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On the Krull Dimension of Domains of Integer-Valued Polynomials

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0. Introduction

Let A be an integral domain with quotient field K. The ring of *integer-valued polynomials* over A, denoted by $\mathrm{Int}(A)$, is the subring of the ring of polynomials $K[X]$ defined as $\{f(X) \in K[X] : f(A) \subseteq A\}$. A crucial problem concerning $\mathrm{Int}(A)$ is to describe its (prime) spectrum and to evaluate its Krull dimension. In this regard, several satisfactory results were obtained for Dedekind domains [Ch1 and Ch2], for Noetherian domains [Ch2], for valuation and pseudo-valuation domains [CH], and recently for pseudo-valuation domains of type n [T]. However, the problem of finding an upper bound, depending on $\dim(A)$, for the Krull dimension of $\mathrm{Int}(A)$ is still open in general. We recall that Seidenberg [S1] proved that $\dim(A[X]) \leq 2\dim(A) + 1$: this inequality is essentially related to the fact that the fiber in $A[X]$ of any prime ideal of A has dimension 1. A major difference between the ring of integer-valued polynomials, $\mathrm{Int}(A)$, and the polynomial ring case, $A[X]$, is that the fiber in $\mathrm{Int}(A)$ of a maximal

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ideal of A may have any dimension (cf. [C3, Example 4.5] or Example 1.14 (b) below). Neverthless, we do not know examples of domains A not satisfying the inequality

$$\dim(\mathrm{Int}(A)) \leq \dim(A[X]). \tag{0.1}$$

This fact leads us to consider the question whether the inequality (0.1) holds for an arbitrary integral domain A.

The purpose of this paper is to review and pursue the study of the prime spectrum of $\mathrm{Int}(A)$ and to establish bounds for its Krull dimension for relevant classes of domains.

Let $A \subseteq B$ be an extension of integral domains and let $I(B,A) := \{f(X) \in B[X] : f(A) \subseteq A\}$ the ring of the A-valued B-polynomials introduced in [AAZ]. Note that $I(K,A) = \mathrm{Int}(A)$, and $A[X] \subseteq I(B,A) \subseteq A + XB[X]$. The inequality

$$\dim(I(B,A)) \leq \dim(A[X]) \tag{0.2}$$

does not hold in general. As a matter of fact, in Example 1.14 (a), we construct explicitly an integral domain A and an overring B of A such that

$$\dim(\mathrm{Int}(A)) < \dim(A[X]) < \dim(I(B,A)).$$

However, our motivation for deepening the study of rings $I(B,A)$ is related to the following considerations: let $A^\bullet = A\setminus\{0\}$, since $K = \varinjlim\{A[a^{-1}] : a \in A^\bullet\}$ (i.e. K is the direct limit of the inductive family of the quotient rings $A[a^{-1}]$, considered as a direct system by setting $a \leq b$ in $A^\bullet$ if $a|b$). It can be easily seen that $\mathrm{Int}(A) = \varinjlim\{I(A[a^{-1}],A) : a \in A^\bullet\}$ and hence

$$\dim(\mathrm{Int}(A)) = \sup\{\dim(I(A[a^{-1}],A)) : a \in A^\bullet\}$$

[DFK, Lemma 2.1]. Thus, in order to prove the inequality (0.1), it would be enough to show that, for each overring $A[a^{-1}]$ of A, where $a \in A^\bullet$,

$$\dim(I(A[a^{-1}],A)) \leq \dim(A[X]). \tag{0.3}$$

Now using the inclusions $A[X] \subseteq I(A[a^{-1}],A) \subseteq A + XA[a^{-1},X]$ and the inequalities $\sup\{\dim(A[a^{-1},X]), \dim(A)+1\} \lneq \dim(A + XA[a^{-1},X]) \leq \dim(A[X])$, in conjunction with the fact $\dim_v(A[X]) \leq \dim_v(A + XA[a^{-1},X])$ (cf. [FK, Proposition 3.1]), we deduce $\dim_v(I(A[a^{-1}],A)) = \dim_v(A) + 1 = \dim_v(\mathrm{Int}(A))$. Thus we are tempted to prove an inequality of the following type:

$$\dim(A + XA[a^{-1},X]) \geq \dim(I(A[a^{-1}],A)). \tag{0.4}$$

Unfortunately there is a countrexample. Take $A := K + ZK(Y)[Z]_{(Z)}$, where K is an infinite field and X, Y, Z are indeterminates over K. For $a = Z$ we get $A[a^{-1}] = A[Z^{-1}] = K[Z^{-1}] + ZK(Y)[Z]_{(Z)}[Z^{-1}] = K(Y,Z) = \mathrm{q.f.}(A)$. Hence $I(A[a^{-1}], A) = I(\mathrm{q.f.}(A), A) = \mathrm{Int}(A)$. Moreover, since A/M is infinite, $\mathrm{Int}(A) = A[X]$ [CC, Corollaires, p. 303]. It follows that $\dim(\mathrm{Int}(A)) = \dim(A[X]) = 2 + \dim(A) = 3$ [HH, Theorem 2.5]. On the other hand, applying [FK, Proposition 3.1] we obtain $\dim(A + XA[a^{-1}, X]) = 1 + \dim(A) = 2$ and hence $\dim(A + XA[a^{-1}, X]) < \dim(I(A[a^{-1}], A)) = \dim(A[X])$. Therefore (0.3) holds but (0.4) fails.

The first section begins with a generalization of [AAZ, Proposition 7.3]. This result shows that, without loss of generality, we can suppose B to be an overring of A, whenever A is not a finite field. Further we show that the inclusion $I(B, A) \subseteq \mathrm{Int}(A)$ always holds, even when B is not an overring of A. We also compute the valuative dimension of $I(B, A)$ and $\mathrm{Int}(A)$ and establish necessary and sufficient conditions for the transfer of the Jaffard property to the ring $I(B, A)$ (hence, in particular, to $\mathrm{Int}(A)$). As a consequence, we show that the inequality (0.2) holds for all extensions of domains $A \subseteq B$ such that the polynomial domain $A[X]$ is a Jaffard domain. In Theorem 1.10 we establish lower and upper bounds for $\dim(I(B, A))$ in the spirit of [C3, Corollaire 1.2 and Proposition 1.4]. Moreover we generalize [C3, Théorème 4.2 and Proposition 2.2] by showing that $\dim(I(B, A)) = \dim(B[X])$, when A is obtained from B by a pullback of a special type. This allows us to construct a pair of domains $A \subset B$ such that $\dim(I(B, A)) > \dim(A[X])$ and hence to exibit failure of (0.2) in general. Finally, we study $I(B, A)$ when $A \subsetneq B$ is a residually algebraic extension of domains.

In the second section, we deal with $\mathrm{Int}(D)$ where D is a pullback of a special type: a major example is when $D = R + I$, where I is a nonzero ideal of a given domain T and R is an infinite subring of T with $\mathrm{q.f.}(R) \subseteq T/I$. We show that, in this case, the spectrum of $\mathrm{Int}(D)$ can be described in terms of the spectrum of $\mathrm{Int}(R)$ and the prime ideals incomparable with $J = ID_I[X] \cap \mathrm{Int}(D)$. This permits, in the present situation, to maintain control over $\dim(\mathrm{Int}(D))$. In particular we can apply the results of this section to the "classical case" where T is a valuation domain and I the maximal ideal of T.

In Section 3 we prove an analogue for $\mathrm{Int}(D)$ of a well-known theorem proved by Seidenberg, concerning the Krull dimension of the ring of polynomials: let $n \geq 1$, for each h, with $n + 1 \leq h \leq 2n + 1$, there exists an integral domain D such that $\dim(D) = n$, $\dim(\mathrm{Int}(D)) = h$ and $\mathrm{Int}(D) \neq D[X]$.

In the fourth section we are concerned with integer-valued polynomials over subsets. Let E be a fractional subset of D. We show that D is a Jaffard domain if and only if $\mathrm{Int}(E, D)$ is a Jaffard domain and $\dim(\mathrm{Int}(E, D)) = 1 + \dim(D)$. This establishes a converse to [C3, Corollaire 1.3]. In Example 4.4 we show that the inequality $\dim(D[X]) - 1 \leq \dim(\mathrm{Int}(E, D))$ does not hold in general when $E \neq D$ (for the case $E = D$, cf. [C3, Proposition 1.4]). The main result of this section deals with $\dim(\mathrm{Int}(E, D))$, where D is a special subring of a Bézout domain; in particular, in the local case, we have the following application: if D is a pseudo-valuation domain with infinite residue field, for each D-submodule E of the field of quotients of D, $\dim(\mathrm{Int}(E, D)) \leq \dim(D[X])$.

1. The A-valued B-polynomials

In this section we suppose that $A \subseteq B$ is an extension of integral domains, such that A is not a finite field, and we denote by K the quotient field of A.

The ring $I(B, A) := \{f(X) \in B[X] : f(A) \subseteq A\}$ of the A-valued B-polynomials was introduced in [AAZ], but not so much explored. It is evident that $I(B, A) = \cap_{a \in A}(A + (X - a)B[X])$ and, therefore, $I(B, A) \subseteq A + XB[X]$.

If $A = \{a_1, \cdots, a_N\}$ is a finite field then $I(B, A) = \cap_{i=1}^{N}(A + (X - a_i)B[X])$ is a finite intersection of domains obtained from pullbacks of the following type:

$$\begin{array}{ccc} A + (X - a_i)B[X] = \varphi^{-1}(A) & \longrightarrow & A \\ \downarrow & & \downarrow \\ B[X] & \xrightarrow{\varphi} & (B[X]/(X - a_i)B[X]) \cong B. \end{array}$$

Therefore this case can be reduced to the study of pullback constructions of a special type [FIK1], [FIK2], and [FIK3].

In this section, we collect more information on this kind of rings especially in order to enlighten their relationship with the integral domains $A[X]$, $B[X]$ and $\mathrm{Int}(A)$. A special result is that the inequality (0.2) does not hold in general (cf. Example 1.14 (a)). On the other hand, it is well known that for any polynomial $f(X) \in \mathrm{Int}(\mathbb{Z})(= I(\mathbb{Q}, \mathbb{Z}))$ of degree n, $n!f(X) \in \mathbb{Z}[X]$. Proposition 7.3 of [AAZ] establishes a similar result for $\mathrm{Int}(D)$, where D is an integral domain. Mimicking the proof of the above-mentioned result, we generalize it to the domain $I(B, A)$, then we point out a kind of independence of $I(B, A)$ from B. In fact, we show that $I(B, A)$ is always a subring of $\mathrm{Int}(A)$ even if B is not an overring of A.

Lemma 1.1. *Given an extension $A \subseteq B$ as above, for each $n \geq 0$, there exists $0 \neq t_n \in A$ such that $t_n f(X) \in A[X]$ for all $f(X) \in I(B, A)$ with $\deg(f(X)) \leq n$.*

Proof. To avoid the trivial case we can assume $n \geq 1$. From the hypothesis that A is not a finite field, we deduce that there exists $r \in A$ such that $(r^n - r^i) \neq 0$ for $i = 0, 1, \ldots, n-1$. Now, let

$$f(X) = a_0 + a_1 X + \cdots + a_n X^n \in I(B, A)$$

be a polynomial of degree n. It is clear that:

$$a_0 + ra_1X + \cdots + r^n a_n X^n = f(rX) \in I(B, A)$$

and

$$r^n a_0 + r^n a_1 X + \cdots + r^n a_n X^n = r^n f(X) \in I(B, A),$$

hence the polynomial:

$$(r^n - 1)a_0 + (r^n - r)a_1 X + \cdots + (r^n - r^{n-1})a_{n-1}X^{n-1} =$$

$$= r^n f(X) - f(rX) =: g(X) \in I(B, A)$$

has degree at most $n - 1$.
By induction, we can find $t_{n-1} \neq 0$, $t_{n-1} \in A$ with $t_{n-1}g(X) \in A[X]$, that is $t_{n-1}(r^n - r^i)a_i \in A$ for $0 \leq i \leq n-1$ and, moreover, $a_0 + a_1 + \cdots + a_n = f(1) \in A$.
Set $t_n := t_{n-1}\prod_{i=0,\cdots,n-1}(r^n - r^i) \in A$. It is clear that: $t_n a_i \in A$ for $0 \leq i \leq n-1$. Furthermore $t_n a_0 + t_n a_1 + \cdots + t_n a_n = t_n f(1) \in A$; hence $t_n a_n \in A$ and thus $t_n f(X) \in A[X]$. □

Proposition 1.2. *Given an extension $A \subsetneq B$ as above, then we have the following inclusions:*

$$A[X] \subseteq I(B, A) \subseteq \mathrm{Int}(A) \subseteq A + XK[X] \subseteq K[X].$$

Proof. Let $f(X) \in I(B, A)$ be a polynomial of degree n, so there exists $0 \neq t_n \in A$ such that $t_n f(X) \in A[X]$ (Lemma 1.1). It follows that $f(X) \in (1/t_n)A[X] \subseteq K[X]$. Since $f(A) \subseteq A$, then $f(X) \in \mathrm{Int}(A)$. The remaining inclusions are obvious. □

The conclusion of Proposition 1.2 does not hold if A is a finite field (see Example 1.18).

