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When is the Dual of an Ideal a Ring?

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1. INTRODUCTION

Throughout this work, R denotes a domain with quotient field K. For a nonzero fractional ideal I of R, the fractional ideal $I^{-1} = (R:I) = \{x \in K | xI \subseteq R\}$ is called the inverse (or dual) of I. In [HuP], Huckaba and Papick studied the question of when I^{-1} is a ring, and this question has received further attention by these authors and by Anderson, Fontana, Heinzer, and Roitman [A], [FHP1], [FHP2], [FHP3], [HP], and [FHPR]. The authors of the present paper have also studied the question in the specific contexts of pullbacks [HKLM1] and polynomial rings [HKLM2]. Our purpose here is to determine when I^{-1} is a ring in much more general situations.

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In the second section, we show that if I^{-1} is a ring, then P^{-1} is a ring for each minimal prime ideal of I. It is known [HuP, Proposition 2.2] that I^{-1} is a ring $\Leftrightarrow I^{-1} = (I_v : I_v)$; thus it is natural to consider connections with divisoriality. In Proposition 2.5, we characterize when the inverse of a nonzero intersection of divisorial ideals is a ring.

It is clear that I^{-1} is a ring when $I^{-1} = (I : I)$, and [A, Proposition 3.3] shows that the converse is true when I is a radical ideal. The third section is devoted to a study of the question for radical ideals. In Theorem 3.1, we give several characterizations of when I^{-1} is a ring for radical I; as a corollary we show that if P is prime, then P^{-1} fails to be a ring $\Leftrightarrow P$ has the form $(aR_R : b)$ and PR_P is principal. One of the characterizations in Theorem 3.1 states that the inverse of a radical ideal I is a ring $\Leftrightarrow$ for each valuation overring V of R with $IV \neq V$, we have $I^{-1} \subseteq V_Q$, where Q is the prime of V which is minimal over IV. This is the first of our extensions of two results from [HuP]. There it is proved that if I is an ideal of a Prüfer domain, and if $\{P_\alpha\}$ and $\{M_\beta\}$ are the set of minimal primes of I and the set of maximal ideals which do not contain I, then $I^{-1} \supseteq (\bigcap R_{P_\alpha}) \cap (\bigcap R_{M_\beta})$ ([HuP, Lemma 3.3]) with equality $\Leftrightarrow I^{-1}$ is a ring ([HuP, Theorem 3.2]). We also obtain several results concerning intersections of radical ideals. We prove, for example, that if I and J are radical ideals, then I^{-1} and J^{-1} are rings $\Leftrightarrow (I \cap J)^{-1}$ and $(I + J)^{-1}$ are rings (Theorem 3.4); we also show that if I is the irredundant intersection of prime ideals P_α, then I^{-1} is a ring $\Leftrightarrow$ each P_α^{-1} is a ring.

Section 4 is devoted to the case of integrally closed R. We give several characterizations of when I^{-1} is a ring in this case, again extending the above-mentioned results of [HuP]; and we apply these ideas to obtain generalizations to Prüfer v-multiplication domains of other results given in [HuP] and [FHPR] for Prüfer domains. We show, for example, that if I is an ideal in an integrally closed domain R, then I^{-1} is a ring $\Leftrightarrow I^{-1} \subseteq V$ for each valuation overring V whose maximal ideal is minimal over IV. We also show that if I is an ideal of a Prüfer v-multiplication domain, then I^{-1} is a ring $\Leftrightarrow I^{-1} = (I : I) = R_{\mathscr{N}} \cap \mathscr{C}_t(R, I)$, where $\mathscr{N}$ is the complement in R of the set of zero divisors on R/I and $\mathscr{C}_t(R, I)$ is the set of maximal t-ideals of R which do not contain I.

Finally, in Section 5, we present examples tending to show that (many of) the results in Sections 2–4 are the best possible. For example, in Example 5.1 we show that it is possible for P^{-1} to be a ring for each minimal prime of a radical ideal I and yet have I^{-1} fail to be a ring, and in Example 5.2 we show that it is possible to have divisorial ideals I and J such that I^{-1} and J^{-1} are rings but such that $(I \cap J)^{-1}$ is not a ring. Many other examples are given.

Most of our notation is standard as in [G1]. We shall often make use of the so-called v-operation. This is defined on the set of nonzero fractional ideals I of a domain R by $I_v = (I^{-1})^{-1}$. The ideal I is said to be *divisorial* or a *v-ideal* if $I = I_v$. For properties of the v-operation, the reader is referred to [G1, Sections 32 and 34].

2. SOME RESULTS IN THE GENERAL CASE

Recall that R denotes a domain with quotient field K. Also recall that if J is a radical ideal of R, then J^{-1} is a ring $\Leftrightarrow J^{-1} = (J : J)$ [A, Proposition 3.3]. We shall often make use of this fact.

PROPOSITION 2.1. *Let I be a nonzero ideal of R for which I^{-1} is a ring. Then*

(1) *$\sqrt{I}^{\,-1}$ is a ring, and (therefore) $\sqrt{I}^{\,-1} = (\sqrt{I} : \sqrt{I})$;*

(2) *P^{-1} is a ring for each minimal prime ideal of I;*

(3) *$I^{-1} = (\sqrt{I} : I) = (Q : I)$ for each prime $Q \supseteq I$;*

(4) *if V is a valuation overring of R with $IV \neq V$, then $I^{-1} \subseteq V_Q$, where Q is the prime ideal of V which is minimal over IV.*

Proof. (1) Let $x \in (\sqrt{I})^{-1}$. It suffices to show that $x\sqrt{I} \subseteq \sqrt{I}$. Let $a \in \sqrt{I}$. Then $a^n \in I$ for some positive integer n. Moreover, since $(\sqrt{I})^{-1} \subseteq I^{-1}$ and I^{-1} is a ring, we have $x^{2n} \in I^{-1}$. Hence $x^{2n}a^n \in R$, whence $(xa)^{2n} \in I$. Since $xa \in R$, this implies that $xa \in \sqrt{I}$. It follows that $\sqrt{I}^{\,-1} = (\sqrt{I} : \sqrt{I})$.

(2) Let $J = \sqrt{I}$. By (1) J^{-1} is a ring. Let $u \in P^{-1}$ and $b \in P$. Since J is a radical ideal, $JR_P = PR_P$. Hence $b \in JR_P$, and we have $sb \in J$ for some $s \in R \setminus P$. Since $u \in P^{-1} \subseteq J^{-1}$, this yields $usb \in J \subseteq P$, whence $ub \in P$. Hence $PP^{-1} \subseteq P$, and P^{-1} is a ring.

(3) Let Q be a prime ideal containing I. Let $x \in I^{-1}$. Then $x^2 \in I^{-1}$, so that $x^2 I \subseteq R$ and $x^2 I^2 \subseteq I \subseteq Q$. Since $xI \subseteq R$, this implies $xI \subseteq Q$. Thus $II^{-1} \subseteq Q$, $I^{-1} \subseteq (Q : I) \subseteq (R : I) = I^{-1}$, and we have $I^{-1} = (Q : I)$. Since this is true for each Q, we have $II^{-1} \subseteq \sqrt{I}$, from which it follows that $I^{-1} = (\sqrt{I} : I)$.

(4) Suppose that $x \in I^{-1} \setminus V_Q$. Then $x^{-1} \in QV_Q = Q$. Since Q is minimal over IV, $x^{-n} \in IV$ for some n. However, since I^{-1} is a ring, $x^n \in I^{-1}$, whence by (3) $1 = x^n x^{-n} \in I^{-1}IV \subseteq \sqrt{I}\,V \subseteq Q$, a contradiction. ∎

In Section 5, we present an example of an ideal I satisfying all four conditions of Proposition 2.1 but for which I^{-1} is not a ring. However, for radical ideals condition (4) characterizes when I^{-1} is a ring (Theorem 3.1 below), and, if R is integrally closed, condition (4) characterizes when I^{-1} is a ring for general I (Theorem 4.4). (The first three conditions together do not imply that I^{-1} is a ring when R is integrally closed—see the remark following Proposition 4.1.)

Thus the conditions in Proposition 2.1 do not characterize when I^{-1} is a ring. The following (admittedly unsatisfying) result is the best characterization we have been able to obtain.

PROPOSITION 2.2. *Let I be a nonzero ideal of the domain R. The following conditions are equivalent*:

(1) I^{-1} *is a ring.*

(2) *I is not invertible, and $(M:I)$ is a ring for each maximal ideal $M \supseteq I$.*

(3) *$I^{-1} = (\sqrt{I} : I)$, and $(P:I)$ is a ring for each minimal prime P of I.*

Proof. Assume (1). Then I is not invertible by [HuP, Proposition 2.2]. Statement (2) now follows from Proposition 2.1. Conversely, if I is not invertible, then $II^{-1} \subseteq M$ for some maximal ideal, and it follows that $I^{-1} = (M:I)$, so that I^{-1} is a ring. Thus (1) and (2) are equivalent. Assume (3), and let P be a minimal prime of I. Then $I^{-1} = (\sqrt{I} : I) \subseteq (P:I) \subseteq I^{-1}$, and $I^{-1} = (P:I)$ is a ring. The converse follows from Proposition 2.1.

