2 leads to the conclusion via Glaz's aforementioned result. □

The second main result establishes the transfer of the fqp property to Nagata and Serre's conjecture rings. For this purpose, recall that an ideal is quasi-projective if it is projective modulo its annihilator; and R is an fqp-ring if every finitely generated ideal of R is quasi-projective [1]. The fqp condition is stable under formation of rings of fractions [1, Lemma 3.6]. However, very recently, Couchot proved that it is not a local property [26, Example 4.6]; and called R an fqf-ring if every finitely generated ideal of R is flat modulo its annihilator (or, equiv., if R is locally an fqp-ring) [26]. Accordingly, we have

$$\text{fqp-ring} \Rightarrow \text{fqf-ring} \Rightarrow \text{Gaussian}$$

and the fqp and fqf conditions coincide in the class of coherent rings [26, Proposition 4.4] or [1, Corollary 3.15]. Recall that R is called a *chained ring* if R is local and arithmetical (i.e., its ideals are linearly ordered with respect to inclusion).

Theorem 2.5.

(1) *Let R be a local ring. Then:*
   (a) *$R(X)$ is an fqp-ring if and only if so is R.*
   (b) *$R\langle X\rangle$ is an fqp-ring if and only if so is R and R_p is a field for every non-maximal prime ideal p of R.*

(2) *Let R be a ring. Then:*
   (a) *$R(X)$ is an fqf-ring if and only if so is R.*
   (b) *$R\langle X\rangle$ is an fqf-ring if and only if so is R and R_p is a field for every non-maximal prime ideal p of R.*

The proof of this theorem involves the next two lemmas which are of independent interest. The first lemma combines and completes [1, Lemma 4.5] and [61, Theorem 2].

Lemma 2.6. *Let R be a local ring which is not a chained ring. Then, the following assertions are equivalent:*

(1) *R is an fqp-ring;*
(2) *R is a Prüfer ring with $\mathrm{Z}(R) = \mathrm{Nil}(R)$ and $(\mathrm{Z}(R))^2 = 0$.*

Proof. (1)⇒ (2) This is handled by [1, Lemma 4.5] and [61, Theorem 2]. Note that the latter result is listed as Lemma 3.12 in [1].

(2)⇒ (1) Let $I := (a_1,...,a_n)$ be a finitely generated ideal of R. We need to prove that I is quasi-projective. Suppose that there exists $i \in \{1,...,n\}$ such that a_i is regular in R. Then, I is projective since R is Prüfer. Next, suppose that $a_i \in \mathrm{Z}(R)$ for all $i \in \{1,...,n\}$. One can check that, for each $i \in \{1,...,n\}$, the combination of the two assumptions $\mathrm{Z}(R) = \mathrm{Nil}(R)$ and $(\mathrm{Nil}(R))^2 = 0$ yields

$$\mathrm{Ann}(a_i) = \mathrm{Nil}(R).$$

It follows that

$$I \cong \bigoplus_{i=1}^{n} \frac{R}{\mathrm{Ann}(a_i)} = \bigoplus_{i=1}^{n} \frac{R}{\mathrm{Nil}(R)}.$$

However, $\dfrac{R}{\mathrm{Nil}(R)}$, being a cyclic R-module, is an R-quasi-projective module [47] (or [1, Lemma 3.4]). Consequently, by [29, Corollary 1.2] (or [1, Lemma 3.5]), I is quasi-projective. □

Remark 2.7. (1) It is worthwhile observing that in the proof of (2)⇒ (1) we do not need the fact that R is local. Precisely, if R is a Prüfer ring with $\mathrm{Z}(R) = \mathrm{Nil}(R)$ and $(\mathrm{Nil}(R))^2 = 0$, then R is an fqp-ring.

(2) Assume R is a total ring of quotients. Then, R is an fqp-ring if and only if R is a chained ring or $(\mathrm{Z}(R))^2 = 0$. The "if" assertion is handled by [1, Theorem 3.2] and [1, Lemma 4.6].