The previous result shows that there is no loss of generality by assuming that B is an overring of A.

Notice that $I(B, A)$ coincides with $\mathrm{Int}(A)$ in several situations. For instance, whenever $\mathrm{Int}(A) \subseteq B[X]$ (in particular when $A \subseteq B$ is a *"bon couple"* in the sense of Cahen [C3]) then, obviously, $I(B, A) = \mathrm{Int}(A)$. Moreover,

Corollary 1.3. (a) *Let $A \subseteq B$ be an extension of integral domains such that A contains an infinite field K_0. Then $I(B, A) = \mathrm{Int}(A) = A[X]$.*

(b) *If $A \subseteq B$ is a flat overring of A with $B_Q[X] = \mathrm{Int}(B_Q)$, for each $Q \in \mathrm{Spec}(B)$, then $I(B, A) = \mathrm{Int}(A)$.*

Proof. (a) Let M be a maximal ideal of A, then $K_0 \cap M = (0)$ and thus K_0 can be canonically embedded in A/M. It follows that A/M is infinite for each maximal ideal of A, therefore $\mathrm{Int}(A) = A[X]$ [CC, Corollaires, p. 303] and, from the previous result, the thesis follows.

(b) Since B is a flat overring of A, then for each prime ideal Q of B, $A_{Q\cap A} = B_Q$ [R, Theorem 2]. Therefore, $\mathrm{Int}(A) \subseteq \cap_{Q\in \mathrm{Spec}(B)}\mathrm{Int}(A_{Q\cap A}) = \cap_{Q\in\mathrm{Spec}(B)}\mathrm{Int}(B_Q) = \cap_{Q\in\mathrm{Spec}(B)}(B_Q[X]) = B[X]$ and hence $I(B,A) = \mathrm{Int}(A)$. □

By Proposition 1.2 we know in particular that $A[X] \subseteq I(B,A) \subseteq K[X]$, hence by [AAZ, §2], the values of elements of $I(B,A)$ at $X = 0$ split this ring into two remarkable parts:

$$M_0 := \{f(X) \in I(B,A) : \ f(0) = 0\}$$

which is a prime ideal of $I(B,A)$, and:

$$S_0 := \{f(X) \in I(B,A) : \ f(0) \neq 0\}$$

which is a multiplicative subset of $I(B,A)$.

More precisely, from [AAZ, §2], it follows that $A = \{f(0) : f(X) \in I(B,A)\} \cong I(B,A)/M_0$ and $I(B,A) = A + M_0$.

With the previous notation we have:

Proposition 1.4. *Given the extension $A \subsetneq B$ of the type described at the beginning of this section, then:*

(a) $S_0^{-1}I(B,A) = I(B,A)_{M_0} = K[X]_{(X)}$.

(b) *M_0 is a height one prime ideal of $I(B,A)$.*

(c) *M_0 is maximal if and only if A is a field. In this case $I(B,A) = A[X]$.*

Proof. (a) By Proposition 1.2, $A[X] \subseteq I(B,A) \subseteq K[X]$. On the other hand, it is easily seen that $M_0 = XK[X] \cap I(B,A)$, and $S_0 = I(B,A) \setminus M_0 = (A \setminus (0)) + M_0 \subseteq K[X] \setminus XK[X]$. Therefore,

$$S_0^{-1}I(B,A) = I(B,A)_{M_0} \subseteq K[X]_{(X)}.$$

Now, let $H(X) := (f(X)/g(X)) \in K[X]_{(X)}$, with $f(X), g(X) \in A[X]$ and $g(0) \neq 0$. It follows that $f(X)$ and $g(X)$ belong to $I(B,A)$ and $g(X) \notin M_0$. Hence $H(X) \in I(B,A)_{M_0}$ and then $K[X]_{(X)} \subseteq I(B,A)_{M_0}$.

(b) We notice that $S = A \setminus (0)$ is a multiplicative subset of both A and $I(B,A)$ such that

$$S^{-1}A[X] = K[X] \subseteq S^{-1}I(B,A) \subseteq K[X].$$

Since $M_0 \cap S = \emptyset$, so $1 \leq \mathrm{ht}_{I(B,A)}(M_0) \leq \dim(K[X]) = 1$.

(c) The first part follows easily from the fact that $I(B,A)/M_0 \cong A$. For the second part, by hypothesis, if A is a field, it is infinite so, from Corollary 1.3 (a), it follows that $I(B,A) = A[X]$. □

Remark 1.5. For each $a \in A$, we can consider

$$M_a := \{f(X) \in I(B,A) : f(a) = 0\}, \quad S_a := I(B,A) \setminus M_a$$

then it is easy to see that $M_a = (X-a)K[X] \cap I(B,A)$ is a prime ideal of $I(B,A)$, $A \cong I(B,A)/M_a$ and $I(B,A) = A + M_a$ and, *mutatis mutandis*, Proposition 1.4 holds if we replace M_a and S_a by M_0 and S_0.

The statement (1) of Proposition 1.4 leads us naturally to investigate the behaviour of $I(B,A)$ under localization. The next lemma gives an answer to this problem.

Lemma 1.6. *Let $A \subseteq B$ be an extension of integral domains with the same quotient field K and S a multiplicative subset of A. Then:*

(a) *$S^{-1}I(B,A) \subseteq I(S^{-1}B, S^{-1}A)$, with the equality occuring when A is Noetherian.*

(b) *Let p be a prime ideal of A with A/p infinite. Then $I(B_p, A_p) = A_p[X](=$ $\mathrm{Int}(A_p))$.*

Proof. It is enough to observe that $I(B,A) = \mathrm{Int}(A) \cap B[X]$, and both statements above hold for $\mathrm{Int}(A)$ [CC, Corollaires, p. 303]. □

Remark 1.7. (a) As mentioned above, for each $a \in A$, $I(B,A) = A + M_a$. We deduce that each prime ideal of $I(B,A)$ which contains M_a is of the form $p + M_a$, where p is a prime ideal of A. For each $p \in \mathrm{Spec}(A)$ and $a \in A$, we set $p_a = \{f(X) \in I(B,A) : f(a) \in p\}$. It is easy to see that p_a is a prime ideal of $I(B,A)$ and, if $p \subset q$ is an inclusion of prime ideals of A, then $p_a \subset q_a$. Therefore, in particular, $1 + \dim(A) \le \dim(I(B,A))$.

(b) Since $S = A \setminus (0)$ is a multiplicative subset of $I(B,A)$, it is easily seen that for each irreducible polynomial $f(X) \in K[X]$, then $P := (f(X)K[X]) \cap I(B,A)$ is a height 1 prime ideal in $I(B,A)$, and $I(B,A)_P = K[X]_{(f(X))}$ is a one dimensional Noetherian valuation domain.

We recall that the *valuative dimension* of an integral domain A is defined as follows:

$$\dim_v(A) := \sup\{\dim(V) : V \quad \text{is a valuation overring of} \quad A\}.$$

A *Jaffard domain* is a domain A such that $\dim_v(A) = \dim(A) < \infty$. Finite dimensional Noetherian domains or Prüfer domains are examples of Jaffard domains [ABDFK].

In the next result we compute the valuative dimension of $I(B,A)$ (and hence, in particular, of $\mathrm{Int}(A)$) and we establish necessary and sufficient conditions for the transfer of the Jaffard property to either $\mathrm{Int}(A)$ or $I(B,A)$.

Proposition 1.8. *Given an extension $A \subsetneq B$ as described at the beginning of this section, then:*

(a) $\dim_v(I(B,A)) = \dim_v(\mathrm{Int}(A)) = 1 + \dim_v(A)$.

(b) *The following conditions are equivalent:*

(i) *A is a Jaffard domain;*

(ii) *$\mathrm{Int}(A)$ is a Jaffard domain and $\dim(\mathrm{Int}(A)) = 1 + \dim(A)$;*

(iii) *$I(B,A)$ is a Jaffard domain and $\dim(I(B,A)) = 1 + \dim(A)$.*

Proof. (a) From Proposition 1.2 we have the following inclusions:

$$A[X] \subseteq I(B,A) \subseteq \mathrm{Int}(A) \subseteq A + XK[X] \subseteq K[X],$$

where $K = \mathrm{q.f.}(A)$. Therefore

$$1 + \dim_v(A) = \dim_v(A + XK[X]) \leq \dim_v(\mathrm{Int}(A)) \leq$$

$$\leq \dim_v(I(B,A)) \leq \dim_v(A[X]) = 1 + \dim_v(A).$$

(b) (i) $\Leftrightarrow$ (iii) We always have the following inequalities (Remark 1.7 (a)):

$$1 + \dim(A) \leq \dim(I(B,A)) \leq \dim_v(I(B,A)) = 1 + \dim_v(A).$$

Therefore, if A is a Jaffard domain, $I(B,A)$ is a Jaffard domain and $\dim(I(B,A)) = 1 + \dim(A)$.

(i) $\Leftrightarrow$ (ii) *Mutatis mutandis* the proof is the same as for (i) $\Leftrightarrow$ (iii). □

As an immediate application of the previous proposition, the next result shows that the inequality (0.2) holds for the integral domains A (possibly non Jaffard) whose polynomial extension $A[X]$ is a Jaffard domain.

For instance, if k is any finite field and X, Y and Z are indeterminates over k, then it is well known that $A = k + Zk(Y)[Z]$ is a 1-dimensional non Jaffard domain, since $\dim_v(A) = 2$. It is not difficult to see that $A[X]$ is a 3-dimensional Jaffard domain [HH, Theorem 2.5].

Corollary 1.9. *Let $A \subsetneq B$ be an extension of integral domains such that either $A[X]$ is a Jaffard domain or A/M is infinite for each maximal ideal M of A. Then $\dim(I(B,A)) \leq \dim(A[X])$.*

Proof. If $A[X]$ is a Jaffard domain, then $\dim(A[X]) = \dim_v(A[X]) = 1 + \dim_v(A) = \dim_v(I(B,A)) \geq \dim(I(B,A))$. If A/M is infinite for each maximal ideal M of A, then, by [CC, Section 4], $\mathrm{Int}(A) = A[X]$ and thus $A[X] = I(B,A)$ (Proposition 1.2). □

The following theorem establishes some bounds for $\dim(I(B,A))$ in the spirit of [C3, Corollaire 1.2 and Proposition 1.4].

Theorem 1.10. *Given an extension $A \subseteq B$ of integral domains where A is not a finite field, then*

$$\sup\{1 + \dim(A);\ \dim(A[X]) - 1\} \leq \dim(I(B, A)) \leq 1 + \dim_v(A).$$

Proof. To avoid the trivial case, we consider the finite dimensional setting.
It is clear that $1 + \dim(A) \leq \dim(I(B, A)) \leq \dim_v(I(B, A)) = 1 + \dim_v(A)$ (Remark 1.7 (a) and Proposition 1.8 (a)).
For the remaining inequality, by the "special chain Theorem" [J, Theorem 3, p. 35], let

$$(0) = Q_0 \subset \cdots \subset Q_{n-2} \subset Q_{n-1} = m[X] \subset Q_n$$

be a chain that realizes $\dim(A[X])$, where $m \in \mathrm{Max}(A)$.
It is easily seen that $p = Q_{n-2} \cap A$ is not maximal, so A/p is infinite and then $\mathrm{ht}_{A[X]}(p[X]) = n - 2$ if $Q_{n-2} = p[X]$, or $n - 3$ if not. On the other hand, $\mathrm{Int}(A_p) = A_p[X]$ [CC, Corollaire 2, p. 303]. Hence, there exists $P \in \mathrm{Spec}(\mathrm{Int}(A))$ such that $P \cap A[X] = p[X]$ and $\mathrm{ht}(P) \geq n - 3$. However, from Lemma 1.6, there exists $P \in \mathrm{Spec}(I(B, A))$, with $P \cap A[X] = p[X]$, and $\mathrm{ht}(P) \geq n - 3$. Now, m is maximal and contains p. Therefore, from Remark 1.7 and the above considerations, there exists a chain of prime ideals of $I(B, A)$ of length at least $n - 1$ and of the form $(0) \subset \cdots \subset P \subset p + M_0 \subset m + M_0$. It follows that $\dim(A[X]) - 1 \leq \dim(I(B, A))$. □

In the Example 1.19 below we will show that, if A is a finite field, it is not true, in general, that $\dim(I(B, A)) \leq 1 + \dim_v(A)$.