PROPOSITION 2.3. *Let S be an overring of R which is also a fractional ideal of R. Then $S_v = (S^{-1} : S^{-1})$; hence S_v is also an overring of R.*

Proof. Let $I = S^{-1}$. Then I is an integral ideal of R, and I is the conductor of the overring S in R. Hence by [B, Proposition 6], $I^{-1} = (I:I)$; that is, $S_v = (S^{-1} : S^{-1})$. ∎

COROLLARY 2.4 ([HuP, Proposition 2.2]). *If I is an ideal of R for which I^{-1} is a ring, then $I^{-1} = (I_v : I_v)$.*

Proof. Set $S = I^{-1}$ in Proposition 2.3. ∎

PROPOSITION 2.5. *Let $\{I_\alpha\}_{\alpha \in \mathscr{A}}$ be a set of divisorial ideals of R for which each I_α^{-1} is a ring and $I = \bigcap I_\alpha$ is nonzero. Let S denote the compositum of the rings I_α^{-1}. Then the following statements are equivalent*:

(1) I^{-1} *is a ring.*

(2) $I^{-1} = (I:I)$.

(3) $S \subseteq I^{-1}$.

(4) $S_v = I^{-1}$.

Proof. The equivalence of (1) and (2) follows from [HuP, Proposition 2.2] and the fact that I is divisorial. Suppose that I^{-1} is a ring. Then, since $I_\alpha^{-1} \subseteq I^{-1}$ for each α, we must have $I_{\alpha_1}^{-1} I_{\alpha_2}^{-1} \cdots I_{\alpha_k}^{-1} \subseteq I^{-1}$ for each finite subset $\{\alpha_1, \dots, \alpha_k\}$ of $\mathscr{A}$. Thus (1) $\Rightarrow$ (3). Now assume (3), and let $x \in S^{-1}$. Then $x I_{\alpha_1}^{-1} I_{\alpha_2}^{-1} \cdots I_{\alpha_k}^{-1} \subseteq R$ for each finite subset $\{\alpha_1, \dots, \alpha_k\}$ of $\mathscr{A}$. In particular, $x I_\alpha^{-1} \subseteq R$ for each α. Since each I_α is divisorial, this gives $x \in \bigcap I_\alpha = I$. Hence $S^{-1} \subseteq I$ and $S_v \supseteq I^{-1}$. On the other hand, since I^{-1} is divisorial, (3) implies that $S_v \subseteq I^{-1}$, yielding (4). Finally, (4) implies (1) by Proposition 2.3. ∎

COROLLARY 2.6. *Let I and J be ideals of R for which I^{-1} and J^{-1} are rings. Then $(I_v \cap J_v)^{-1}$ is a ring $\Leftrightarrow I^{-1}J^{-1} \subseteq (I_v \cap J_v)^{-1}$.*

Remark. It is possible to have $(I_v \cap J_v)^{-1}$ be a ring even though $(I \cap J)^{-1}$ is not—see Example 5.3 below.

3. RADICAL IDEALS

In this section, we consider (intersections of) radical ideals. We begin by characterizing when the inverse of a radical ideal is a ring.

THEOREM 3.1. *Let I be a radical ideal of R. The following statements are equivalent*:

(1) *I^{-1} is a ring.*

(2) *$I^{-1} \subseteq \bigcap \{R_P \mid P$ is a minimal prime of I and IR_P is principal$\}$.*

(3) *There does not exist a minimal prime P of I and an element $x \in K$ for which IR_P is principal and $I \subseteq (R :_R x) \subseteq P$.*

(4) *For each valuation overring V of R with $IV \neq V$, we have $I^{-1} \subseteq V_Q$, where Q is the prime of V which is minimal over IV.*

(5) *For each minimal prime P of I, there is a valuation overring V of R centered on P with $I^{-1} \subseteq V$.*

(6) *For each $x \in I^{-1}$, $x^2 \in I^{-1}$.*

Proof. (1) $\Rightarrow$ (2). By [A, Proposition 3.3(1)], $I^{-1} = (I : I)$. If P is a minimal prime of I with $IR_P = aR_P$, $a \in I$, then $I^{-1} = (I : I) \subseteq (IR_P : IR_P) = (aR_P : aR_P) = R_P$.

(2) $\Rightarrow$ (3). Let P and x be as described in (3). Then $(R :_R x) \subseteq P$ implies $x \notin R_P$, and $I \subseteq (R :_R x)$ implies $x \in I^{-1}$. Hence $I^{-1} \not\subseteq \bigcap \{R_P \mid P$ is minimal over I and IR_P is principal$\}$.

(3) $\Rightarrow$ (1). If I^{-1} is not a ring, then $II^{-1} \not\subseteq I$, whence $II^{-1} \not\subseteq P$ for some minimal prime P of I. It follows that IR_P is principal. Choose $x \in I^{-1}$ with $xI \not\subseteq P$. Of course, $I \subseteq (R :_R x)$. If $a \in (R :_R x)$, then $ax \in R$. Thus

$axI \subseteq I \subseteq P$; since $xI \nsubseteq P$, we have $a \in P$. Thus $(R:_R x) \subseteq P$. This shows that (3) implies (1).

(1) $\Rightarrow$ (4). This is true for general I by Proposition 2.1.

(4) $\Rightarrow$ (5). This is clear.

(5) $\Rightarrow$ (1). Suppose that I^{-1} is not a ring. Then we may choose $x \in I^{-1}$ and $a \in I$ with $xa \notin P$ for some minimal prime P of I. Let (V, M) be a valuation overring of R centered on P. Since $xa \notin P$, we have $xa \notin M$. It follows that $x \notin V$. Hence $I^{-1} \nsubseteq V$.

(1) $\Rightarrow$ (6). Clear.

(6) $\Rightarrow$ (1). Let $x \in I^{-1}$. It suffices to show that $xI \subseteq I$. By hypothesis, $x^2 \in I^{-1}$. Hence $x^2 I \subseteq R$, and $(xI)^2 \subseteq I$. Since $xI \subseteq R$ and I is radical, we have $xI \subseteq I$. ∎

We observe, as a consequence of Theorem 3.1, that if I is a radical ideal of R and IR_P is nonprincipal for each minimal prime P of I, then I^{-1} is a ring.

COROLLARY 3.2. *Let P be a prime ideal of R. The following statements are equivalent*:

(1) P^{-1} *is a ring.*

(2) *Either PR_P is not principal or P is not of the form $(aR:_R b)$ for $a, b \in R$.*

(3) *$P^{-1} \subseteq V$ for each valuation overring V of R whose maximal ideal is minimal over PV.*

For convenience, we state (without proof) a straightforward variation of Theorem 3.1.

THEOREM 3.3. *Let I be a nonzero radical ideal of R, and let $\{P_\alpha\}_{\alpha \in \mathscr{A}}$ be a set of minimal primes of I for which $I = \bigcap P_\alpha$. The following statements are equivalent*:

(1) I^{-1} *is a ring.*

(2) $I^{-1} \subseteq \bigcap\{R_{P_\alpha} \mid \alpha \in \mathscr{A}$ *and* IR_{P_α} *is principal*$\}$.

(3) *There does not exist an $\alpha \in \mathscr{A}$ and an element $x \in K$ for which IR_{P_α} is principal and $I \subseteq (R:_R x) \subseteq P_\alpha$.*

(4) *For each $\alpha \in \mathscr{A}$, there is a valuation domain (V_α, M_α) with $R \subseteq V_\alpha$, $M_\alpha \cap R = P_\alpha$, and $I^{-1} \subseteq V_\alpha$.*

THEOREM 3.4. *Let I and J be radical ideals of R. Then the following statements are equivalent*:

(1) I^{-1} *and* J^{-1} *are rings.*

(2) $(I \cap J)^{-1}$ *and* $(I + J)^{-1}$ *are rings.*

Moreover, if (either of) these statements hold, then $(I+J)^{-1} = ((I+J):(I+J))$.

Proof. Assume (1). Then $(I+J)^{-1} = I^{-1} \cap J^{-1}$ is a ring. Let $t \in (I \cap J)^{-1}$, $r \in (I \cap J)$, $a \in I$, and $b \in J$. Since $tab \in R$, $ta \in J^{-1} = (J:J)$, and $tb \in I^{-1} = (I:I)$. Hence $tar, tbr \in (I \cap J)$. Thus $t^2ar, t^2br \in R$, and we have $t^2r \in I^{-1} \cap J^{-1}$. It follows that $t^2r^2 \in I \cap J$, and since $I \cap J$ is a radical ideal and $tr \in R$, we have $tr \in I \cap J$. Hence $t \in ((I \cap J):(I \cap J))$. Therefore, $(I \cap J)^{-1} = ((I \cap J):(I \cap J))$ is also a ring, and (2) holds.

Now assume (2). It suffices to show that I^{-1} is a ring. Let $x \in I^{-1}$, $a \in I$, and $b \in J$. Since $I^{-1} \subseteq (I \cap J)^{-1}$ and $(I \cap J)^{-1}$ is a ring, we have $x^2 \in (I \cap J)^{-1}$. Hence $x^2ab \in R$, and so $x^2a \in J^{-1}$. Since $xa \in R$, $x \in I^{-1}$, and I^{-1} is an R-module, we obtain $x^2a \in I^{-1}$. Thus $x^2a \in I^{-1} \cap J^{-1} = (I+J)^{-1}$, and, since $(I+J)^{-1}$ is a ring, we have $x^4a^2 \in I^{-1} \cap J^{-1}$. Thus $x^4a^3 \in R$ and $x^4a^4 \in I$. Since I is a radical ideal of R and $xa \in R$, this yields $xa \in I$. Hence $x \in (I:I)$. It follows that $I^{-1} = (I:I)$, as desired.