Lemma 2.8. *Let R be a ring. Then, the following assertions are equivalent:*

(1) $\mathrm{Z}(R) = \mathrm{Nil}(R)$;
(2) $\mathrm{Z}(R[X]) = \mathrm{Nil}(R[X])$;
(3) $\mathrm{Z}(R(X)) = \mathrm{Nil}(R(X))$;
(4) $\mathrm{Z}(R\langle X\rangle) = \mathrm{Nil}(R\langle X\rangle)$.

Proof. (1)⇒ (2) ⇒ (3) ⇒ (1) Suppose that $\mathrm{Z}(R) = \mathrm{Nil}(R)$. Then, we have

$$\begin{array}{rcl} \mathrm{Z}(R[X]) & \subseteq & \mathrm{Z}(R)[X] \\ & = & \mathrm{Nil}(R)[X] \\ & = & \mathrm{Nil}(R[X]) \\ & \subseteq & \mathrm{Z}(R[X]). \end{array}$$

Therefore, $\mathrm{Z}(R[X]) = \mathrm{Nil}(R[X])$ and hence, for $S := \left\{f \in R[X] \mid c(f) = R\right\}$, we get

$$\begin{array}{rcl} S^{-1}\,\mathrm{Z}(R[X]) & = & S^{-1}\,\mathrm{Nil}(R[X]) \\ & = & \mathrm{Nil}(R(X)) \\ & \subseteq & \mathrm{Z}(R(X)) \\ & \subseteq & S^{-1}\,\mathrm{Z}(R[X]). \end{array}$$

That is, $\mathrm{Z}(R(X)) = \mathrm{Nil}(R(X))$. Now, suppose that $\mathrm{Z}(R(X)) = \mathrm{Nil}(R(X))$. Then, by the fact $R \subseteq R(X)$ and [5, Theorem 2.2(1)], we obtain

$$\begin{array}{rcl} \mathrm{Z}(R) & \subseteq & R \cap \mathrm{Z}(R(X)) \\ & = & R \cap \mathrm{Nil}(R(X)) \\ & = & R \cap \mathrm{Nil}(R)(X) \\ & = & \mathrm{Nil}(R) \\ & \subseteq & \mathrm{Z}(R). \end{array}$$

That is, $\mathrm{Z}(R) = \mathrm{Nil}(R)$.

(2) ⇒ (4) ⇒ (1) Conclude using the above arguments applied to the Serre's conjecture ring with $S := \left\{f \in R[X] \mid f \text{ is monic}\right\}$. □

Proof of Theorem 2.5. (1)(a) In view of [5, Theorem 3.1(1)] and the one-to-one correspondence between the maximal ideals of R and those of $R(X)$, R is a chained ring if and only if so is $R(X)$. Hence, it remains to prove the theorem for non-chained local rings. For this purpose, note that if R (resp., $R(X)$) is an fqp-ring, then by Theorem 2.2 $R(X)$ (resp., R) is Gaussian and, a fortiori, Prüfer. Further, notice that the fact $\mathrm{Nil}(R(X)) = \mathrm{Nil}(R)(X)$ and [5, Theorem 2.2(1)] yield $(\mathrm{Nil}(R))^2 = 0$ if and only if $\left(\mathrm{Nil}(R(X))\right)^2 = 0$. Consequently, a combination of Lemmas 2.6 and 2.8 lead to the conclusion, completing the proof of (1).

(b) Notice that, similarly to Nagata rings, the fact $\mathrm{Nil}(R\langle X\rangle) = \mathrm{Nil}(R)\langle X\rangle$ and [5, Theorem 2.2(1)] yield $(\mathrm{Nil}(R))^2 = 0$ if and only if $\left(\mathrm{Nil}(R\langle X\rangle)\right)^2 = 0$. Next, assume that R is an fqp-ring. If R is a chained ring, we appeal to [5, Theorem 3.1.(2)]. If R is not a chained ring, as above, we conclude via a combination of Theorem 2.2, Remark 2.7, and Lemma 2.8. Note that the use of Remark 2.7 is incumbent here since $R\langle X\rangle$ is not necessarily local. Conversely, assume that $R\langle X\rangle$ is an fqp-ring. Then, $R(X)$, being a localization of $R\langle X\rangle$, is an fqp-ring [1, Lemma 3.6] and so is R by (1). By Theorem 2.2(2), R_p is a field for every non-maximal prime ideal p of R.