Corollary 1.11. [C3, Corollaire 1.2 and Proposition 1.4].
Let A be an integral domain with quotient field K. Then

$$\sup\{1 + \dim(A);\ \dim(A[X] - 1)\} \leq \dim(\mathrm{Int}(A)) \leq 1 + \dim_v(A). \quad \square$$

As we mentioned in the introduction, the problem of computing the Krull dimension of $\mathrm{Int}(A)$ is still open. Up to now, there are no examples of domains A such that the inequality $\dim(\mathrm{Int}(A)) > \dim(A[X])$ holds. However, we shall give an example of an extension of domains $A \subset B$ such that $\dim(I(B, A)) > \dim(A[X])$. First we need to generalize some results proved in [C3], in the case of $\mathrm{Int}(A)$.

Proposition 1.12. *Let $A \subseteq B$ be an extension of integral domains, and let J be a common nonzero ideal of A and B with A/J finite. Then:*

(a) *every chain of prime ideals in $I(B, A)$ lifts in $B[X]$.*

(b) $\dim(I(B, A)) = \dim(B[X])$.

(c) *For each prime ideal m of A with $m \supseteq J$, then* $\dim(I(B, A)/mI(B, A)) = \dim((B/mB)[X])$.

The proof of this result is partially based on the following lemma, which can be proved by adapting the arguments used in [C3, Proposition 2.2].

Lemma 1.13. *Given an extension of integral domains $A \subseteq B$, let J be a nonzero ideal of A such that A/J is finite. If M is a prime ideal of $I(B,A)$ which contains $J^* := \{f \in I(B,A) : f(A) \subseteq J\}$, then:*

(a) *M lies over a maximal ideal m of A that contains J.*

(b) *M contains the ideal $m^* := \{f \in I(B,A) : f(A) \subseteq m\}$.*

(c) *M is maximal.*

(d) *$I(B,A)/M \cong A/m$.* □

Proof of Proposition 1.12. *Mutatis mutandis*, after remarking that J^* is a common ideal of $I(B,A)$ and $B[X]$, we can use the proof of [C3, Théorème 4.2 and Corollaire 4.4] replacing $\mathrm{Int}(A)$ and [C3, Proposition 2.2] with $I(B,A)$ and Lemma 1.13, respectively. □

Examples 1.14. **(a)** Let k be a finite field, let Z_1, Z_2, Z_3 be indeterminates over k, $K := k(Z_1, Z_2)$ and $M := Z_3K[Z_3]_{(Z_3)}$. Consider the following domains:

$$V := K + M = K[Z_3]_{(Z_3)}, \qquad B := k[Z_1] + M \qquad \text{and} \qquad A := k + M.$$

Then we have the following pullback diagrams:

$$\begin{array}{ccc} A & \longrightarrow & A/M = k \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/M = k[Z_1] \\ \downarrow & & \downarrow \\ V & \longrightarrow & V/M = K \end{array}$$

where V is a rank one discrete valuation domain, A is a one dimensional pseudo-valuation domain of maximal ideal M and residue field k (a finite field). Moreover, since $\mathrm{t.d.}_k(K) > 0$, $\dim(A[X]) = 2 + \dim(A) = 3$ [HH] and $\dim(B[X]) = \dim(k[Z_1, X]) + \dim(V) + 1 = 4$ [ABDFK, Corollary 2.8], $\dim(\mathrm{Int}(A)) = 1 + \dim(A) = 2$ [T, Corollary 1.4] and by Proposition 1.12, $\dim(I(B,A)) = \dim(B[X]) = 4$. It follows that

$$\dim(\mathrm{Int}(A)) < \dim(A[X]) < \dim(I(B,A)).$$

We note that $I(B,A)$ is a Jaffard domain since $\dim_v(I(B,A) = \dim_v(A) + 1 = 3 + 1 = 4$.

(b) (Cahen [C3]). Let k be a finite field, $k \subset K$ a field extension, $X_1, X_2, \cdots, X_{n+1}$ a finite family of indeterminates over K, $n \geq 1$.

Set $B := K[X_1, X_2, \cdots, X_{n+1}]$ and let Q be a prime ideal of height 2 in B, e.g. $Q = (X_n, X_{n+1})B$, and $\varphi : B \to B/Q$ the natural projection. We can consider the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} A := \varphi^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/Q \cong K[X_1, X_2, \cdots, X_{n-1}] \end{array}$$

In this situation, $Q = Q \cap A$ becomes a maximal ideal in A; however, $\dim(A) = \dim(B) = n+1$ since every chain of distinct prime ideals of B

$$(0) = Q_0 \subset Q_1 \subset \cdots \subset Q_{n+1}$$

with $Q_i \not\supseteq Q$ contracts into a chain of distinct prime ideals of A.

Furthermore, since $\mathrm{ht}_B(Q) = 2$ and B is a Krull domain, then $A \subset B$ is a "bon couple" in the sense of Cahen [C3, Proposition 3.2, Definition 4.1], hence $\mathrm{Int}(A) \subseteq B[X]$ (and thus $I(B, A) = \mathrm{Int}(A)$, Section 1) and

$$\dim(I(B,A)) = \dim(\mathrm{Int}(A)) = \dim(B[X]) = n+2 = \dim(A)+1$$

(Proposition 1.12 or [C3, Théorème 4.2]). However if we consider the fiber of the maximal ideal Q of A in $I(B, A) = \mathrm{Int}(A)$, then by Proposition 1.12 (3) we have:

$$\dim(I(B,A)/QI(B,A)) = \dim(\mathrm{Int}(A)/Q\mathrm{Int}(A)) = \dim(B[X]/QB[X]) = n-1+1 = n.$$

The following result compares the Krull dimension of $B[X]$, $I(B,A)$ and $A + XB[X]$ for residually algebraic extensions of domains. We recall that $A \subseteq B$ is *residually algebraic* if for each prime ideal Q of B, B/Q is algebraic over $A/(Q \cap A)$.

Theorem 1.15. *Given a residually algebraic extension of integral domains $A \subseteq B$, then*

$$\dim(B[X]) \leq \dim(A + XB[X]) \leq \dim(I(B,A)).$$

Proof. We already noticed that

$$A[X] \subseteq I(B,A) \subseteq A + XB[X] \subseteq B[X].$$

Now, let be given a chain of prime ideals in $A + XB[X]$ of the following type:

$$(0) = P'_0 \subset \cdots \subset P'_{k-1} \subset P'_k \subset P'_{k+1} = p_{k+1} + XB[X] \subset \cdots \subset P'_n = p_n + XB[X].$$

where $X \notin P'_r$ for $0 \leq r \leq k$, and where p_i is a prime ideal of A, for $k+1 \leq i \leq n$. The chain

$$(1) \qquad (0) = P'_0 \subset \cdots \subset P'_{k-1} \subset P'_k$$

lifts to a unique chain of $B[X]$ of the same length

$$(2) \qquad (0) = P''_0 \subset \cdots \subset P''_{k-1} \subseteq P''_k$$

[FIK1, Lemma 1.1 (b)]. Since $A \hookrightarrow B$ is residually algebraic, $A[X] \hookrightarrow B[X]$ is incomparable ([FIK2, Lemma 1.6 (b)] and [DF1, Theorem 2]), whence the chain (1) or, equivalently, the chain (2) contracts in $I(B,A)$ to a chain of the same length.
On the other hand, $P'_i = p_i + XB[X]$ contracts to the ideal $p_i + M_0$ of $I(B,A)$, for $i > k$. Thus $P'_i \cap I(B,A) \subset P'_{i+1} \cap I(B,A)$. From the previous argument it follows that each chain of prime ideals of $A + XB[X]$ contracts itself to a chain of the same length in $I(B,A)$. Then the second inequality follows. For the first inequality see [FIK1, Theorem 2.1 (a)]. $\square$

In particular, since the natural injection $A \hookrightarrow S^{-1}A$, where S is a multiplicative subset of A, is residually algebraic, then, from the previous theorem we have

Corollary 1.16. *Let A be any integral domain and S a multiplicative subset of A. Then*

$$\dim(S^{-1}A[X]) \leq \dim(A + XS^{-1}A[X]) \leq \dim(I(S^{-1}A, A)). \quad \square$$

We notice that for $S = A \setminus \{0\}$, from the previous result, we reobtain that

$$1 + \dim(A) \leq \dim(\mathrm{Int}(A)).$$

Note that, in general, the second inequality of Theorem 1.15 does not hold. In fact, the following example (which is a slight modification of Example 1.14 (a)) shows that it may happen $\dim(I(B,A)) < \dim(B[X]) \leq \dim(A + XB[X])$.

Example 1.17 Let K be an infinite field, Z_1, Z_2 and Z_3 indeterminates over K. Set $V := K(Z_1, Z_2)[Z_3]_{(Z_3)} = K(Z_1, Z_2) + M$ where $M := Z_3K(Z_1, Z_2)[Z_3]_{(Z_3)}$, $B := K[Z_1] + M$ and $A := K + M$. Then, as in Example 1.14 (a), $\dim(A[X]) = 3$ and $\dim(B[X]) = 4$. In this case, since K is infinite, by Corollary 1.3 we have $I(B,A) = A[X]$, thus $\dim(I(B,A)) = 3 < \dim(B[X]) = 4 \leq \dim(A + XB[X]) \leq 5$ [FIK1, Theorem 2.1 (a)]. Since $\dim_v(I(B,A)) = \dim_v(A) + 1 = 4$, in this case $I(B,A)$ is not a Jaffard domain. Moreover, since $A + XB[X]$ is an overring of $I(B,A)$, then necessarily $\dim(A + XB[X]) = 4$.

All over this section, we assumed A to be infinite. If A is a finite field, the inclusion $I(B,A) \subsetneq \mathrm{Int}(A)$ of Proposition 1.2 does not hold.

Example 1.18. Let $A = \mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$ and $B := \mathbb{F}_4 \cong \mathbb{F}_2[X]/(X^2+X+1)$, i.e., $\mathbb{F}_4 = \mathbb{F}_2[\omega]$, where $\omega^2+\omega+1=0$.

Now, let $f(X) = \omega X^2 + \omega X + 1$, clearly $f(X) \in \mathbb{F}_4[X] \setminus \mathbb{F}_2[X]$. Moreover, $f(0) = 1 \in \mathbb{F}_2$ and $f(1) = 1+\omega+\omega^2 = 0 \in \mathbb{F}_2$. That is, $f(X) \in I(\mathbb{F}_4, \mathbb{F}_2) \setminus \mathbb{F}_2[X]$. On the other hand, $\mathrm{Int}(\mathbb{F}_2) = \mathbb{F}_2[X]$, hence $\mathrm{Int}(\mathbb{F}_2) \subset I(\mathbb{F}_4, \mathbb{F}_2)$. In this example, the composite cover, in the sense of [AAZ], of $I(\mathbb{F}_4, \mathbb{F}_2)$ is $\mathbb{F}_2 + X\mathbb{F}_4[X]$ because there is no integral domain between $\mathbb{F}_2$ and $\mathbb{F}_4$. Therefore it follows that $\dim_v(I(\mathbb{F}_4, \mathbb{F}_2)) = \dim_v(\mathbb{F}_2 + X\mathbb{F}_4[X]) = 1$, thus $I(\mathbb{F}_4, \mathbb{F}_2)$ is a 1-dimensional Jaffard domain with quotient field $\mathbb{F}_4(X)$.

Example 1.19. Let $A = \mathbb{F}_2$, T and X two indeterminates over A and set $B := A[T]$. In this situation, $I(B,A) = (\mathbb{F}_2 + X\mathbb{F}_2[T,X]) \cap (\mathbb{F}_2 + (X-1)\mathbb{F}_2[T,X]) = (A + XB[X]) \cap (A + (X-1)B[X])$. Let $J := X(X-1)B[X]$. We have that $I(B,A)/J \cong A \times A$, J is a common ideal of the rings $A + J \subseteq I(B,A) \subseteq B[X]$. Now, $\dim_v(A+J) = 2 \geq \dim_v(I(B,A))$ [FIK3, Theorem 2.1]. Since each chain of prime ideals of $B[X]$ not containing J is preserved by contraction to $I(B,A)$, then $\dim(I(B,A)) = 2$. Therefore, $2 = \dim(I(B,A)) = \dim_v(I(B,A)) > \dim_v(A) + 1 = 1$.