To prove the last statement, note that $(I+J)^{-1} = I^{-1} \cap J^{-1} = (I:I) \cap (J:J)$ (since I and J are radical ideals). It is straightforward to show that this latter ideal is equal to $((I+J):(I+J))$. ∎

Remarks. (1) Although the implication (1) ⇒ (2) can be easily extended to an intersection of any finite number of radical ideals, we have not been able to extend it to infinite irredundant intersections. (In Proposition 3.13 we do show that if a nonzero ideal I is an irredundant intersection of prime ideals P with each P^{-1} a ring, then I^{-1} is also a ring.)

(2) The radical assumptions are necessary. In Example 5.2, we show that it is possible for $(I \cap J)^{-1}$ to fail to be a ring even though I and J are divisorial ideals with I^{-1} and J^{-1} both rings; and in Example 5.3, we exhibit (non-divisorial) ideals I and J for which $I^{-1} = (I:I)$, $J^{-1} = (J:J)$ (so that I^{-1} and J^{-1} are rings), $(I_v \cap J_v)^{-1}$ is a ring, but $(I \cap J)^{-1}$ is not a ring.

(3) The implication (2) ⇒ (1) may not hold for an intersection of more than two radical ideals, as the following example shows.

EXAMPLE 3.5. Let X be an indeterminate over $\mathbb{Q}$, and set $T = \mathbb{Q}[X] = \mathbb{Q} + X\mathbb{Q}[X]$ and $R = \mathbb{Z} + X\mathbb{Q}[X]$. Consider the ideals I, J, K of R given by $I = 2\mathbb{Z} + X\mathbb{Q}[X] = 2R$, $J = 3\mathbb{Z} + X\mathbb{Q}[X] = 3R$, and $L = X\mathbb{Q}[X]$. Then I, J, L are prime ideals with $I \cap J \cap L = L$. It is easy to see that $(I \cap J \cap L)^{-1} = L^{-1} = (L:L) = T$. Since I and J are comaximal, $I + J + L = R$, so that $(I+J+L)^{-1} = R$. Hence $(I \cap J \cap L)^{-1}$ and $(I+J+L)^{-1}$ are rings, but, since I and J are principal ideals, I^{-1} and J^{-1} are not rings.

COROLLARY 3.6. *If I and J are ideals of R for which I^{-1} and J^{-1} are rings, then $(\sqrt{I} \cap \sqrt{J})^{-1}$ is a ring.*

Proof. This follows from Propositions 2.1(1) and 3.4. ∎

THEOREM 3.7. *Let I and J be ideals of R for which $I^{-1} \cap J^{-1} = R$. Then the following statements are equivalent.*

(1) *I^{-1} and J^{-1} are rings.*

(2) *$(I \cap J)^{-1}$ is a ring.*

(3) *$(I_v \cap J_v)^{-1}$ is a ring.*

Moreover, if the statements hold, then $(I \cap J)^{-1} = (I_v \cap J_v)^{-1} = (IJ)^{-1} = (I_v J_v)^{-1}$.

Proof. (1) ⇒ (2). Let $x, y \in (I \cap J)^{-1}$, $z \in I \cap J$, $a, b \in I$, and $c, d \in J$. Then $xac, ybd \in R$. Hence $xa, yb \in J^{-1}$ and $xc, yd \in I^{-1}$. Thus $zxyab, zxycd \in R$, whence $zxya \in I^{-1}$ and $zxyc \in J^{-1}$. Since $xa \in J^{-1}$, $xc \in I^{-1}$, $zy \in R$, and I^{-1} and J^{-1} are R-modules, we have $zxya, zxyc \in I^{-1} \cap J^{-1} = R$. It now follows that $zxy \in I^{-1} \cap J^{-1} = R$, whence $xy \in (I \cap J)^{-1}$. Therefore, $(I \cap J)^{-1}$ is a ring.

(2) ⇒ (1). We show that I^{-1} is a ring. Let $x, y \in I^{-1}$, $a \in I$, and $b \in J$. Since $I^{-1} \subseteq (I \cap J)^{-1}$ and $(I \cap J)^{-1}$ is a ring, we have $xy \in (I \cap J)^{-1}$, whence $xyab \in R$. Thus $xya \in J^{-1}$. Since $x \in I^{-1}$, $ya \in R$, and I^{-1} is an R-module, we also have $xya \in I^{-1}$. Hence $xya \in R$. Therefore, $xy \in I^{-1}$, as desired.

The equivalence of (1) and (3) follows from the equivalence of (1) and (2) and the fact $A^{-1} = (A_v)^{-1}$ for any ideal A. To prove the last statement, we first note that it is clear that $(I_v \cap J_v)^{-1} \subseteq (I \cap J)^{-1} \subseteq (IJ)^{-1}$. Let $x \in (IJ)^{-1}, z \in I_v \cap J_v$. Then $xIJ \subseteq R$, from which it follows that $xI_v \subseteq J^{-1}$ and $xJ_v \subseteq I^{-1}$. Hence $xz \in I^{-1} \cap J^{-1} = R$. Thus $x \in (I_v \cap J_v)^{-1}$. It follows that $(I_v \cap J_v)^{-1} = (I \cap J)^{-1} = (IJ)^{-1}$. The remaining equality follows from standard facts about star operations [G1, Proposition 32.2]. ∎

The following example shows that Theorem 3.7 cannot be extended to the case of an arbitrary finite number of ideals.

EXAMPLE 3.8. Let D be a domain with quotient field k, $D \neq k$, and let X, Y, Z be indeterminates over k. Let $T = D[X, Y, Z] = D[X, Y] + P$, where $P = ZT$, and let $R = D + P$. Consider the ideals I, J, L of R given by $I = Z(D[X] + P)$, $J = Z(D[Y] + P)$, and $L = aD + P$, where a is any nonzero nonunit of D. We shall show that $I^{-1} = J^{-1} = T$ and that $L^{-1} = R$. Since $IT = P$, we have $I^{-1} \subseteq (T : IT) = (T : P) = Z^{-1}T$. Let $f \in I^{-1}$, and write $f = h/Z$ with $h \in T$. Write $h = h_0 + m$, where $h_0 \in$

$D[X, Y]$ and $m \in P$. Since $ZX \in I$, we have $h_0 X + mX = hX = fZX \in R$, from which it follows that $h_0 X \in D$. Hence $h_0 = 0$, and we have $h \in P$, i.e., $f \in T$. Thus $I^{-1} \subseteq T$. The reverse inclusion follows from the fact that $IT = P \subseteq R$. Thus $I^{-1} = T$, and, similarly, $J^{-1} = T$. Now by [HKLM, Theorem 1], we have $P^{-1} = (P : P) = T$ (P is being considered as an ideal of R). Since $P \subseteq L$, we have $L^{-1} \subseteq P^{-1} = T$. Let $f \in L^{-1}$. Then $f \in T$, and we may write $f = f_0 + n$ with $f_0 \in D[X, Y]$ and $n \in P$. Since $a \in L$, $fa \in R$, whence $f_0 a \in D$. Hence $f_0 \in D$, and $f \in R$. It follows that $L^{-1} = R$. Thus $I^{-1} \cap J^{-1} \cap L^{-1} = R$, and I^{-1}, J^{-1}, and L^{-1} are rings. However, $I \cap J \cap L = I \cap J$ is the principal ideal ZR, so that $(I \cap J \cap L)^{-1}$ is not a ring.

COROLLARY 3.9. *Let I and J be ideals of R for which I_v and J_v are comaximal. Then the following statements are equivalent.*

(1) *I^{-1} and J^{-1} are rings.*
(2) *$(I \cap J)^{-1}$ is a ring.*
(3) *$(I_v \cap J_v)^{-1}$ is a ring.*

Proof. We have $I^{-1} \cap J^{-1} = (I_v)^{-1} \cap (J_v)^{-1} = (I_v + J_v)^{-1} = R$. ∎

COROLLARY 3.10. *Let R be a completely integrally closed domain, and let I and J be ideals of R. Then the following statements are equivalent:*

(1) *I^{-1} and J^{-1} are rings.*
(2) *$I^{-1} = J^{-1} = R$*
(3) *$(I \cap J)^{-1}$ is a ring.*
(4) *$(I \cap J)^{-1} = R$.*

Proof. The implications (1) ⇒ (2) and (3) ⇒ (4) follow from [A, Corollary 2.4], and the implication (2) ⇒ (3) follows from Theorem 3.7. It is straightforward to show that (4) (⇒ (2)) ⇒ (1). ∎

PROPOSITION 3.11. *Let I be a radical ideal of R such that I^{-1} is a ring. If $I = A \cap B$ for ideals A and B, then $(I :_R A)$ is a radical ideal, $(I :_R A)^{-1}$ is a ring, and $I = (I :_R A) \cap A$.*

Proof. Set $B' = (I :_R A)$, and let $r \in B' \cap A$. Then $rA \subseteq I$ and $r \in A$ together imply $r^2 \in I$; since I is a radical ideal, we have $r \in I$. Hence $I = B' \cap A$. Now let $u \in (B')^{-1}$. Then, since $(B')^{-1} \subseteq I^{-1}$, we have $uI \subseteq I$. Thus if $y \in B'$, then $yA \subseteq I$, and $uyA \subseteq I$; that is, $uy \in (I :_R A) = B'$. Thus $(B')^{-1} = (B' : B')$. Finally, to see that B' is a radical ideal, observe that $z^n \in B'$ implies $z^n A \subseteq I$, whence $(zA)^n = z^n A^n \subseteq I$. Since I is a radical ideal, this yields $zA \subseteq I$, and $z \in B'$. ∎

COROLLARY 3.12. *Let I be a radical ideal of R such that $I = A \cap B$ for ideals A and B. Then I^{-1} is a ring ⇔ there are radical ideals $A_1 \supseteq A$ and $B_1 \supseteq B$ of R such that $I = A_1 \cap B_1$ and $(A_1)^{-1}$ and $(B_1)^{-1}$ are rings.*

Proof. Suppose that I^{-1} is a ring, and set $B_1 = (I :_R A)$. Then $B_1 \supseteq B$, and by Proposition 3.11, we have $I = A \cap B_1$ with B_1 a radical ideal and $(B_1)^{-1}$ a ring. Now set $A_1 = (I :_R B_1)$, and apply Proposition 3.11 again. The converse follows from Theorem 3.4. ∎

Remark. In the notation of Corollary 3.12, it need not be the case that A^{-1} is a ring. A counterexample is presented in Example 5.7 below.