(2) This is a global version of (1) which is readily obtained via the correspondence between the maximal ideals of $R(X)$ (resp., $R\langle X\rangle$) and the maximal ideals of R (resp., the maximal ideals and non-maximal prime ideals of R, as exhibited in the proof of ($\Leftarrow$) of Theorem 2.2(2)). □

Next, we address [38, Open Question 12] about the transfer of the locally Prüfer condition to Nagata and Serre's conjecture rings. In this vein, observe that this notion transfers readily from $R(X)$ or $R\langle X\rangle$ to R, as shown below.

Remark 2.9. Let R be a ring. If $R(X)$ (resp., $R\langle X\rangle$) is locally Prüfer, then so is R. Indeed, for any prime ideal p of R, $R(X)_{p(X)}$ (resp., $R\langle X\rangle_{p\langle X\rangle}$) is isomorphic to $R_p(X)$; and then conclude, via [5, Theorem 3.2(1)], that R_p is strongly Prüfer and hence Prüfer; that is, R is locally Prüfer. The converse is not true, in general. Indeed, let k be a field, X_1, X_2 two indeterminates over k, $A := k[X_1, X_2]$, and $B := \oplus(A/p)$, where p ranges over the set of height-one prime ideals of A. Let $R := A \ltimes B$ be the trivial ring extension of A by B. Then, R is the ring of Example 27.19 of [43], where it was shown that it is Prüfer but not strongly Prüfer. Moreover, R is also the ring of Example 13 of [46], where it was shown that, in fact, it is locally Prüfer. However, $R(X)$ and, a fortiori, $R\langle X\rangle$ are not locally Prüfer by [5, Theorem 3.2(1)].

We close this paper by discussing the second part of [38, Open Question 11] about how the locally Prüfer condition and strongly Prüfer condition relate to each other. (The first part of this open question asks about the possible transfer of the strongly Prüfer condition to pullbacks which is not the object of this paper.) We, now, know for sure that the class of locally Prüfer rings and the class of strongly Prüfer rings are distinct. Indeed, on one hand, as seen above, [46, Example 13] features a ring which is locally Prüfer but not strongly Prüfer. On the other hand, one can build numerous examples of strongly Prüfer rings which are not locally Prüfer via trivial ring extensions of local rings by vector spaces over their residue fields, as shown below.

Example 2.10. Let $(A, \mathfrak{m})$ be a local ring and E a nonzero $\frac{A}{\mathfrak{m}}$-vector space. Let $R := A \ltimes E$ be the trivial ring extension of A by E. Then R is always a local strongly Prüfer ring since the annihilator of its maximal ideal $\mathfrak{m} \ltimes E$ contains $0 \ltimes E$ (so that there is no faithful ideal in R). If, in addition, A is supposed to be Prüfer, then R is locally Prüfer if and only if so is A: Indeed, let $p \subsetneq \mathfrak{m}$ be a prime ideal of A. Necessarily, $E_p = 0$. By [26, Lemma 1.3], $R_{p \ltimes E} = A_p \ltimes E_p = A_p \ltimes 0 \cong A_p$, leading to the conclusion. Now, let k be a field and X_1, X_2, X_3, X_4 indeterminates over k. Let

$$B := \frac{k[X_1, X_2, X_3, X_4]}{(X_1^2, X_1X_2, X_1X_3, X_1X_4)}$$

and $\mathfrak{m} := (x_1, x_2, x_3, x_4)$, where $x_i := \overline{X_i}$ for $i := 1, ..., 4$. Then $A := B_{\mathfrak{m}}$ is a local Prüfer ring that is not locally Prüfer [18, Example 2.4] (see also [38, Example 3.16]). Consequently, for any nonzero $\frac{A}{\mathfrak{m}}$-vector space E, $R := A \ltimes E$ is a local strongly Prüfer ring that is not locally Prüfer. Notice that A itself is strongly Prüfer (since its maximal ideal has nonzero annihilator).

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