2. Integer-valued polynomials and pullbacks

The description of the prime spectrum of the ring of the integer-valued polynomials is a crucial problem, and it has been settled for just a few classes of domains. The purpose of this section is to evaluate the Krull dimension of $\mathrm{Int}(D)$ and to describe $\mathrm{Spec}(\mathrm{Int}(D))$ for domains D arising from a particular type of pullback constructions.

In this section, we assume that

($\bullet$) *T is any integral domain, I is a nonzero ideal of T, R is an infinite integral domain such that $q.f.(R) = K \subseteq T/I$, $\varphi : T \to T/I$ is the canonical projection,*

and we consider the following pullback:

$$\begin{array}{ccc} D := \varphi^{-1}(R) & \longrightarrow & R \\ \downarrow & & \downarrow \\ S := \varphi^{-1}(K) & \longrightarrow & K \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & T/I \end{array}$$

In the setting ($\bullet$), the following inequality for the Krull dimension of the polynomial rings holds (cf. for instance [C1, Théorème 1 et Corollaire 1]):

$$\dim(D[X]) \leq \dim(T[X]) + \dim(R[X]). \tag{2.1}$$

The first aim of this section is to prove that an analogous inequality holds if we take the integer-valued polynomial ring.

Recall that, given an integral domain D, a *divided prime ideal* P of D is a prime ideal such that $PD_P = P$. If we denote by $\mathbf{k}(P)$ the residue field of D in P, it is well known that $\mathbf{k}(P)$ is canonically isomorphic to the quotient field of D/P. Therefore it is is easy to see that P is divided in D if and only if the following diagram of canonical homomorphisms is a pullback:

$$\begin{array}{ccc} D & \longrightarrow & D/P \\ \downarrow & & \downarrow \\ D_P & \longrightarrow & \mathbf{k}(P) \end{array}$$

A *divided domain* is an integral domain such that each prime ideal is divided. Next we recall a result proved by Tartarone [T, Corollary 2.2], that will be used several times in the sequel.

Lemma 2.1. *Let P be a divided prime of an integral domain D. Then*

$$\dim(\mathrm{Int}(D)) = \dim(D_P[X]) - 1 + \dim(\mathrm{Int}(D/P)). \qquad \square$$

The previous result may be generalized in several ways as will show Proposition 2.3 and its corollaries.

Let H be a nonzero ideal of an integral domain R and let $S_H = S := \{x \in R : x + H$ is a nonzero divisor of $R/H\}$. We notice that the total ring of fractions of R/H, $\mathrm{Tot}(R/H)$, is canonically isomorphic to $S^{-1}R/S^{-1}H$ and $S^{-1}H \cap R = H$. Consider the overring $R(H)$ of R defined by the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R(H) := \varphi^{-1}(R/H) & \longrightarrow & R/H \\ \downarrow & & \downarrow \\ S^{-1}R & \xrightarrow{\varphi} & S^{-1}R/S^{-1}H = \mathrm{Tot}(R/H) \end{array} \tag{2.2}$$

It is obvious that if $H = P$ is a prime ideal of R, then $R(P)$ coincides with R if and only if P is a divided prime.

Proposition 2.3. *Let R, H and $R(H)$ be as above. Then there exists a canonical isomorphism*

$$I(S^{-1}R, R(H)) \cong S^{-1}R[X] \times_{\mathrm{Tot}(R/H)[X]} I(\mathrm{Tot}(R/H), R/H).$$

Proof. By passing to polynomial extensions in (2.2), we obtain the pullback:

$$\begin{array}{ccc} R(H)[X] & \longrightarrow & (R/H)[X] \\ \downarrow & & \downarrow \\ I(S^{-1}R, R(H)) & \longrightarrow & I(\mathrm{Tot}(R/H), R/H) \\ \downarrow & & \downarrow \\ S^{-1}R[X] & \xrightarrow{\Phi} & \mathrm{Tot}(R/H)[X] \end{array}$$

where Φ is the canonical homomorphism associated to φ.
It is straightforward that $\Phi\mid_{I(S^{-1}R,R(H))}: I(S^{-1}R, R(H)) \to I(\mathrm{Tot}(R/H), R/H)$, this fact implies $\Phi^{-1}(I(\mathrm{Tot}(R/H), R/H)) = I(S^{-1}R, R(H))$. □

Corollary 2.4. *With the notation of Proposition 2.3, assume that, for each prime ideal P of R such that $P \cap S = \emptyset$, R/P is infinite, we have the following canonical isomorphism*

$$\mathrm{Int}(R(H)) \cong S^{-1}R[X] \times_{\mathrm{Tot}(R/H)[X]} I(\mathrm{Tot}(R/H), R/H).$$

In particular, if $H = P$ is a prime nonmaximal ideal of R, then we have the following canonical isomorphism

$$\mathrm{Int}(R(P)) \cong R_P[X] \times_{k(P)[X]} \mathrm{Int}(R/P).$$

Proof. In order to apply Corollary 1.3 (b), we notice that $S^{-1}R(H) = S^{-1}R$, because $R(H) \subseteq S^{-1}R$, hence $S^{-1}R$ is a flat overring of $R(H)$. Moreover, $\mathrm{Int}(R_P) = R_P[X]$ for each P such that R/P is infinite [CC, Corollaires, p. 303]. Therefore $I(S^{-1}R, R(H)) = \mathrm{Int}(R(H))$, hence the first statement follows from Proposition 2.3. □

Corollary 2.5. *With the notation and hypotheses of Corollary 2.4, we have:*

(a) $\dim(\mathrm{Int}(R(H))) \leq \dim(S^{-1}R[X]) + \dim(I(\mathrm{Tot}(R/H), R/H))$;
(b) $\dim(\mathrm{Int}(R(P))) = \dim(R_P[X]) + \dim(\mathrm{Int}(R/P)) - 1$.

Proof. (a) is a consequence of [C1, Corollaire 1, p. 509].

(b) From the second statement of Corollary 2.4, we deduce that $\mathrm{Int}(R/P) = \mathrm{Int}(R(P)/PR_P) \cong \mathrm{Int}(R/P)/PR_P[X]$. Since $\mathrm{Spec}(\mathrm{Int}(R(P)))$ is homeomorphic to the amalgamated sum $\mathrm{Spec}(R_P[X]) \coprod_{\mathrm{Spec}(k(P)[X])} \mathrm{Spec}(\mathrm{Int}(R/P))$ [F, Theorem 1.4], then

$$\dim(\mathrm{Int}(R(P))) = \max\{\dim(R_P[X]), \mathrm{ht}(PR_P[X]) + \dim(\mathrm{Int}(R/P))\} =$$
$$= \max\{\dim(R_P[X]), \dim(R_P[X]) - 1 + \dim(\mathrm{Int}(R/P))\} =$$
$$= \dim(R_P[X]) - 1 + \dim(\mathrm{Int}(R/P))$$

since $\dim(\mathrm{Int}(R/P)) \geq 1$, because P is nonmaximal prime by assumption. □

Proposition 2.6. *In the setting (*) described above, assume, in addition, that T is local. Then:*

$$\dim(\mathrm{Int}(D)) \leq \dim(T[X]) + \dim(\mathrm{Int}(R)), \tag{2.6.1}$$

and moreover,

$$\mathrm{Int}(D) = D[X] \Leftrightarrow \mathrm{Int}(R) = R[X]. \tag{2.6.2}$$

Proof. To avoid the trivial case, we can assume $R \neq K$. Let $\Sigma := R\backslash\{0\}$ and $\bar{\Sigma} := \varphi^{-1}(\Sigma)$. It is easy to see that the multiplicative subset $\bar{\Sigma}$ of D coincides with $D\backslash I$, where I is a prime ideal of D and D/I is isomorphic to R.

Claim. *Let T be as in (*) (thus, T is not necessarily local) and suppose that $Q \cap D \subsetneq I$ for each $Q \in \mathrm{Spec}(T)$, then T satisfies (2.6.1) and (2.6.2).*

Under the previous hypothesis, S is local with maximal ideal I and hence $S = \varphi^{-1}(K) = \bar{\Sigma}^{-1}D = D_I$, whence I is a divided prime ideal of D and $\mathbf{k}(I) \cong K$.

By applying Lemma 2.1 we obtain

$$\dim(\mathrm{Int}(D)) = \dim(D_I[X]) - 1 + \dim(\mathrm{Int}(D/I)). \tag{2.3}$$

Since the following diagram is also a pullback:

$$\begin{array}{ccc} D_I[X] & \longrightarrow & (D_I/ID_I)[X] \cong \mathbf{k}(I)[X] \\ \downarrow & & \downarrow \\ T[X] & \longrightarrow & (T/I)[X] \end{array}$$

it follows that $\dim(D_I[X]) \leq \dim(T[X]) + 1$ [C1, Théorème 1]. Therefore,

$$\dim(\mathrm{Int}(D)) \leq \dim(T[X]) + 1 - 1 + \dim(\mathrm{Int}(R)) = \dim(T[X]) + \dim(\mathrm{Int}(R)).$$

Moreover I is a prime nonmaximal divided ideal in D, so $I[X] = ID_I[X]$ and $\mathrm{Int}(D) \subseteq \bar{\Sigma}^{-1}\mathrm{Int}(D) \subseteq \mathrm{Int}(D_I) = D_I[X]$.

We claim that the following diagram of canonical maps:

$$\begin{array}{ccc} D[X] & \longrightarrow & (D/I)[X] \cong R[X] \\ \downarrow & & \downarrow \\ \mathrm{Int}(D) & \longrightarrow & \mathrm{Int}(D)/I[X] \cong \mathrm{Int}(R) \\ \downarrow & & \downarrow \\ D_I[X] & \longrightarrow & (D_I/ID_I)[X] \cong \mathbf{k}(I)[X] \end{array}$$

is a pullback. In fact, if $f(X) \in \mathrm{Int}(R) \subseteq \mathrm{k}(I)[X]$, then for each $F(X) \in \varphi^{-1}(f(X)) \in D_I[X]$ and for each $d \in D$, we have $\varphi(F(d)) = f(\varphi(d)) \in R$. Thus $F(d) \in D$. We conclude that $F(X) \in \mathrm{Int}(D)$. It is straightforward that $ker(\varphi : \mathrm{Int}(D) \to \mathrm{Int}(R)) = I[X]$, and hence $\mathrm{Int}(D) = D[X]$ if and only if $\mathrm{Int}(R) = R[X]$. □

Next goal is to study the case in which T is not necessarily local.

Theorem 2.7. *In the setting (*) described above, let $J := ID_I[X] \cap \mathrm{Int}(D)$. Then:*

(a) *we have the following canonical homeomorphisms of topological spaces:*

(i) $\mathrm{Spec}(\mathrm{Int}(R)) \to \{P \in \mathrm{Spec}(\mathrm{Int}(D)) : P \supseteq J\}$, $Q \mapsto \varphi^{-1}(Q)$;

(ii) $\mathrm{Spec}(D_I[X]/ID_I[X]) \to \{P \in \mathrm{Spec}(\mathrm{Int}(D)) : P \subseteq J\}$, $Q \mapsto Q \cap \mathrm{Int}(D)$.