PROPOSITION 3.13. *Let $\{P_\alpha\}_{\alpha \in \mathscr{A}}$ be a set of prime ideals in R, and let $I = \bigcap_{\alpha \in \mathscr{A}} P_\alpha$ be a nonzero irredundant intersection. The following statements are equivalent*:

(1) *I^{-1} is a ring.*

(2) *P_α^{-1} is a ring for each $\alpha \in \mathscr{A}$.*

(3) *$(\bigcap_{\beta \in \mathscr{B}} P_\beta)^{-1}$ is a ring for each subset $\mathscr{B}$ of $\mathscr{A}$.*

Proof. (1) ⇒ (2). This follows from Proposition 2.1, in view of the fact that irredundancy forces each P_α to be minimal over I.

(2) ⇒ (3). Put $J = \bigcap_{\beta \in \mathscr{B}} P_\beta$, and let $z \in J^{-1}$. Fix $\beta \in \mathscr{B}$, let $\mathscr{C} = \mathscr{B} \setminus \{\beta\}$, and choose $b \in (\bigcap_{\gamma \in \mathscr{C}} P_\gamma) \setminus P_\beta$. Then $bP_\beta \subseteq J$. Hence $zbP_\beta \subseteq R$ and $zb \in P_\beta^{-1}$. Since P_β^{-1} is a ring, we have $zbP_\beta \subseteq P_\beta$. Thus $zbJ \subseteq P_\beta$; since $zJ \subseteq R$ and $b \notin P_\beta$, we have $zJ \subseteq P_\beta$. Since this is true for each β, $zJ \subseteq J$, and $J^{-1} = (J : J)$ is a ring.

(3) ⇒ (1). Clear. ∎

PROPOSITION 3.14. *Let I be a radical ideal of R, and let $\{P_\alpha\}_{\alpha \in \mathscr{A}}$ be a set of minimal primes of I with $I = \bigcap_{\alpha \in \mathscr{A}} P_\alpha$. Then I^{-1} is a ring ⇔ $(\bigcap_{\beta \in \mathscr{B}} P_\beta)^{-1}$ is a ring for each proper subset $\mathscr{B}$ of $\mathscr{A}$.*

Proof. (⇒) Set $J = (\bigcap_{\beta \in \mathscr{B}} P_\beta)$. Applying Theorem 3.3 we have $J^{-1} \subseteq I^{-1} \subseteq \bigcap\{R_{P_\alpha} \mid \alpha \in \mathscr{A}$ and IR_{P_α} is principal$\} \subseteq \bigcap\{R_{P_\beta} \mid \beta \in \mathscr{B}$ and JR_{P_β} is principal$\}$, the last inequality following from the fact that $IR_{P_\beta} = JR_{P_\beta} = P_\beta R_{P_\beta}$ for each β. Invoking Theorem 3.3 again, we have that J^{-1} is a ring.

(⇐) Pick $\alpha \in \mathscr{A}$, and set $\mathscr{B} = \mathscr{A} \setminus \{\alpha\}$ and $J = \bigcap_{\beta \in \mathscr{B}} P_\beta$. By hypothesis, P_α^{-1} and J^{-1} are rings, and since P_α and J are radical ideals, Theorem 3.4 assures that $I^{-1} = (P_\alpha \cap J)^{-1}$ is a ring. ∎

Remarks. (1) In spite of the preceding two results, we present in Section 5 an example of a radical ideal I for which I^{-1} is not a ring while P^{-1} is a ring for each minimal prime P of I (Example 5.1). We show in Proposition 3.15 below, however, that divisoriality of the P_α does force I^{-1} to be a ring.

(2) From Proposition 3.14, one might suspect that if I is a radical ideal for which I^{-1} is a ring, and if J is a radical ideal trapped between I and some minimal prime of I, then J^{-1} should also be a ring. Example 5.7 below, however, shows that this is not necessarily the case.

(3) The simplest way to ensure that each P_α^{-1} be a ring in Propositions 3.13 and 3.14 is to have each $P_\alpha^{-1} = R$. However, even though for an ideal I equal to the irredundant intersection of such P_α, we must have that I^{-1} is a ring, it need not be the case that $I^{-1} = R$. For such an I, see Example 5.4 below.

PROPOSITION 3.15. *Let I be a nonzero ideal of R, and suppose that $I = \bigcap I_\alpha$, where each I_α is a divisorial radical ideal with I_α^{-1} a ring. Then I^{-1} is a ring.*

Proof. Let $\{I_{\alpha_1}, \dots, I_{\alpha_t}\}$ be a finite subset of $\{I_\alpha\}$. By Theorem 3.4 $(I_{\alpha_1} \cap \cdots \cap I_{\alpha_t})^{-1}$ is a ring. By Proposition 2.5, this implies that $I_{\alpha_1}^{-1} \cdots I_{\alpha_t}^{-1} \subseteq (I_{\alpha_1} \cap \cdots \cap I_{\alpha_t})^{-1} \subseteq I^{-1}$. Another application of Proposition 2.5 shows that I^{-1} is a ring. ∎

4. THE INTEGRALLY CLOSED CASE

In this section, we characterize when I^{-1} is a ring when I is an ideal in an integrally closed domain R (Theorem 4.4); we then study the situation in Prüfer v-multiplication domains. We begin with a result in the seminormal case.

PROPOSITION 4.1. *If I is a nonzero ideal of the seminormal domain R for which I^{-1} is a ring, then $I^{-1} \subseteq \bigcap\{R_P \mid P$ is minimal over I and PR_P is principal$\}$.*

Proof. Let P be minimal over I with PR_P principal. Then by [FHPR, Corollary 3.4(1)], we have $I^{-1} = (\sqrt{I} : \sqrt{I}) \subseteq (\sqrt{I}R_P : \sqrt{I}R_P) = (PR_P : PR_P) = R_P$. ∎

Remark. The converse of Proposition 4.1 is false. For an example, let V be a valuation domain of the form $V = K + M$, where M is the maximal ideal of V and K is a field; we further assume that M is nonprincipal in V and that M is branched (so that M is minimal over a principal ideal of V). Let F be a subfield of K which is algebraically closed in K, and set $R = F + M$. Then R is integrally closed. Choose $a \in M$ with M minimal over Va, and let $I = Ma$. Then I is a divisorial ideal of R (but is not divisorial in V). Note that $I^{-1} = M^{-1}a^{-1} = Va^{-1}$, whence I^{-1} is not a ring. Now MR_M is not principal, so that $\bigcap\{R_P \mid P$ is minimal over I and PR_P is principal$\}$ is the quotient field of R. Thus $I^{-1} \subseteq \bigcap\{R_P \mid P$ is minimal over I and PR_P is principal$\}$, but I^{-1} is not a ring. Also note that $I^{-1}I = Va^{-1}Ma = M$; thus $I^{-1} = (M : I) = (\sqrt{I} : I)$. Since $M = \sqrt{I}$ and $M^{-1} = V$ is a ring, we see that the first three conditions of Proposition 2.1 are satisfied.

In [FHPR, Lemma 3.5], the following result is proved: if R is seminormal, I is an ideal of R for which I^{-1} is a ring, and J is an ideal of R with $I \subseteq J \subseteq \sqrt{I}$, then $J^{-1} = I^{-1}$ (so that J^{-1} is also a ring). We observe that the seminormal hypothesis cannot be removed. To see this, let F be a field, let X be an indeterminate over F, and let $R = F[X^3, X^4]$. Let $I = (X^6, X^7, X^8)$ and $J = (X^3, X^6, X^7, X^8)$. Then $I \subseteq J \subseteq \sqrt{I} = (X^3, X^4)$. It is easy to check that $I^{-1} = F[X]$, so that I^{-1} is a ring. However, $X \in J^{-1}$, but $X^2 \notin J^{-1}$.

We next turn our attention to some results in the integrally closed case. Recall that for an ideal I of an integrally closed domain R, the *completion* of I is the ideal $I^* = \bigcap IV$, where the intersection is taken over all valuation overrings V of R. (For a discussion of completion, see [G1, Sect. 24].)

For convenience, we state without proof several (probably well-known) easily verified facts about completions.