(b) *Let $\Gamma := \{q \in \mathrm{Spec}(T) : q$ is incomparable with $I\}$. Assume that T/q is infinite for each $q \in \Gamma$. Then we have the following homeomorphism:*

$$\cup_{q\in\Gamma}\mathrm{Spec}(T_q[X]) \to \{P \in \mathrm{Spec}(\mathrm{Int}(D)) : P \quad \textit{is incomparable with} \quad J\},$$

$$Q \mapsto Q \cap \mathrm{Int}(D).$$

Proof. To avoid the trivial case we can assume $D \neq K$. As above, let $\hat{\Sigma} := D \setminus I$ and set $\hat{S} := \hat{\Sigma}^{-1}S$, $\hat{T} := \hat{\Sigma}^{-1}T$, and $\hat{I} := \hat{\Sigma}^{-1}I$. Since I is an ideal both in S and T, and it is maximal in S, then every element of $\hat{\Sigma}$ becomes a unit in S/I (and hence also in T/I). Consequently, $\hat{I}$ is also an ideal of $\hat{T}$, $\hat{S}/\hat{I} \cong S/I$, and $\hat{T}/\hat{I} \cong T/I$. Let $\hat{\varphi} : \hat{T} \to \hat{T}/\hat{I} \cong T/I$ be the natural projection associated canonically to φ. We consider the following diagram:

$$\begin{array}{ccccc}
D & \longrightarrow & \hat{D} := \hat{\varphi}^{-1}(R) & \longrightarrow & R \\
\downarrow & & \downarrow & & \downarrow \\
S & \longrightarrow & \hat{S} & \longrightarrow & \hat{S}/\hat{I} \cong K \\
\downarrow & & \downarrow & & \downarrow \\
T & \longrightarrow & \hat{T} & \xrightarrow{\hat{\varphi}} & \hat{T}/\hat{I} \cong T/I
\end{array}$$

In this situation, the argument used in the proof of Proposition 2.6 can be applied. In fact each prime ideal of $\hat{T}$ contracts to a prime ideal of $\hat{D}$ contained in $\hat{I}$, and $\hat{I}$ is a divided prime ideal of $\hat{D}$. Moreover $D_I = \hat{D}_{\hat{I}} = \hat{S}$. Thus

$$\mathrm{Int}(D) \subseteq \hat{\Sigma}^{-1}\mathrm{Int}(D) \subseteq \mathrm{Int}(D_I) = D_I[X] = \hat{D}_{\hat{I}}[X] = \hat{S}[X],$$

and we have the following diagram of canonical homomorphisms:

$$\begin{array}{ccccc} D[X] & \longrightarrow & \tilde{D}[X] & \longrightarrow & R[X] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Int}(D) & \longrightarrow & \mathrm{Int}(\tilde{D}) & \longrightarrow & \mathrm{Int}(R) \\ & & \downarrow & & \downarrow \\ & & \tilde{S}[X] & \longrightarrow & K[X] \end{array}$$

where $\mathrm{Int}(D) \subseteq \mathrm{Int}(\tilde{D})$. Indeed, if $F \in \mathrm{Int}(D) \subseteq \tilde{S}[X]$, then $f := F + I[X] \in K[X]$. Since $F \in \mathrm{Int}(D)$ and $D = \varphi^{-1}(R)$, we have $f(r) = \varphi(F(d)) = F(d)+I \in D/I \cong R$ for each $r \in R$ and $d \in \varphi^{-1}(\{r\})$. Therefore, $f \in \mathrm{Int}(R)$. Similarly, $\tilde{\varphi}(F(\tilde{d})) = f(\tilde{\varphi}(\tilde{d})) \in R$ for each $\tilde{d} \in \tilde{D}$. Hence $F(\tilde{d}) \in \tilde{D}$. It follows that $\mathrm{Int}(D) \subseteq \mathrm{Int}(\tilde{D})$.
Recall that $J = \tilde{I}[X] \cap \mathrm{Int}(D)$, hence $J \cap D[X] = I[X]$. Moreover, from the canonical isomorphisms $\mathrm{Int}(\tilde{D})/\tilde{I}[X] \cong \mathrm{Int}(R)$ and $\mathrm{Int}(\tilde{D})_{\tilde{D}\setminus I} = \tilde{D}_I[X]$, we deduce immediately the isomorphisms $\mathrm{Int}(D)/J \cong \mathrm{Int}(R)$ and $\mathrm{Int}(D)_{D\setminus I} = D_I[X]$, thus the statement (a) holds.
For (b), we note first that $J = \mathrm{rad}_{\mathrm{Int}(D)}(I[X]\mathrm{Int}(D))$. Therefore, if P is a prime ideal of $\mathrm{Int}(D)$ incomparable with J, then $P \cap D[X]$ is incomparable with $I[X]$. By a well known property of pullbacks [F1, Theorem 1.4 (c)], there exists a unique prime ideal $Q \in \mathrm{Spec}(T[X])$, such that Q is incomparable with $I[X]$, $Q \cap D[X] = P \cap D[X]$ and $D[X]_{P\cap D[X]} = T[X]_Q$. Further, if $p = P \cap D$ and $q = Q \cap T$, then p is a prime ideal of D incomparable with I and $D_p = T_q$. So that,

$$D_p[X] \subseteq \mathrm{Int}(D)_p \subseteq \mathrm{Int}(D_p) = \mathrm{Int}(T_q) = T_q[X] = D_p[X].$$

Here $\mathrm{Int}(T_q) = T_q[X]$ since T/q is infinite by assumption. We conclude that the map

$$\cup_{q\in\Gamma}\mathrm{Spec}(T_q[X]) \to \{P \in \mathrm{Spec}(\mathrm{Int}(D)) : P \quad \text{is incomparable with} \quad J\}$$

$$Q \mapsto Q \cap \mathrm{Int}(D)$$

is an homeomorphism of topological spaces. □

Corollary 2.8. (a) *With the same notation and hypotheses of Theorem 2.7, let* $\Delta := \{q \in \mathrm{Spec}(T) : q \supseteq I\}$ *and* $\mathcal{N} := \{q \in \Delta : q \text{ is minimal in } \Delta\}$. *Then:*

$$\dim(\mathrm{Int}(D)) \geq \dim(\mathrm{Int}(R)) + \dim(D_I[X]) - 1 \geq \dim(\mathrm{Int}(R)) + \sup\{\mathrm{ht}(q[X]) : q \in \mathcal{N}\}.$$

(b) *With the same notation and hypotheses of Theorem 2.7 (b), let* $\mathcal{M}$ *be the set of maximal primes of* Γ. *Then*

$$\dim(\mathrm{Int}(D)) = \sup\{\dim(\mathrm{Int}(R)) + \dim(D_I[X]) - 1; \quad \dim(T_m[X]) : m \in \mathcal{M}\}$$

where

$$\dim(D_I[X]) \leq \sup\{\dim(T[X]); \sup\{\mathrm{ht}(q[X]) + 1 + \inf\{1, \mathrm{t.d.}_K(T/q) : q \in \Delta\}\}.$$

Proof. (a) By Theorem 2.7 (a), we deduce easily that

$$\mathrm{ht}_{\mathrm{Int}(D)}(J) + \dim(\mathrm{Int}(R)) \leq \dim(\mathrm{Int}(D)).$$

On the other hand $\tilde{\Sigma}^{-1}J = ID_I[X]$, and since $D/I \cong R$ is infinite, $\tilde{\Sigma}^{-1}\mathrm{Int}(D) \subseteq \mathrm{Int}(\tilde{\Sigma}^{-1}D) = D_I[X]$ [CC, Corollaire 4, p.303].
Therefore $\mathrm{ht}_{\mathrm{Int}(D)}(J) = \mathrm{ht}_{D_I[X]}(ID_I[X]) = \dim(D_I[X]) - 1$. The second inequality follows from [C1, Théorème 1].

(b) The stated relations follow easily from Theorem 2.7 (b) and [C2, Lemme 4], recalling that $D_I[X] = \hat{D}_{\hat{I}}[X]$, $\hat{T} = \tilde{\Sigma}^{-1}T$ and $\hat{T}/\hat{I} \cong T/I$. □

Corollary 2.9. *In the situation (*) described above, assume, in addition, that T is local and $I = M$ is the maximal ideal of T. Let $d := \mathrm{t.d.}_K(T/M)$. We have:*

(a) $\dim(\mathrm{Int}(D)) = \dim(\mathrm{Int}(R)) + \dim(D_M[X]) - 1$.

(b) *If $d = 0$, then*

$$\dim(\mathrm{Int}(D)) = \dim(\mathrm{Int}(R)) + \dim(T[X]) - 1.$$

(c) *If $d \geq 1$, then*

$$\dim(\mathrm{Int}(R)) + \dim(T) + \inf\{1, d\} \leq \dim(\mathrm{Int}(D)) \leq \dim(\mathrm{Int}(R)) + \dim(T[X]).$$

Proof. (a) It follows from Corollary 2.8 (b).

(b) Since $d = 0$, it is well known that T is an integral extension of $S = D_M$: therefore $\dim(D_M[X]) = \dim(T[X])$.

(c) From [ABDFK, Proposition 2.7], we have

$$\dim(T) + \inf\{1, d\} + 1 \leq \dim(D_M[X]).$$

Hence the first inequality is a consequence of (a). The second inequality follows from (a) and [C2, Proposition 3 (a)]. □

We notice that the previous corollary implies that if T is a local Jaffard domain and $I = M$ is the maximal ideal of T, then $\dim(\mathrm{Int}(D)) = \dim(\mathrm{Int}(R)) + \dim(T[X]) - 1$ if $d = 0$, and $\dim(\mathrm{Int}(D)) = \dim(\mathrm{Int}(R)) + \dim(T[X])$ if $d \geq 1$.

Corollary 2.10. *In the situation* $(*)$ *described above, assume that* $T = V$ *is a valuation domain and* $\mathrm{rad}_V(I) = M$ *is the maximal ideal of* V. *Then*

$$\dim(\mathrm{Int}(D)) = \dim(V) + \inf\{1, \mathrm{t.d.}_K(V/M)\} + \dim(\mathrm{Int}(R)).$$

Proof. From [C2, Proposition 3] or [ABDFK Corollary 2.8] we can deduce that

$$\dim(D_I[X]) = \dim(V) + 1 + \inf\{1, \mathrm{t.d.}_K(V/M)\},$$

and the conclusion follows from Corollary 2.9 (a). □

It is of special interest the case in which K is a subring of T and, *a fortiori*, K is naturally embedded in T/I. Under this assumption,

Proposition 2.11. *Let* T *be an integral domain,* I *an ideal of* T, *and* R *an infinite integral domain such that* $K = \mathrm{q.f.}(R) \subseteq T$. *Let* $K \hookrightarrow T/I$ *be the natural embedding and* $\varphi : T \twoheadrightarrow T/I$ *the canonical projection. Set* $S := \varphi^{-1}(K) = K + I$, $D := \varphi^{-1}(R) = R + I$, *and* $J := ID_I[X] \cap \mathrm{Int}(D)$. *Then*

$$\mathrm{Int}(D) = \mathrm{Int}(R) + J.$$

Proof. Since $D = R + I$ and $S = K + I$, $KI \subseteq I$; then it is easily seen that $\mathrm{Int}(R) \subseteq \mathrm{Int}(D) \subseteq D_I[X]$. We already noticed, in a more general setting, that $\varphi : \mathrm{Int}(D) \twoheadrightarrow \mathrm{Int}(R)$ defines a canonical surjective homomorphism having kernel equal to J (cf. Proof of Theorem 2.7). We conclude immediately that $\mathrm{Int}(D) = \mathrm{Int}(R) + J$. □

Next result generalizes Corollary 2.10 under the assumptions of Proposition 2.11.

Corollary 2.12. *With the same notation and hypotheses of Proposition 2.11, assume in addition that* $T = V$ *is a valuation domain. Set* $P := \mathrm{rad}_V(I)$. *Then*

$$\dim(\mathrm{Int}(D)) = \dim(V_P) + \inf\{1, \mathrm{t.d.}_K(V/P)\} + \dim(\mathrm{Int}(R)).$$

Proof. From [FIK3, Théorème 1.13] we know that

$$\dim(D_I[X]) = \dim(V_P) + \inf\{1, \mathrm{t.d.}_K(V/P)\} + \dim(K[X]).$$

The conclusion follows from Corollary 2.8 (b). □

The following Example shows that the inequality in Corollary 2.8 (a) may be strict.

Example 2.13. Let T be a semilocal Prüfer domain with two maximal ideals M of height 1 and N of height 4. Assume that T/M and T/N have characteristic zero. In the situation $(*)$ of the beginning of this section, let $I := M$, $R := \mathbb{Z}$, $K := \mathbb{Q}$ and $\varphi : T \to T/M$ the natural projection. Since $\mathbb{Q}$ is canonically embedded in T/M, we can consider $D := \varphi^{-1}(\mathbb{Z})$. It is easy to check that $\dim(D) = 4$ and $\mathrm{ht}_D(M) = 1$; so that $\dim(D_M[X]) \leq 3$. Since $\dim(\mathrm{Int}(\mathbb{Z})) = 2$, $\dim(D_M[X]) + \dim(\mathrm{Int}(\mathbb{Z})) - 1 \leq 3 + 2 - 1 = 4$. It follows that $\mathrm{Int}(T) = T[X]$ [CC, Corollaire 2, p.303] and $\dim(T[X]) = \dim(T_N[X]) = 5$. Therefore, $\dim(\mathrm{Int}(D)) = \dim(T[X])$ (Corollary 2.8 (b)) and $\dim(D_M[X]) + \dim(\mathrm{Int}(\mathbb{Z})) - 1 < \dim(T[X]) = \dim(T_N[X]) = 5$.

3. An analogue of the Seidenberg Theorem

A celebrated result by Seidenberg [S2] ensures that: every integer h, with $n+1 \leq h \leq 2n+1$, is equal to the Krull dimension of a polynomial ring with coefficients in a suitable n-dimensional ring. The aim of this section is to show explicitly that a similar result holds for the integer-valued polynomial ring.

We start by recalling the definition of a P^rVD. Given a valuation domain $(V, M, \varphi : V \to V/M = k)$ and a subfield k_0 of the residue field k of V, a *pseudo-valuation domain* D or, for short, a PVD is an integral domain defined by a pullback of canonical homomorphisms of the following type:

$$
(3.1) \qquad \begin{array}{ccc} D := \varphi^{-1}(k_0) & \longrightarrow & k_0 \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & k \end{array}
$$

Given $r \geq 0$, by induction we define a P^rVD in the following way: if $r = 0$, a P^0VD is a PVD, and if $r \geq 1$, a P^rVD is an integral domain D obtained from a pullback diagram of the following type:

$$
(3.2) \qquad \begin{array}{ccc} D := \varphi^{-1}(R_{r-1}) & \longrightarrow & R_{r-1} \\ \downarrow & & \downarrow \\ T_{r-1} & \xrightarrow{\varphi} & k_{r-1} \end{array}
$$

where $(T_{r-1}, M_{r-1}, \varphi : T_{r-1} \to T_{r-1}/M_{r-1} = k_{r-1})$ is a $P^{r-1}VD$ and R_{r-1} is a PVD with k_{r-1} as its quotient field.

It is easy to see that a PVD is a local domain D with the same maximal ideal M of V; moreover $\dim(D) = \dim(V)$, and the valuation overring $V = (D: M) = (M: M)$ is called the *valuation domain associated* to D (cf. [AD], [HH], [F2]).

To each pseudo-valuation domain D, we can associate a pair of parameters $(\alpha;\delta)$, where $\alpha := \dim(V)$ and $\delta := \text{t.d.}_{k_0}(k)$.

By induction to each P^rVD we can associate a family of parameters $(\alpha_0, \cdots, \alpha_r; \delta_0, \cdots, \delta_r)$, where $(\alpha_0, \cdots, \alpha_{r-1}; \delta_0, \cdots, \delta_{r-1})$ (respectively, $(\alpha_r; \delta_r)$) are the parameters associated to the $P^{r-1}VD$ T_{r-1} (respectively, to the PVD R_{r-1}). We recall that if D is a PVD with parameters $(\alpha;\delta)$, then

$$\dim(D) = \alpha, \quad \dim_v(D) = \alpha + \delta, \tag{3.3}$$

$$\dim(D[X]) = \alpha + 1 + \inf\{1, \delta\}, \tag{3.4}$$

$$\dim(\text{Int}(D)) = \alpha + 1, \quad \text{if } k_0 \text{ is a finite field;} \tag{3.5}$$

(cf. [HH] and [ABDFK], or [DF2] for (3.3) and (3.4); [CH] or [T] for (3.5)). Recall that $\text{Int}(D) = D[X]$, when k_0 is infinite [CH].

Lemma 3.1. *Let D be a P^rVD with parameters $(\alpha_0, \cdots, \alpha_r; \delta_0, \cdots, \delta_r)$, $r \geq 0$. Then:*

(a) $\dim(D) = \sum_{i=0}^{r} \alpha_i$

(b) $\dim_v(D) = \sum_{i=0}^{r} \alpha_i + \delta_i$

(c) $\dim(D[X]) = \sum_{i=0}^{r} \alpha_i + 1 + \sum_{i=0}^{r} \inf\{1, \delta_i\}$

(d) $\dim(\text{Int}(D)) = \sum_{i=0}^{r} \alpha_i + 1 + \sum_{i=0}^{r-1} \inf\{1, \delta_i\}$, *if* $\text{Int}(D) \neq D[X]$,

where, obviously, $\sum_{i=0}^{r-1} \inf\{1, \delta_i\} = 0$ *if* $r = 0$.

Proof. (a), (b) and (c) were proved in [DF2] by using (3.3) and (3.4) and general properties of pullbacks (cf. [F1] and [ABDFK]).

(d) Since $\text{Int}(D) \neq D[X]$, the residue field of D is finite. We proceed by induction on r. If $r = 0$, the result holds by (3.5). Let $r \geq 1$. Since a P^rVD domain is a divided domain [F3], if $P := \ker(\varphi : T_{r-1} \to k_{r-1})$, then we have the following pullback diagram:

$$\begin{array}{ccc} D & \longrightarrow & D/P = R_{r-1} \\ \downarrow & & \downarrow \\ D_P = T_{r-1} & \longrightarrow & k_{r-1} = \text{q.f.}(R_{r-1}) \end{array}$$

Thus, by Lemma 2.1,

$$\dim(\text{Int}(D)) = \dim(T_{r-1}[X]) + \dim(\text{Int}(R_{r-1})) - 1.$$

By (c), we know that:

$$\dim(T_{r-1}[X]) = \sum_{i=0}^{r-1} \alpha_i + 1 + \sum_{i=0}^{r-1} \inf\{1, \delta_i\}.$$

Since the residue field of D is naturally isomorphic to the residue field of R_{r-1}, we can apply (3.5) to R_{r-1} and we obtain $\dim(\mathrm{Int}(R_{r-1})) = \alpha_r + 1$. The conclusion is now straightforward. □

Proposition 3.2. *Let n, m two positive integers with $n < m$. For each integer t, $1 \leq t \leq \inf\{n, m-n+1\}$ there exists an integral domain D with $\mathrm{Int}(D) \neq D[X]$, and:*

(a) $\dim(D) = n$;

(b) $\dim_v(D) = m$;

(c) $\dim(\mathrm{Int}(D)) = n + t$;

(d) $\dim(D[X]) = n + t^*$;

where $t^* \in \{t, t+1\}$, if $1 < t < m-n+1$, $t^* = t$, if $t = m-n+1$ (hence $t \neq 1$), and $t^* = t+1$, if $t = 1$.

Proof. We can assume $n \geq 2$ (for the case $n = 1$ an appropriate 1-dimensional PVD will satisfy the statements). In this case, an appropriate $P^{n-1}VD$ will satisfy the claim. Let D be a $P^{n-1}VD$ with parameters $(\alpha_0, \cdots, \alpha_{n-1}; \delta_0, \cdots, \delta_{n-1})$, with $\alpha_i = 1$ for $0 \leq i \leq n-1$. Assume that D has finite residue field. Clearly, $\dim(D) = n$ and $\mathrm{Int}(D) \neq D[X]$ [CH, Lemme 1.1].

In order to verify the other conditions, we only need to solve the following system:

$$m = n + \textstyle\sum_{i=0}^{n-1} \delta_i,$$
$$n + t = n + 1 + \textstyle\sum_{i=0}^{n-2} \inf\{1, \delta_i\},$$
$$n + t^* = n + 1 + \textstyle\sum_{i=0}^{n-1} \inf\{1, \delta_i\}.$$

If $t = 1$, then the given system has a solution for $\delta_i = 0$, when $0 \leq i \leq n-2$, and $\delta_{n-1} = m - n$.

Assume that $2 \leq t = m - n + 1 \leq n$. In this case, we fix a subset $\{i_1, i_2, \cdots, i_{t-1}\}$ of $\{0, \cdots, n-2\}$ and we see that the given system has a solution for $\delta_{i_1} = \cdots = \delta_{i_{t-1}} = 1$, $\delta_i = 0$ when $i \in \{0, \cdots, n-1\} \setminus \{i_1, i_2, \cdots, i_{t-1}\}$.

Let $1 < t < \inf\{n, m-n+1\}$, also in this case, we fix a subset $\{i_1, i_2, \cdots, i_{t-1}\}$ of $\{0, \cdots, n-2\}$. The given system will have a solution for $t^* = t$ (respectively, for $t^* = t+1$) by taking $\delta_{i_1} = \cdots = \delta_{i_{t-2}} = 1$, $\delta_{i_{t-1}} = m - n - (t-2)$, $\delta_i = 0$ when $i \in \{0, \cdots, n-1\} \setminus \{i_1, i_2, \cdots, i_{t-1}\}$ (respectively, $\delta_{i_1} = \cdots = \delta_{i_{t-1}} = 1$, $\delta_i = 0$ when $i \in \{0, \cdots, n-2\} \setminus \{i_1, i_2, \cdots, i_{t-1}\}$ and $\delta_{n-1} = m - n - (t-1)$). □

The goal of the remaining part of this section is to show that, as for the polynomial case, for each $n \geq 1$ and for each h where $n + 1 \leq h \leq 2n + 1$, there exists an integral domain D such that $\dim(D) = n$, $\dim(\mathrm{Int}(D)) = h$, and $\mathrm{Int}(D) \neq D[X]$.

We start proving this result for $n = 1$. If $h = 2$, one may apply Proposition 3.2 with $m = 2$ and $t = 1$. The following example establishes the case $h = 3$.

Example 3.3: Example of 1-dimensional domain D such that $\dim(\mathrm{Int}(D)) = 3$ and $\mathrm{Int}(D) \neq D[X]$.

Let k be a finite field. We denote an algebraic closure of k by K. Let U, V be two indeterminates over K. In the principal ideal domain $A := K(U)[V]$ we consider two maximal ideals $M_1 := (V)$, $M_2 := (V-1)$ and the multiplicatively closed set $S := K(U)[V] \backslash (M_1 \cup M_2)$. The maximal ideals of the principal ideal domain $T := S^{-1}A$ are $M' := S^{-1}M_1$ and $M'' := S^{-1}M_2$. Let $J(T) := M' \cap M''$ and let $\varphi : T \to T/J(T) \cong K(U) \times K(U)$ be the canonical projection. We can consider the following pullback diagram of canonical homomorphisms

$$\begin{array}{ccc} D := \varphi^{-1}(k \times K) & \longrightarrow & k \times K \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K(U) \times K(U). \end{array}$$

By general properties of pullbacks, D is a 1-dimensional semilocal domain with two maximal ideals $N' := M' \cap D$ and $N'' := M'' \cap D$, such that $D/N' \cong k$ and $D/N'' \cong K$. Moreover, $D_{N'}$ (respectively, $D_{N''}$) is a PVD with associated valuation domain $T_{M'}$ (respectively, $T_{M''}$). Therefore

$$\dim_v(D) = \sup\{\dim_v(D_{N'}), \dim_v(D_{N''})\} =$$

$$= \sup\{\dim(T_{M'}) + \mathrm{t.d.}_k(K(U)); \dim(T_{M''}) + \mathrm{t.d.}_K(K(U))\} = 2.$$

Hence, $\dim(\mathrm{Int}(D)) \leq \dim_v(\mathrm{Int}(D)) \leq \dim_v(D) + 1 = 3$.

Moreover, since $D/N'' = K$ is infinite and $\mathrm{t.d.}_K(K(U)) > 0$, we have $D_{N''}[X] \subseteq \mathrm{Int}(D)_{N''} \subseteq \mathrm{Int}(D_{N''}) = D_{N''}[X]$ and $\dim(D_{N''}[X]) = 3$. From the previous considerations, we deduce that $\dim(\mathrm{Int}(D)) = 3$. In order to prove that $\mathrm{Int}(D) \neq D[X]$, we consider the ideal $J(T)$ which is a common ideal of D and T which is principal in T. Let $y \in T$ such that $J(T) = yT$. Fix a set of elements $\{u_0 = 0, \cdots, u_q\}$ inside D such that their canonical images describe the finite field k. Pick an element $x \in N'' \backslash N'$ then $x^2/y \notin D$. If we consider $f(X) := (x^2/y)(X(X-u_1)\cdots(X-u_q))^2$ then $f(X) \notin D[X]$. Moreover, for each $d \in D$, there exists j, $0 \leq j \leq q$ such that $(d - u_j) \in N'$. Therefore $f(d) \in (x^2/y)N'^2 \subseteq (N''^2 N'^2)/y = J(T)^2/y = J(T) \subset D$, hence $f(X) \in \mathrm{Int}(D)$.

We are now ready to prove the main result of this section.