LEMMA 4.2. *Let I be an ideal of the integrally closed domain R. Then*

(1) $(I^* : I^*) = \bigcap (IV : IV)$, *where the intersection is taken over all valuation overrings V of R*;

(2) $(I : I) \subseteq (I^* : I^*)$;

(3) *if I is a radical ideal, then I is complete*; *and*

(4) $(\sqrt{I} : \sqrt{I}) = ((\sqrt{I})^* : (\sqrt{I})^*) = \bigcap (\sqrt{I}V : \sqrt{I}V)$.

For an ideal I of a domain R, set $\mathscr{V}(I) = \{(V, M) | (V, M)$ is a valuation overring of R whose maximal ideal M is minimal over $IV\}$ and $\mathscr{W}(I) = \{W | W$ is a valuation overring of R with $IW = W\}$. Observe that $I^{-1} \subseteq W$ for each $W \in \mathscr{W}(I)$. When no confusion is likely, we will write $\mathscr{V}$ for $\mathscr{V}(I)$ and $\mathscr{W}$ for $\mathscr{W}(I)$.

LEMMA 4.3. *If I is an ideal of the integrally closed domain R, then* $I^{-1} \supseteq (\bigcap_{V \in \mathscr{V}} V) \cap (\bigcap_{W \in \mathscr{W}} W)$.

Proof. Let x be an element of the given intersection, and let $a \in I$. Let U be any valuation overring of R. If $U \in \mathscr{W}$, then $x \in U$, whence $xa \in U$. If $U \notin \mathscr{W}$, then $IU \neq U$. Let Q be the prime of U minimal over IU. Then $U_Q \in \mathscr{V}$, whence $x \in U_Q$. It follows that $xa \in QU_Q = Q \subseteq U$. Thus xa is in every valuation overring of R; since R is integrally closed, this implies that $xa \in R$. Thus $x \in I^{-1}$, as desired. ∎

THEOREM 4.4. *Let I be an ideal of the integrally closed domain R. Then the following statements are equivalent.*

(1) I^{-1} *is a ring.*

(2) $I^{-1} \subseteq (II^{-1}V : II^{-1}V)$ *for each valuation overring V of R.*

(3) $I^{-1} \subseteq (I_v V : I_v V)$ *for each valuation overring* V *of* R.

(4) $\exists$ *an ideal* J *of* R *for which* $J \supseteq I$ *and* $I^{-1} \subseteq (JV : JV)$ *for each valuation overring* V *of* R.

(5) $I^{-1} \subseteq \bigcap_{V \cap \mathscr{V}} V$.

(6) $I^{-1} = (\bigcap_{V \in \mathscr{V}} V) \cap (\bigcap_{W \in \mathscr{W}} W)$.

Moreover, if I^{-1} *is a ring, then* $I^{-1} = J^{-1}$ *for each ideal* $J \supseteq I$ *such that* $I^{-1} \subseteq (JV : JV)$ *for each valuation overring* V *of* R.

Proof. By [HuP, Proposition 2.2], (1) implies (2) and (3), and it is clear that both (2) and (3) imply (4). Let J be an ideal as given in (4). By Lemma 4.2, we have $(J^* : J^*) = \bigcap(JV : JV)$. Hence $I^{-1} \subseteq (J^* : J^*) \subseteq (J^*)^{-1} \subseteq J^{-1} \subseteq I^{-1}$. Thus I^{-1} is a ring, and we have $I^{-1} = J^{-1}$ for each such J. Thus (4) $\Rightarrow$ (1). Therefore, statements (1)–(4) are equivalent, and the "moreover" statement has been proved. The equivalence of (5) and (6) follows easily from Lemma 4.3, and it is obvious that (6) $\Rightarrow$ (1). Hence it suffices to show that (4) $\Rightarrow$ (5). Again, let J be as given in (4). Let $V \in \mathscr{V}(I)$. If $JV = V$, then $I^{-1} \subseteq (JV : JV) = V$. If $JV \neq V$, then, since the maximal ideal of V is minimal over IV, it is also minimal over JV. Hence $J^{-1} \subseteq V$ by Proposition 2.1. As shown above, $I^{-1} = J^{-1}$, and hence $I^{-1} \subseteq V$. ∎

Remark. Lemma 4.3 may be regarded as an extension of [HuP, Lemma 3.3] to the integrally closed case; similarly, the equivalences (1), (5), and (6) represent an extension of [HuP, Theorem 3.2].

In Example 5.6 below, we use an example of Heinzer and Papick to show the necessity of the v's in statement (3) of Theorem 4.4; that is, we show that I^{-1} a ring does not imply $I^{-1} \subseteq (IV : IV)$ for every valuation overring V.

We now wish to generalize the above-mentioned results of [HuP] to Prüfer v-multiplication domains. We first recall the t-operation: for a nonzero fractional ideal I of a domain R, set $I_t = \bigcup\{J_v | J$ is a nonzero finitely generated subideal of $I\}$; I is called a t-ideal if $I = I_t$. Of course, the t-operation is an example of a star-operation (see [10] or [13]). Of particular importance are the well-known facts that every t-ideal is contained in a maximal t-ideal, that maximal t-ideals are prime, and that any prime minimal over a t-ideal is a prime t-ideal (t-prime). Recall that a domain R is a Prüfer v-multiplication domain (PVMD) $\Leftrightarrow$ R_M is a valuation domain for each (maximal) t-prime M of R [Gr, Theorem 5].

THEOREM 4.5. *Let* I *be an ideal of the PVMD* R. *Let* $\{P_\alpha\}$ *denote the set of minimal primes of* I, $\{Q_\beta\}$ *the set of minimal primes of* I_t, *and* $\{M_\gamma\}$ *the set of maximal* t*-ideals of* R *which do not contain* I. *The following statements are* *equivalent*:

(1) I^{-1} *is a ring*.

(2) $I^{-1} = (\bigcap R_{P_\alpha}) \cap (\bigcap R_{M_\gamma})$.

(3) $I^{-1} = (\bigcap R_{Q_\beta}) \cap (\bigcap R_{M_\gamma})$.

Proof. We first observe that $Q_\beta \in \{P_\alpha\}$ for each β. To see this, note that $Q_\beta \supseteq Q$ for some prime Q minimal over I. As a minimal prime of a t-ideal, Q_β is a t-prime. Hence R_{Q_β} is a valuation domain. It follows that R_Q is also a valuation domain, and it is well known that this implies that Q is itself a t-prime. Thus $Q \supseteq I_t$, and we have that $Q_\beta = Q$ is minimal over I. Now suppose that I^{-1} is a ring, and let $x \in I^{-1}$. We wish to show that $x \in R_{P_\alpha}$ for each α. If $(P_\alpha)_t = R$, then there is a finitely generated ideal $A \subseteq P_\alpha$ with $A_v = R$. Since P_α is minimal over I, there is an element $s \in R \setminus P_\alpha$ and a positive integer n for which $sA^n \subseteq I$. Hence $xsA^n \subseteq R$. Since $(A^n)_v = R$, this gives $xs \in R$, whence $x \in R_{P_\alpha}$. If $(P_\alpha)_t \neq R$, then as in the argument given above for Q, we have that R_{P_α} is a valuation domain. By Theorem 4.4, $I^{-1} \subseteq R_{P_\alpha}$ in this case as well. It follows that $I^{-1} \subseteq (\bigcap R_{P_\alpha}) \cap (\bigcap R_{M_\gamma}) \subseteq (\bigcap R_{Q_\beta}) \cap (\bigcap R_{M_\gamma})$. Now let $y \in (\bigcap R_{Q_\beta}) \cap (\bigcap R_{M_\gamma})$ and $a \in I$. To show that $ya \in R$, it suffices to show that $ya \in R_M$ for each maximal t-ideal M of R. This is clear if $M = M_\gamma$ for some γ. If $I \subseteq M$, then $M \supseteq Q_\beta$ for some β. Since $y \in R_{Q_\beta}$, we have $ya \in IR_{Q_\beta} \subseteq Q_\beta R_{Q_\beta} = Q_\beta R_M$ (using the fact that $Q = QV_Q$ for each prime ideal Q in a valuation domain V). Thus $ya \in R_M$. It follows that (1) implies (2) and (3). Of course, it is clear that either (2) or (3) implies (1). ∎

Remark 4.6. (1) In Theorem 4.5, although each minimal prime of I_t is in fact minimal over I, a minimal prime of I need not contain I_t, even when I^{-1} is a ring properly containing R. Example 5.8 below is an example of a PVMD R containing an ideal I and a minimal prime M of I such that I^{-1} is a ring but $M_t = R$.

(2) For a general integrally closed domain R and t-ideal I of R, I^{-1} need not be contained in $\bigcap R_P$, where the intersection is taken over the minimal (necessarily t-) primes P of I. For an example, let $F \subseteq K$ be fields with F algebraically closed in K, let (V, M) be a valuation domain of the form $K + M$, and let $R = F + M$. Then M is divisorial (and therefore a t-ideal), but $M^{-1} = V \nsubseteq R_M = R$.

(3) For any ideal I of a domain R, if P is minimal over I with $I^{-1} \neq R$ but $I^{-1} \nsubseteq R_P$, we have $P_t \neq R$. To see this, suppose that A is a finitely generated ideal contained in P with $A^{-1} = R$. As in the proof of Theorem 4.5, we have $sA^n \subseteq I$ for some $s \notin P$. Then $sA^nI^{-1} \subseteq R$, and since

$(A^n)_v = R$, this implies that $I^{-1} \subseteq s^{-1}R \subseteq R_P$, a contradiction. Hence $A^{-1} \neq R$. Thus $P_t \neq R$.