Theorem 3.4. *Let $n \geq 1$. For each h, with $n+1 \leq h \leq 2n+1$, there exists an integral domain D such that $\dim(D) = n$, $\dim(\mathrm{Int}(D)) = h$, and $\mathrm{Int}(D) \neq D[X]$.*

Proof. If $h = n+1$, then a PVD with parameters $(\alpha; \delta) = (n; 0)$ and finite residue field proves the statement. Let $n + 1 < h \leq 2n$. If we take $m \geq h - 1$ and $t = h - n$

in Proposition 3.2, we can find a local ($P^{n-1}VD$) domain satisfying the statement. It remains to prove the case $h = 2n+1$.
Take R to be equal to the 1-dimensional semilocal domain D considered in Example 3.3. Set $F := K(U,V)$ the quotient field of R. It is not difficult to construct a $P^{n-2}VD$ T, with $\dim(T) = n-1$ (and $\dim_v(T) \geq 2n-2$), such that $\dim(T[X]) = 2(n-1)+1$ and the residue field of T in its maximal ideal M is isomorphic to F. Let $\varphi : T \to F$ be the canonical projection and set $D := \varphi^{-1}(R)$. By construction, it is easy to see that M is a prime ideal of D, $D_M = T$, and $D/M \cong R$. Therefore Lemma 2.1 yields $\dim(\mathrm{Int}(D)) = \dim(T[X]) - 1 + \dim(\mathrm{Int}(R))$ with $\mathrm{Int}(D) \neq D[X]$ (since $\mathrm{Int}(R) \neq R[X]$). The conclusion follows immediately since $\dim(T[X]) = 2n-1$ and $\dim(\mathrm{Int}(R)) = 3$. We note also that $\dim_v(D) = \dim_v(T) + \dim_v(R) \geq 2n-2+2 = 2n$; this is a necessary condition for having $\dim(\mathrm{Int}(D)) = 2n+1$. □

4. Integer-valued polynomials on a subset

Let D be any integral domain with quotient field K and E a nonempty subset of K. Following [C4], we define the ring of integer-valued polynomials over E to be $\mathrm{Int}(E,D) := \{f(X) \in K[X] : f(E) \subseteq D\}$. We recall that $\mathrm{Int}(E,D)$ may be equal to D in several cases. For instance, if the integral closure of D is finitely generated over D, then $D = \mathrm{Int}(E,D)$ if and only if E is not a fractional subset of D [Mc, Lemma 2.0] (where E is said *fractional* of D if E is a subset of K and $dE \subseteq D$ for some nonzero $d \in D$). Cahen has established in [C4] some properties of the prime ideals of $\mathrm{Int}(E,D)$. He also proved that $\dim(D) \leq \dim(\mathrm{Int}(E,D)) \leq \dim_v(D) + 1$; and if, in addition, E is fractional, then $\dim(D) + 1 \leq \dim(\mathrm{Int}(E,D)) \leq \dim_v(D) + 1$ [C4, Proposition 1.3 and Corollary 1.4]. In this section we pursue this study. In particular, we compute the valuative dimension of $\mathrm{Int}(E,D)$ when E is a fractional subset of D and, as a consequence, we recover [C4, Corollary 1.4]. We first consider the case in which E is finite.

Lemma 4.1. *Let D be any integral domain of quotient field K and E a finite subset of K. Then:*

(a) $\dim(\mathrm{Int}(E,D)) = 1 + \dim(D)$
(b) $\dim_v(\mathrm{Int}(E,D)) = 1 + \dim_v(D)$
(c) *D is a Jaffard domain if and only if so is* $\mathrm{Int}(E,D)$.

Proof. Let $a \in E$, it is easily seen that $\mathrm{Int}(\{a\},D) = D + (X-a)K[X]$. Hence, $\mathrm{Int}(E,D) = \cap_{a\in E}\mathrm{Int}(\{a\},D) = \cap_{a\in E}(D+(X-a)K[X])$. Let $I := \Pi_{a\in E}(X-a)K[X]$; the rings $D + I \subset \mathrm{Int}(E,D) \subset K[X]$ share the ideal I. In particular, $\mathrm{Int}(E,D)/I \cong B := \Pi_{a\in E}D$, and $\dim(B) = \dim(D)$. Since $\dim(K[X]/I) = 0$, from [C1] the point (a) holds and, from [FIK3, Lemma 2.2], (b) and (c) follow. □

Proposition 4.2. *Let D be any integral domain of quotient field K and E a fractional subset of D. Then:*

(a) $\dim_v(\mathrm{Int}(E, D)) = \dim_v(D) + 1$

(b) *D is a Jaffard domain if and only if $\mathrm{Int}(E, D)$ is a Jaffard domain and* $\dim(\mathrm{Int}(E, D)) = \dim(D) + 1$.

Proof. (a) Since E is a fractional ideal, there exists $d \in D$ such that $dE \subseteq D$. Hence we have the following inclusions:

$$D[dX] \subseteq \mathrm{Int}(E, D) \subseteq \mathrm{Int}(F, D) \subseteq K[X],$$

where F is any finite subset of E. Therefore,

$\dim_v(D) + 1 = \dim_v(\mathrm{Int}(F, D)) \leq \dim_v(\mathrm{Int}(E, D)) \leq \dim_v(D[dX]) = \dim_v(D) + 1.$

(b) Assume that $\mathrm{Int}(E, D)$ is a Jaffard domain with $\dim(\mathrm{Int}(E, D)) = \dim(D) + 1$. From (a) it follows that $\dim(D) = \dim_v(D)$, so that D is a Jaffard domain. The converse is a straightforward consequence of (a). □

It is well known that $\dim(D[X]) - 1 \leq \dim(\mathrm{Int}(D))$ [C3, Proposition 1.4]. If $A \subseteq B$ is any extension of integral domains, we proved that the same inequality holds for $I(B, A)$ (Theorem 1.10). So we may ask whether a similar inequality does hold for $\mathrm{Int}(E, D)$ where E is a subset of the quotient field of D. Next Example 4.4 gives a negative answer to this question.

Remark 4.3. If E is just a non empty subset of K, the equality in Proposition 4.2 (a) does not hold in general. However, in any case, we have

$$\dim_v(\mathrm{Int}(E, D)) \leq \dim_v(D) + 1 \tag{4.1}$$

since from the inclusions $D \subseteq \mathrm{Int}(E, D) \subseteq K[X]$ we can deduce that $\dim_v(\mathrm{Int}(E, D)) \leq \dim_v(D) + \mathrm{t.d.}_D(\mathrm{Int}(E, D))$ [J, IV Proposition 5] and [G, Theorem 20.7]. The inequality (4.1) may be strict even when E is a D-submodule of K; for instance, if $D = \mathbb{Z}$ and $E = S^{-1}\mathbb{Z}$ for some multiplicative subset S of $\mathbb{Z}$, then $\mathrm{Int}(S^{-1}\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ [Mc, Lemma 2.0]; thus $\dim_v(\mathrm{Int}(S^{-1}\mathbb{Z}, \mathbb{Z})) = \dim_v(\mathbb{Z}) = 1$.

Example 4.4. Let k be a field and X, Y, Z and W four indeterminates over k. Set

$V_1 := k(X) + Yk(X)[Y]_{(Y)} = k(X) + M_1$,

$D_1 := k + M_1$,

$V := F(Z) + WF(Z)[W]_{(W)} = F(Z) + M$, where $F := \mathrm{q.f.}(D_1) = k(X, Y)$, and

$D := D_1 + M$.

Then V_1 and V are rank one valuation domains, and by wellknown properties of $D+M$ constructions, D_1 and D are respectively of dimension 1 and 2 (cf. [ABDFK] and

[BG]). Moreover, $\dim(D_1[X]) = 3$ by [HH], and $\dim(D[X]) = \dim(V) + \dim(D_1[X]) + \inf\{1, \text{t.d.}_{D_1}(F(Z))\} = 5$ by [ABDFK, Corollary 2.8]. It follows that $\dim(D[X]) - 1 = 4$. However, if E is a finite subset of q.f.(D), then by Lemma 4.1 $\dim(\text{Int}(E, D)) = \dim(D) + 1 = 3 < 4$. So that the inequality $\dim(D[X]) - 1 \leq \dim(\text{Int}(E, D)$ fails to be true.

Lemma 4.5. *Let D be an integral domain such that $D[X] = \text{Int}(D)$. Let K be the quotient field of D and I an ideal of D such that D/I is not a finite field. Then:*

(a) *For each nonzero principal fractional ideal aD of D,*

$$\text{Int}(aD, D) = D[X/a] \qquad \text{and} \qquad \text{Int}(aD, I) = I[X/a].$$

(b) *For each nonzero D-submodule E of K, let $E^* := E \setminus \{0\}$, and set*

$$D[X/E^*] := \cap_{a \in E^*} D[X/a], \qquad \text{and} \qquad I[X/E^*] := \cap_{a \in E^*} I[X/a].$$

Then

$$\text{Int}(E, D) = D[X/E^*], \qquad \text{Int}(E, I) = I[X/E^*].$$

Proof. (a) For the first equality, let $f(X) \in K[X]$, then

$$f(aD) \subseteq D \Leftrightarrow f(aX) \in \text{Int}(D) = D[X] \Leftrightarrow f(X) = f(aX/a) \in D[X/a].$$

It is obvious that $I[X/a] \subseteq \text{Int}(aD, I)$. Let $f(X) \in \text{Int}(aD, I)$. Then, in particular, $f(X) \in \text{Int}(aD, D) = D[X/a]$. Thus $g(X) := f(aX) \in D[X] \cap \text{Int}(D, I)$. If we consider the canonical image $\overline{g}(X) = \overline{f}(aX) \in (D/I)[X]$, then $\overline{g}(X) = 0$ because D/I is infinite and $\overline{g}(D/I) = 0$. Hence, $f(aX) \in I[X]$, whence $f(X) \in I[X/a]$.

(b) Since $\text{Int}(E, D) = \cap_{a \in E^*} \text{Int}(aD, D)$ and $\text{Int}(E, I) = \cap_{a \in E^*} \text{Int}(aD, I)$, the conclusion follows from (a). $\square$

Proposition 4.6. *Let T be an integral domain, I a nonzero ideal of T, $\varphi : T \to T/I$ the canonical projection and k an infinite field embedded in T/I. Set $D := \varphi^{-1}(k)$ and $K = $ q.f.(D). Let E be a D-submodule of K such that ET is a principal T-module with $ET = eT$ for some $e \in E$. Assume $D[X] = \text{Int}(D)$. Then*

$$\dim(\text{Int}(E, D)) \leq \inf\{\dim(T[X]) + 1, \dim(D[X]) + 1, \dim(\text{Int}(T)) + 1\}.$$

Proof. Without loss of generality we may assume that $E \neq 0$. From the assumptions and from Lemma 4.5, we have:

$$\text{Int}(E, D) = \cap_{a \in E^*} D[X/a] \subseteq \cap_{a \in E^*} T[X/a] \subseteq T[X/e] \subseteq \text{Int}(ET, T),$$

and

$$\mathrm{Int}(E,I) = \cap_{a \in E^*} I[X/a] \supseteq \mathrm{Int}(ET,I) \supseteq I[X/e].$$

Therefore $\mathrm{Int}(E,I) = \mathrm{Int}(ET,I) = I[X/e]$ is a common ideal of $\mathrm{Int}(E,D)$ and $\mathrm{Int}(ET,T)$. Hence, the following diagram of canonical homomorphisms is a pullback

$$\begin{array}{ccc}
\mathrm{Int}(E,D) & \longrightarrow & \mathrm{Int}(E,D)/I[X/e] \\
\downarrow & & \downarrow \\
D[X/e] & \longrightarrow & (D/I)[X/e] \cong k[X/e] \\
\downarrow & & \downarrow \\
T[X/e] & \longrightarrow & (T/I)[X/e] \\
\downarrow & & \downarrow \\
\mathrm{Int}(ET,T) & \longrightarrow & \mathrm{Int}(ET,T)/I[X/e].
\end{array}$$

Moreover, we note that $k \cong D/I \subseteq \mathrm{Int}(E,D)/I[X/e] \subseteq k[X/e]$. Since $\mathrm{t.d.}_k(k[X/e]) = 1$ and $I[X/e]$ is a prime ideal of $D[X/e]$ and $\mathrm{Int}(E,D)$, we deduce $\dim(\mathrm{Int}(E,D)/I[X/e]) \leq 1$ (cf. [G, Theorems 20.7 and 30.9]). By [C1, Corollaire 1, p. 509] we have

$$\dim(\mathrm{Int}(E,D)) \leq \dim(D[X/e]) + \dim(\mathrm{Int}(E,D)/I[X/e]) \leq \dim(D[X/e]) + 1$$

$$\dim(\mathrm{Int}(E,D)) \leq \dim(T[X/e]) + \dim(\mathrm{Int}(E,D)/I[X/e]) \leq \dim(T[X/e]) + 1$$

$$\dim(\mathrm{Int}(E,D)) \leq \dim(\mathrm{Int}(ET,T)) + \dim(\mathrm{Int}(E,D)/I[X/e]) \leq \dim(\mathrm{Int}(ET,T)) + 1.$$

The conclusion follows immediately since $D[X/e] \cong D[X]$, $T[X/e] \cong T[X]$, and $\mathrm{Int}(eT,T) \cong \mathrm{Int}(T)$. □

Corollary 4.7. *Let D be a PVD with quotient field K and E a D-submodule of K. Assume that the valuation overring canonically associated to D is a discrete valuation domain. Then*

$$\dim(\mathrm{Int}(E,D)) \leq \dim(D[X]) \leq \dim(D) + 2.$$

Proof. Let M be the maximal ideal of D, $V := (M : M)$, $k_0 := D/M$ and $k := V/M$. If $\mathrm{t.d.}_{k_0}(k) = 0$, then D is a 1-dimensional Jaffard domain. By Remark 4.3, $\dim_v(\mathrm{Int}(E,D)) \leq \dim_v(D) + 1 = \dim(D) + 1 = \dim(D[X])$.
If $\mathrm{t.d.}_{k_0}(k) > 0$, D is a 1-dimensional domain with $\dim(D[X]) = \dim(D) + 2$ [HH]. By Proposition 4.6 $\dim(\mathrm{Int}(E,D)) \leq \dim(V[X]) + 1 = \dim(V) + 2 = \dim(D) + 2 = \dim(D[X])$. □

Next goal is to show that Corollary 4.7 holds in a more general context (in particular, for every finite dimensional PVD).