We continue to study when I^{-1} is a ring, where I is a nonzero ideal of a PVMD R. In particular, we wish to generalize a theorem of Fontana–Huckaba–Papick–Roitman [FHPR, Theorem 4.11]. We shall use the following notation and notions from [FHPR]:

$$\mathscr{Z}(R, I) = \text{the set of zero divisors on the } R\text{-module } R/I,$$
$$\mathbf{Z}(R, I) = \{P \in \mathrm{Spec}(R) | I \subseteq P \subseteq \mathscr{Z}(R, I)\},$$
$$\mathscr{N}(R, I) = R \setminus \mathscr{Z}(R, I),$$
$$\mathscr{C}_t(R, I) = \bigcap\{R_M | M \text{ is a maximal } t\text{-ideal of } R \text{ with } I \not\subseteq M\}.$$

We say that I *has no embedded primes* if each element of $\mathbf{Z}(R, I)$ is minimal over I. Finally, we note that $\mathscr{N} = \mathscr{N}(R, I)$ is a multiplicatively closed subset of R, and we let $\mathbf{M}(\mathscr{N})$ denote the set of maximal elements in the set of ideals which have empty intersection with $\mathscr{N}$.

THEOREM 4.7 (cf. [FHPR, Theorem 4.11]). *Let R be a PVMD, and let I be an ideal of R with no embedded primes. Then*

(1) $(I : I) = (\sqrt{I} : \sqrt{I}) = R_{\mathscr{N}} \cap \mathscr{C}_t(R, I)$ *and*

(2) I^{-1} *is a ring* $\Leftrightarrow I^{-1} = (I : I)$.

Proof. (1) By [FHPR, Theorem 3.1], $(I : I) \subseteq (\sqrt{I} : \sqrt{I})$. Now let $y \in (\sqrt{I} : \sqrt{I})$, let $M \in \mathscr{C}_t(R, I)$, and let $b \in \sqrt{I} \setminus M$. Then $yb \in \sqrt{I} \subseteq R$, whence $y \in R_M$. Thus $(\sqrt{I} : \sqrt{I}) \subseteq \mathscr{C}_t(R, I)$. By [G1, Corollary 4.6], $R_{\mathscr{N}} = \bigcap\{R_Q | Q \in \mathbf{M}(\mathscr{N})\}$. Let $Q \in \mathbf{M}(N)$. By [FHPR, Lemma 4.6], $I \subseteq Q$, and, since Q is prime, we have $Q \in \mathbf{Z}(R, I)$. Since I has no embedded primes, Q is minimal over I. Since Q is minimal over I, we have $\sqrt{I}R_Q = QR_Q$. Thus $y \in (\sqrt{I} : \sqrt{I}) \subseteq (\sqrt{I}R_Q : \sqrt{I}R_Q) = (QR_Q : QR_Q)$. If $Q = Q_t$, then R_Q is a valuation ring, and $(QR_Q : QR_Q) = R_Q$, and we have $y \in R_Q$. If $Q \neq Q_t$, then, since maximal primes of t-ideals are t-primes, we have $I_t \not\subseteq Q$. Since $(\sqrt{I} : \sqrt{I}) \subseteq (\sqrt{I})^{-1} \subseteq I^{-1} = (I_t)^{-1}$, $y \in (I_t)^{-1}$. If $s \in I_t \setminus Q$, then $sy \in R$, whence, again, $y \in R_Q$. Thus $(\sqrt{I} : \sqrt{I}) \subseteq R_{\mathscr{N}}$, and we have $(\sqrt{I} : \sqrt{I}) \subseteq R_{\mathscr{N}} \cap \mathscr{C}_t(R, I)$. Now let $z \in R_{\mathscr{N}} \cap \mathscr{C}_t(R, I)$ and $a \in I$. We claim that $za \in R$. For this it suffices to show that $za \in R_M$ for each maximal t-ideal M of R [Gr, Theorem 5]. Let M be a maximal t-ideal. If $I \not\subseteq M$, then $za \in \mathscr{C}_t(R, I) \subseteq R_M$. Suppose $I \subseteq M$ and that $za \notin R_M$. Then, since R_M is a valuation ring, $z^{-1}a^{-1} \in R_M$. Since $z \in R_{\mathscr{N}}$, $\exists t \in N$ with $tz \in R$. Hence $tza \in I \subseteq IR_M$, and we have $t = (za)^{-1}tza \in IR_M$. This produces $u \in R \setminus M$ with $ut \in I$. But then, since $t \in N$, we have $u \in I \subseteq M$, a contradiction. Hence $za \in R_M$. Thus $za \in R$, as claimed. Since $tza \in I$

and $t \in N$, we have $za \in I$. Hence $z \in (I:I)$, and we have $R_{\mathcal{N}} \cap \mathscr{C}_t(R, I) \subseteq (I:I)$. This gives (1).

(2) If I^{-1} is a ring, then by Theorem 4.5, $I^{-1} = (\bigcap R_{P_\alpha}) \cap \mathscr{C}_t(R, I)$, where $\{P_\alpha\}$ is the set of minimal primes of I. Recall that for $Q \in \mathbf{M}(N)$, Q is minimal over I, whence $I^{-1} \subseteq R_Q$. Thus $I^{-1} \subseteq R_{\mathcal{N}}$, and by (1) we have $I^{-1} \subseteq R_{\mathcal{N}} \bigcap \mathscr{C}_t(R, I) = (I:I)$. ∎

COROLLARY 4.8 (cf. [HuP, Corollary 3.4]). *If M is a maximal ideal of a PVMD R, then either M is invertible or $M^{-1} = R$.*

Proof. Suppose that M is not invertible, so that $M^{-1} = (M:M)$. By Theorem 4.5 or 4.7, this gives $M^{-1} = R_M \cap \mathscr{C}_t(R, M)$. If M is a t-ideal this yields $M^{-1} = R$ [Gr, Theorem 5]. Of course, if M is not a t-ideal, then $M_v \supseteq M_t = R$, and again we have $M^{-1} = R$. ∎

In Example 4.9 below, we show that it is not enough to assume in Corollary 4.8 that M is a maximal t-ideal.

Corollary 3.2 asserts that P is a prime ideal of a domain R such that PR_P is not principal, then P^{-1} is a ring. The following two examples show that it is possible to have PR_P principal with P^{-1} a ring or not, where P is a maximal t-ideal of a PVMD.

EXAMPLE 4.9. An example of a PVMD R containing a maximal t-ideal P such that P is not invertible, PR_P is principal, and $P^{-1} = R$.

Let R be an almost Dedekind domain which is not a Dedekind domain. Then R is a PVMD (since it is a Prüfer domain). Since R is not a Dedekind domain, there is a maximal ideal P of R which is not invertible. Since P is maximal and has height 1, P is a maximal t-ideal. Of course, PR_P is principal by definition. Finally, $P^{-1} = R$ by Corollary 4.8.

EXAMPLE 4.10. An example of a PVMD R containing a maximal t-ideal P such that P is not invertible, PR_P is principal, and P^{-1} is not a ring.

Let $T = \mathbb{Q}[Y] = \mathbb{Q} + M$, where $M = Y\mathbb{Q}[Y]$, and let $S = \mathbb{Z} + M$. By [CMZ, Theorem 4.43], S is a PVMD. Hence $R = S[X]$ is also a PVMD. Let $f = YX + (1/2)Y \in R$, and let $P = f\mathbb{Q}(Y)[X] \cap R$. Then P is an upper to zero, and by [Q, Lemma 1], $P = f(Y, (1/2)Y)^{-1}R$. It is easy to see that $(Y, (1/2)Y)^{-1} = M^{-1} = T$, so that $(Y, (1/2)Y)^{-1}(Y, (1/2)Y) \subseteq MM^{-1} = M$, and $(Y, (1/2)Y)^{-1}$ is not invertible in S. Hence P is not invertible in R. By [HMM, Proposition 2.6] and [HZ, Theorem 1.4], P is a maximal t-ideal and $(PP^{-1})_t = R$. Thus $PP^{-1} \not\subseteq P$, and P^{-1} is not a ring. Finally, that PR_P is principal follows from the well-known fact that R_P is a DVR.

5. EXAMPLES

In this section, we give several examples tending to show that (many of) the results in Sections 2–4 are the best possible. In what follows, we use F to denote a field and (possibly subscripted) capital letters X, Y, Z, and W to denote indeterminates over F.

EXAMPLE 5.1. An example of a domain R containing a radical ideal I for which I^{-1} is not a ring but P^{-1} is a ring for each minimal prime of I. Let R denote the semigroup ring $\mathbb{Q}[\mathbb{Q}_0] = \mathbb{Q}[\{X^\alpha | \alpha \in \mathbb{Q}_0\}]$. (Here, $\mathbb{Q}_0$ denotes the set of non-negative rational numbers.) Set $I = (X-1)R$. Since I is principal, I^{-1} is not a ring. However, we shall show that I is a radical ideal and that $P^{-1} = R$ for each minimal prime P of R. By [G2, Theorem 13.5], R is a Bézout domain. For $n \geq 1$, set $R_n = \mathbb{Q}[X^{1/n}]$. Then $R_n \simeq R_1$ is a PID, and $R = \bigcup R_n$. Let $I_n = (X-1)R_n$. The fact that I_1 does not ramify in R_n implies that I_n is a radical ideal of R_n. (That I_1 does not ramify in R_n means that each irreducible factor of $X - 1$ in R_n occurs to the first power. After an application of the isomorphism $X^{1/n} \mapsto X$ from R_n to R_1, this means that each irreducible factor of $X^n - 1$ occurs to the first power in R_1. Of course, this follows from the well-known fact that the factors of $X^n - 1$ are just the cyclotomic polynomials g_d for $d|n$.) It follows easily that $I = \bigcup I_n$ is a radical ideal of R. Now let P be a prime ideal of R containing I. By [G2, Theorems 17.1 and 21.4], R is one dimensional. Hence P is maximal, and to show that P^{-1} is a ring, we need only show that P is not invertible. Thus, since R is Bézout, we need only show that P is not principal. Granting that P^{-1} is a ring, we have $P^{-1} = (PP^{-1})^{-1} = R$, since a one-dimensional Bézout domain is completely integrally closed. We proceed to show that P is not principal. Suppose, on the contrary, that $P = fR$. Then f is a principal prime of R_m for each m for which $f \in R_m$. Write $X - 1 = fg$. Choose n with $f, g \in R_n$, and set $P_n = P \cap R_n$. Then $P_n = fR_n$, and f is one of the irreducible factors of $X - 1$ in R_n. Via the isomorphism $X^{1/n} \to X$ from R_n to R_1, we get an equation $X^n - 1 = f(X^n)h$ for some $h \in R_1$. Thus $f(X^n)$ is an irreducible factor of $X^n - 1$ in R_1, so that $f(X^n)$ is a cyclotomic polynomial. Let $p > n$ be a prime number. We have $X^{np} - 1 = f(X^{np})h(X^p)$. Therefore, $f(X^{np})$ is irreducible in R_1, so that it must also be a cyclotomic polynomial. Thus $\deg(f(X^{np})) = \phi(r)$ for some positive integer $r \,|\, np$. If $p \nmid r$, then $r \,|\, n$, and $\phi(r) \leq r \leq n < p \leq \deg(f(X^{np}))$, a contradiction. If $p \,|\, r$, then $r = ps$ for some $s \,|\, n$. In this case $\phi(r) = \phi(p)\phi(s)$, contradicting that $\deg(f(X^{np}))$ is divisible by p. Hence $f(X^{np})$ is not irreducible in R_1, whence f is not irreducible in R_{np}, a contradiction. Therefore, P is not principal, as claimed.

EXAMPLE 5.2. An example of a domain R containing divisorial ideals I and J, such that I^{-1} and J^{-1} are rings but $(I \cap J)^{-1}$ is not a ring (cf. Theorem 3.4). Let $R = F[\{X^nZ, Y^nZ | n \geq 0\}]$, and let I and J denote the ideals generated by the sets $\{X^nZ\}$ and $\{Y^nZ\}$, respectively. We make the following claims:

(1) $I^{-1} = F[X, Z, \{Y^nZ | n \geq 0\}] = R[X]$.

(2) $J^{-1} = F[Y, Z, \{X^nZ | n \geq 0\}] = R[Y]$.

(3) I and J are divisorial.

(4) $(I \cap J)^{-1} = I^{-1} + J^{-1} = R[X] + R[Y]$. In particular, $X, Y \in (I \cap J)^{-1}$, but $XY \notin (I \cap J)^{-1}$.

Proof. (1) It is clear that $I^{-1} \supseteq F[X, Z, \{Y^nZ | n \geq 0\}]$. Let $f \in I^{-1}$. Since $Z \in I$, we may write $f = g/Z$ for some $g \in R$, and we may assume that g is a monomial, say $g = X^nY^mZ^k$. We wish to show that $f = X^nY^mZ^{k-1} \in F[X, Z, \{Y^nZ | n \geq 0\}]$. If $k > 1$, then clearly $f \in F[X, Z, \{Y^nZ | n \geq 0\}] = R[X]$. From $fXZ = gX \in R$, we infer that $k \geq 1$ and that if $k = 1$, then $m = 0$. Again, we have $f \in F[X, Z, \{Y^nZ | n \geq 0\}] = R[X]$.

(2) This is similar to (1).

(3) It suffices to show that if $h \in R$ and $hF[X, Z, \{Y^nZ | n \geq 0\}] \subseteq R$, then $h \in I$. We may assume that $h = X^rY^sZ^t$. It is clear that $h \in I$ if $s = 0$. If $s \neq 0$, then, since $hX = X^{r+1}Y^sZ^t \in R$, we have $t \geq 2$, whence again $h \in I$. This shows that $I = I_v$. Similarly, $J = J_v$.

(4) It suffices to show that $(I \cap J)^{-1} = R[X] + R[Y]$. Since $(I^{-1} + J^{-1})^{-1} = I_v \cap J_v = I \cap J$, we have $R[X] + R[Y] = I^{-1} + J^{-1} \subseteq (I \cap J)^{-1}$. Now let $f = g/Z \in (I \cap J)^{-1}$, where $g = X^iY^jZ^k$ is a monomial in R. Clearly, $k \geq 1$. If $i = 0$, then $f = Y^jZ^{k-1} \in R[Y]$. Similarly, if $j = 0$, then $f \in R[X]$. Finally, if $i, j \geq 1$, then $g \in R$ implies $k \geq 2$, and so, in this case, we have $f \in R[X] \cap R[Y]$. It follows that $(I \cap J)^{-1} \subseteq R[X] + R[Y]$, and the proof is complete. ∎

Remark. In Theorem 3.1, we showed that to determine whether the inverse of a radical ideal A of a domain R is a ring, it suffices to check that A^{-1} is closed under squares. We can use (a slight modification of) Example 5.2 to show that this is not true for general (non-radical) A. First, however, we observe that if A is an ideal of a domain R in which 2 is a unit, then A^{-1} is a ring $\Leftrightarrow A^{-1}$ is closed under squares. This follows from the equation $2xy = (x + y)^2 - x^2 - y^2$, in view of the fact that A^{-1} is a fractional ideal of R.

Now suppose that R is the ring of Example 5.2 and that the characteristic of F is 2. Let $A = I \cap J$. Since $A^{-1} = R[X] + R[Y]$, A^{-1} contains the square of each of its monomial elements. For an arbitrary element $f \in A^{-1}$, let $f = f_1 + \cdots + f_k$ be the representation of f as a sum of monomials.

Then, since $\operatorname{char}(R) = 2$, $f^2 = f_1^2 + \cdots + f_k^2 \in A^{-1}$. Thus A^{-1} is closed under squares but is not a ring.

EXAMPLE 5.3. An example of a domain R with ideals I and J such that $I^{-1} = (I:I)$, $J^{-1} = (J:J)$, $(I_v \cap J_v)^{-1}$ is a ring, but $(I \cap J)^{-1}$ is not a ring.

Let $R = F[X, Y, WXY, W^2XY, \{W^kX^2Y, W^kXY^2 | k \geq 0\}]$, $I = (Y^2, WXY, \{W^kX^2Y, W^kXY^2 | k \geq 0\})$, and $J = (X^2, WXY, \{W^kX^2Y, W^kXY^2 | k \geq 0\})$.

We claim that

(1) $I^{-1} = (I:I) = F[X, Y, \{W^kX | k \geq 0\}]$,

(2) $J^{-1} = (J:J) = F[X, Y, \{W^kY | k \geq 0\}]$,

(3) $I \cap J = (WXY, \{W^kX^2Y, W^kXY^2 | k \geq 0\})$,

(4) $I_v = (Y^2, XY, WXY, W^2XY, \{W^kX^2Y, W^kXY^2 | k \geq 0\})$

(5) $J_v = (X^2, XY, WXY, W^2XY, \{W^kX^2Y, W^kXY^2 | k \geq 0\})$,

(6) $I_v \cap J_v = (XY, WXY, W^2XY, \{W^kX^2Y, W^kXY^2 | k \geq 0\})$,

(7) $I^{-1}J^{-1} \subseteq (I_v \cap J_v)^{-1}$.

Proof.

(1) Easy calculations show that $F[X, Y, \{W^kX | k \geq 0\}] \subseteq (I:I)$. Let $f \in I^{-1}$. Since $fY^2, fXY \in R \subseteq F[X, Y, W]$, we have $fY \in F[X, Y, W]$. Write $f = g/Y$ with $g \in F[X, Y, W]$. We may assume that g is a monomial, say $g = X^iY^jW^k$. Since $fXYW = X^{i+1}Y^jW^{k+1} \in R$, we have $j \geq 1$, so that $f = X^iY^{j-1}W^k \in F[X, Y, W]$. Suppose $k \geq 1$. Then since $fY^2 \in R$, we must have $i \geq 1$. It follows that $I^{-1} \subseteq F[X, Y, \{W^kX | k \geq 0\}]$. Hence $I^{-1} = (I:I) = F[X, Y, \{W^kX | k \geq 0\}]$.

(2) Similar to (1).

(3) Clear.

(4) It is easy to see that $XY, W^2XY \in I_v$. Hence $I' = (Y^2, XY, WXY, W^2XY, \{W^kX^2Y, W^kXY^2 | k \geq 0\}) \subseteq I_v$. Let $g = X^rY^sW^t \in I_v$. If $r = t = 0$, then $gW^3X \in R$ implies $s \geq 2$, and we have $g \in I'$. If $t = 0$ and $r > 0$, then $gWX \in R$ implies $s \geq 1$, and again $g \in I'$. If $t > 0$, then $gX, gY \in R$ together imply $g \in I'$. It follows that $I' = I_v$, as desired.

(5) Similar to (4).

(6) Clear.

(7) Straightforward.

Now by (1) and (2), $I^{-1} = (I:I)$ and $J^{-1} = (J:J)$. That $(I_v \cap J_v)^{-1}$ is a ring follows from (7), in view of Corollary 2.6. Finally, from (3) it is easy to see that $W \in (I \cap J)^{-1}$ but that $W^2 \notin (I \cap J)^{-1}$, so that $(I \cap J)^{-1}$ is not a ring. ∎

EXAMPLE 5.4. An example of a domain R containing a radical ideal I and a set $\{P_\alpha\}$ of minimal primes of I with I the irredundant intersection of the P_α, $P_\alpha^{-1} = R$ (so that P_α is a ring) for each α, I^{-1} a ring, but $I^{-1} \neq R$. (See the third remark following Proposition 3.14.) Denote by S the set of all double sequences (k_i, m_i) of non-negative integers with $k_i \geq m_i \geq 0$ and $k_i \geq 1$ for infinitely many i; and for $s \in S$, denote by W_s the formal infinite product $\Pi_{i=1}^{\infty} X_i^{k_i} Y_i^{m_i}$. Let $R = F[\{X_i, X_iY_i | i \geq 1\}, \{Z^n W_s | n \geq 0, s \in S\}]$, $I = (\{Z^n W_s | n \geq 0, s = (k_i, m_i)$ with $k_i \geq 1$ for each $i\})$, and $P_i = (X_i, X_iY_i)$. Then

(1) for each i, P_i is prime and $P_i^{-1} = R$;

(2) each P_i is minimal over I, and I is the irredundant intersection of the P_i;

(3) $I^{-1} = R[Z]$;

(4) $I_v = (\{Z^n W_s | n \geq 0, s \in S\})$, which is a prime ideal.

Proof. (1) It is easy to see that P_i is prime. Suppose that $f \in P_i^{-1}$, and write $f = g/X_i$ for some $g \in R$. Since $gY_i \in R$, each monomial in g must contain $X_i^k Y_i^m$ with $k \geq m$; it follows that $f \in R$.

(2) If $h \in P_j$, then each infinite product in each monomial of h must contain a positive power of X_j. Hence each infinite product in a monomial contained in $\bigcap P_i$ must contain positive powers of each X_j. It follows that $I = \bigcap P_i$. Since $\Pi_{i \neq j} X_i \in \bigcap_{i \neq j} P_i \setminus I$, the intersection is irredundant (from which it follows that each P_i is minimal over I).

(3) Note that I^{-1} is a ring by Proposition 3.13. We show, in fact, that $I^{-1} = R[Z]$. It is clear that $I^{-1} \supseteq R[Z]$. Let $f \in I^{-1}$; as usual we may assume that f is a monomial. Write $f = g/\Pi X_i$ for some monomial $g \in R$. Since $\Pi_{i=1}^{\infty} X_iY_i \in I$, we have $g\Pi_{i=1}^{\infty} Y_i \in R$. Thus $g = Z^n \Pi_{i=1}^{\infty} X_i^{k_i} Y_i^{m_i}$, with $k_i > m_i \geq 0$ and $n \geq 0$. It follows that $f = Z^n \Pi_{i=1}^{\infty} X_i^{k_i - 1} Y_i^{m_i} \in R[Z]$. Hence $I^{-1} = R[Z]$.

(4) That $I_v = (\{Z^n W_s | n \geq 0, s \in S\})$ follows from the fact that no finite product of the X_iY_i is multiplied into R by Z, but every infinite product is. It is easy to check that this ideal is prime. Note that it is not minimal over I. ∎

EXAMPLE 5.5. An example of a domain R containing an ideal I which satisfies the four conditions of Proposition 2.1 but for which I^{-1} is not a ring. Let $R = F[X, XY, Y^3, Y^4, Y^5]$, $I = (X, Y^3)$, and $M = (X, XY, Y^3, Y^4, Y^5)$. It is easy to see that $M = \sqrt{I}$. The integral closure of R is $R' = F[X, Y]$. Thus R is a two-dimensional Noetherian ring, and $\mathrm{ht}(M) = 2$. It follows that I cannot be invertible. Since M is the only prime containing I, we have $II^{-1} \subseteq M$, and condition (3) of Proposition 2.1 is satisfied. Since $I^{-1} \subseteq R'$, condition (4) is automatically satisfied. For conditions (1) and (2), we need only show that M^{-1} is a ring, and this follows from Corollary 3.2. On the other hand, I^{-1} is not a ring, since (as is easily checked) $Y \in I^{-1}$ but $Y^2 \notin I^{-1}$.

EXAMPLE 5.6. An example of a (Prüfer) domain R containing ideals I and J for which (1) I^{-1} is a ring, but $I^{-1} \not\subseteq (IV : IV)$ for some valuation overring V of R, and (2) J^{-1} is also a ring, but $(I \cap J)$ is principal (so that $(I \cap J)^{-1}$ is not a ring).

Let R be the domain of [HP, Example 2.6]. Thus R is a two-dimensional Prüfer domain with two maximal ideals M_1 and M_2, both of height two, and a (unique) prime ideal P contained in $M_1 \cap M_2$ with R_P a DVR. By localizing, if necessary, we may assume that M_1 and M_2 are the only maximal ideals of R. Let $x \in P$ be such that $PR_P = xR_P$, and let $I = xR_{M_1}$ and $J = xR_{M_2}$. Since $P \subseteq M_1 \cap M_2$, $PR_P = PR_{M_1} = PR_{M_2}$. It follows that $P = RP_P$, that $P^{-1} = R_P$, and that $I, J \subseteq P$, so that I and J are ideals of R. We shall show that $I^{-1} = J^{-1} = P^{-1} = R_P$. Now $I^{-1} = (R :_R xR_{M_1}) = x^{-1}(R :_R R_{M_1})$ and $(R :_R xR_{M_2}) = x^{-1}(R :_R xR_{M_2})$. Since R_{M_1} and R_{M_2} are valuation rings and R is seminormal, we have by [DF, Lemma 2.10] that $(R :_R R_{M_1})$ and $(R :_R R_{M_2})$ are nonzero prime ideals of R. We claim that $(R_R : R_{M_1}) \subsetneqq M_1$. If not, pick $a \in M_2 \setminus M_1$, so that $a^{-1} \in R_{M_1}$; then $a^{-1}M_1 \subseteq R$, and $M_1 \subseteq aR \subseteq M_2$, a contradiction. It follows that $(R :_R R_{M_1}) = P = (R :_R R_{M_2})$. Hence $I^{-1} = x^{-1}P = x^{-1}xR_P = R_P$, and similarly, $J^{-1} = R_P$. Hence $I^{-1} \not\subseteq (IV : IV)$ for $V = R_{M_1}$, and statement (1) follows. Finally, it is easy to see that $I \cap J = xR$; this gives statement (2).

EXAMPLE 5.7. An example of a Prüfer domain D containing radical ideals I and J such that I is the intersection of radical ideals A and B, I^{-1} is a ring, but A^{-1} is not a ring (see Proposition 3.11 and Corollary 3.12) and such that J is a radical ideal between I and a minimal prime of I, but J^{-1} is not a ring (see the second remark following Proposition 3.14). Let D be a Prüfer domain with exactly two maximal ideals M_1 and M_2 with M_2 principal, $\mathrm{ht}(M_2) = 2$, M_1 not invertible, and $\mathrm{ht}(M_1) = 1$; and let Q denote the (unique) height one prime ideal contained in M_2. Since Q is not maximal, it cannot be invertible, and by [Hu, Theorem 3.8], Q^{-1} and M_1^{-1} are both rings. Clearly, M_2^{-1} is not a ring. If $I = M_1 \cap Q$, then Theorem 3.4 shows that I^{-1} is a ring. However, if we set $A = M_1 \cap M_2$ and $B = Q$, then $I = A \cap B$, and A^{-1} is not a ring by Proposition 3.13. Finally, if $J = M_1 \cap M_2$, then J is trapped between I and the minimal prime M_1 of I, but J^{-1} is not a ring, again by Proposition 3.13.

EXAMPLE 5.8. An example of a PVMD R containing an ideal I and a minimal prime M of I for which I^{-1} is a ring properly containing R but $M_t = R$. Let D denote the Prüfer domain of Example 5.7, and let M_1, M_2, and Q be as defined there. By [HuP, Theorem 3.2] $Q^{-1} = D_{M_1} \cap D_Q$. Thus if $b \in M_2 \setminus (M_1 \cup Q)$, then $b^{-1} \in Q^{-1}$. Now let $R = D[X]$. It is well known that R is a PVMD. Let $P = Q[X]$, and let $M = M_1 + XR$. Then M is not a t-prime. As an ideal of a Prüfer domain, Q is a t-prime; thus P is a t-prime of R. Moreover, $P^{-1} = Q^{-1}[X]$, and P^{-1} is a ring which properly contains R. Let $I = M \cap P$. Since M is not a t-prime and M is maximal in R, we have $M_t = R$. By Theorem 3.4, I^{-1} is a ring. From above $b^{-1} \in Q^{-1} \subseteq Q^{-1}[X] = P^{-1} \subseteq I^{-1}$, and I^{-1} properly contains R.

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