Let D be an integral domain with quotient field K and S a non empty subset of $K^* := K\setminus\{0\}$. For each ideal $I \subseteq D$ we extend the notation introduced in Lemma 4.5 setting:

$$I[X/S] := \cap_{s\in S} I[X/s].$$

Lemma 4.8. *Let D be a Prüfer domain with quotient field K, S a non empty subset of K^* and I an ideal of D.*

(a) $\mathrm{Int}(SD, I) = \mathrm{Int}(\cup_{s\in S} sD, I) = \cap_{s\in S}\mathrm{Int}(sD, I)$.

Assume, in addition, that $\mathrm{Int}(D) = D[X]$. *Then*

(b) $\mathrm{Int}(SD, D) = D[X/S]$ *and* $\mathrm{Int}(SD, I) = I[X/S]$ *whenever D/I is infinite.*

Proof. (a) It is obvious that, for each ideal I of D

$$\mathrm{Int}(SD, I) \subseteq \mathrm{Int}(\cup_{s\in S} sD, I) = \cap_{s\in S}\mathrm{Int}(sD, I).$$

For the converse, let $f \in K[X]$. Since D_M is a valuation domain for each $M \in \mathrm{Max}(D)$, we have

$$f(sD) \subseteq I, \forall s \in S \Leftrightarrow f(sD_M) \subseteq I_M, \forall s \in S, \forall M \in \mathrm{Max}(D)$$

$$\Leftrightarrow f((s_1, \cdots, s_n)D_M) \subseteq I_M, \forall n \geq 1, \forall \{s_1, \cdots, s_n\} \subseteq S, \forall M \in \mathrm{Max}(D)$$

$$\Leftrightarrow f(xD_M) \subseteq I_M, \forall x \in SD_M, \forall M \in \mathrm{Max}(D) \Leftrightarrow f(SD) \subseteq I.$$

(b) Follows from (a) and Lemma 4.5. □

Lemma 4.9. *Let D be a Bézout domain with quotient field K, E a nonzero D-submodule of K and P a prime ideal of D such that D/P is infinite. Then*

(a) $P[X/E^*] = \mathrm{Int}(E, P) \cap D[X/E^*]$ *is a prime ideal of* $D[X/E^*]$

(b) *if* $\dim(D) < \infty$, *then there exists an element $a \in E^*$ such that*

$$P[X/a] \cap D[X/E^*] = P[X/E^*].$$

Proof. (a) From Lemma 4.8, we have

$$P[X/E^*] \subseteq \mathrm{Int}(E, P) \cap D[X/E^*] = \cap_{a\in E^*}(\mathrm{Int}(aD, P) \cap D[X/a]).$$

For the converse, since D/P is infinite, by the same argument of the proof of Lemma 4.5 (a), we have for each $a \in E^*$

$$\mathrm{Int}(aD, P) \cap D[X/a] = P[X/a].$$

Since D is a Bézout domain, for each finitely generated D-submodule $F = (f_1, f_2 \cdots, f_n)D$ of K, there exists $f \in F$ such that $F = fD$, so that:

$$P[X/F^*] = \mathrm{Int}(F, P) \cap D[X/F^*] = \mathrm{Int}(fD, P) \cap D[X/f] = P[X/f].$$

Let $\mathfrak{F}$ be the set of all finitely generated D-submodules of E.
Since $\mathrm{Int}(E, P) = \cap_{F \in \mathfrak{F}} \mathrm{Int}(F, P)$, $D[X/E^*] = \cap_{F\in\mathfrak{F}} D[X/F^*]$, and $P[X/E^*] = \cap_{F\in\mathfrak{F}} P[X/F^*]$, then

$$P[X/E^*] = \mathrm{Int}(E, P) \cap D[X/E^*].$$

Next we show that $P[X/E^*]$ is a prime ideal of $D[X/E^*]$.
Let $\alpha, \beta \in D[X/E^*]\setminus P[X/E^*]$. There exist $F(\alpha), F(\beta) \in \mathfrak{F}$ such that $\alpha \in D[X/E^*]\setminus P[X/F(\alpha)^*]$, $\beta \in D[X/E^*]\setminus P[X/F(\beta)^*]$. Set $F(\alpha) + F(\beta) =: F = fD$, for some $f \in F$. Then $\alpha, \beta \in D[X/F^*]\setminus P[X/F^*] = D[X/f]\setminus P[X/f]$, hence $\alpha\beta \notin P[X/f]$ and $\alpha\beta \in D[X/E^*]\setminus P[X/E^*]$.

(b) Since a finite dimensional Bézout domain is a Jaffard domain and D_P is a valuation domain, from the inclusions $D_P \subsetneq D[X/E^*]_P \subsetneq \mathrm{Int}(E, D)_P$ we get $\dim(D[X/E^*]_P) \leq \dim(D_P) + 1$ (Remark 4.3).
For each $a \in E^*$ and for each $\alpha \in aD$, we have $P[X/E^*] \subseteq P[X/a] \cap D[X/E^*] \subseteq P_\alpha = \{f \in D[X/E^*] : f(\alpha) \in P\}$. Notice that $P_\alpha := (P + (X - \alpha)K[X]) \cap D[X/E^*]$ is a prime ideal of $D[X/E^*]$ and it is a maximal ideal above P. In fact, using the projection $\pi : D[X/E^*] \to D/P$, $f \mapsto f(\alpha)(\mathrm{mod} P)$, we have that $D[X/E^*]/P_\alpha \cong D/P$ (being $ker(\pi) = P_\alpha$).
Now, if $P[X/a] \cap D[X/E^*] \subsetneq P_\alpha$, for some $\alpha \in aD$, we conclude that $P[X/E^*] = P[X/a] \cap D[X/E^*]$; otherwise we would have a chain of prime ideals of $D_P[X/E^*]$ longer than $\dim(D_P) + 1$.
Let us suppose, on the contrary, that for each $a \in E^*$, and for each $\alpha \in aD$, $P[X/a] \cap D[X/E^*] = P_\alpha$. Now, $0 \in aD$ for every $a \in E^*$, so that $P_\alpha = P_0 = P_\beta$ for each $\alpha, \beta \in E$. Since $P[X/E^*] = \cap_{a\in E^*}(P[X/a] \cap D[X/E^*]) = \cap_{\alpha\in E} P_\alpha$, it holds that $P[X/E^*] = P[X/a] \cap D[X/E^*] = P_\alpha$ for every $a \in E^*$ and $\alpha \in E$. □

Theorem 4.10. *Let T be a finite dimensional Bézout domain with quotient field K, Q a prime ideal of T, k a field contained in T/Q and $\varphi : T \to T/Q$ the canonical projection. Set $D := \varphi^{-1}(k)$. Assume that $D[X] = \mathrm{Int}(D)$. For each D-submodule E of K,*

$$\dim(\mathrm{Int}(E, D)) \leq \dim(T) + 2.$$

Moreover, if Q is maximal in T or if $\mathrm{t.d.}_k(T/Q) = 0$, then

$$\dim(\mathrm{Int}(E, D)) \leq \dim(D[X]).$$

Proof. We note that $\mathrm{Int}(0, D) = D + XK[X]$. Hence $\dim(\mathrm{Int}(0, D)) = \dim(D) + 1 \leq \dim(D[X]) \leq \dim(T[X]) + \dim(k[X]) = \dim(T) + 2$. Without loss of generality, we may assume that E is nonzero, k is infinite (since $\mathrm{Int}(D) = D[X]$, [C3, Lemma 3.1]) and then T/Q is also infinite. It is easy to see that Q is a maximal ideal of D. By Lemma 4.5 we have

$$\mathrm{Int}(E, D) = D[X/E^*] \quad \text{and} \quad \mathrm{Int}(E, Q) = Q[X/E^*].$$

Moreover by Lemma 4.8, we have

$$\mathrm{Int}(E, T) \supseteq \mathrm{Int}(ET, T) = \cap_{a \in E^*} \mathrm{Int}(aT, T) \supseteq \cap_{a \in E^*} T[X/a] = T[X/E^*],$$

$$\mathrm{Int}(E, Q) \supseteq \mathrm{Int}(ET, Q) = \cap_{a \in E^*} \mathrm{Int}(aT, Q) \supseteq \cap_{a \in E^*} Q[X/a] = Q[X/E^*].$$

Hence $\mathrm{Int}(E, Q) = \mathrm{Int}(ET, Q) = Q[X/E^*]$. By Lemma 4.9, for each $a \in E^*$,

$$Q[X/E^*] = Q[X/a] \cap T[X/E^*] = Q[X/a] \cap D[X/E^*].$$

Let us consider the following pullback of canonical homomorphisms:

$$\begin{array}{ccc}
\mathrm{Int}(E, D) = D[X/E^*] & \longrightarrow & D[X/E^*]/Q[X/E^*] \cong k[X/E^*] \\
\downarrow & & \downarrow \\
T[X/E^*] & \longrightarrow & T[X/E^*]/Q[X/E^*] \cong (T/Q)[X/E^*] \\
\downarrow & & \downarrow \\
\mathrm{Int}(ET, T) & \longrightarrow & \mathrm{Int}(ET, T)/Q[X/E^*].
\end{array}$$

Since $k \cong D/Q \subseteq D[X/E^*]/Q[X/E^*] \subseteq D[X/a]/Q[X/a] \cong k[X/a]$, then
$\dim(D[X/E^*]/Q[X/E^*]) \leq 1$ (cf. proof of Proposition 4.6).
Therefore, by [C1, Corollaire 1, p. 509] and Remark 4.3 $\dim(\mathrm{Int}(E, D)) \leq \dim(\mathrm{Int}(ET, T)) + \dim(D[X/E^*]/Q[X/E^*]) \leq \dim(\mathrm{Int}(ET, T)) + 1 \leq \dim(T) + 2$.
Note that if $\mathrm{t.d.}_k(T/Q) = 0$, then D is a Jaffard domain. By Remark 4.3, $\dim(\mathrm{Int}(E, D) \leq \dim(D) + 1 = \dim(D[X])$.
If Q is a maximal ideal of T, then $\dim(D) = \dim(T)$; and if D is not a Jaffard domain (i.e. $\mathrm{t.d.}_k(T/Q) > 1$), then $\dim(D[X]) > \dim(D) + 1 = \dim(T) + 1$. □

Corollary 4.11. *Let V be a finite dimensional valuation domain with quotient field K and residue field $\mathbf{k}(V)$. Let $\varphi : V \twoheadrightarrow \mathbf{k}(V)$ be the canonical projection. Let k be an infinite subfield of $\mathbf{k}(V)$ and $D := \varphi^{-1}(k)$. For each D-submodule E of K, we have*

$$\dim(\mathrm{Int}(E, D)) \leq \dim(D[X]) \leq \dim(D) + 2.$$

Proof. Since D is a PVD with an infinite residue field k, then $D[X] = \mathrm{Int}(D)$. The conclusion follows from Theorem 4.10. ☐

In the situation of the previous corollary, if $k \subset \mathbf{k}(V)$ is a finite extension of fields and E is not a fractional subset of D, then $\mathrm{Int}(E, D) = D$ [Mc, Lemma 2.0], so that $\dim(\mathrm{Int}(E, D)) = \dim(D)$.

BIBLIOGRAPHY

[AAZ] D.D. ANDERSON - D.F. ANDERSON - M. ZAFRULLAH, *Rings between D[X] and K[X]*, Houston J. Math. **17** (1991), 109–129.

[AD] D.F. ANDERSON - D.E. DOBBS, *Pair of rings with the same prime ideals*, Canad. J. Math. **32** (1980), 362–384.

[ABDFK] D.F. ANDERSON - A. BOUVIER - D.E. DOBBS - M. FONTANA - S. KABBAJ, *On Jaffard domains*, Expo. Math. **6** (1988), 145–175.

[BG] E. BASTIDA - R. GILMER, *Overrings and divisorial ideals of the form D+M*, Michigan Math. J. **20** (1973), 79–95.

[C1] P.-J. CAHEN, *Couples d'anneaux partageant un idéal*, Arch. Math. **51** (1988), 505–514.

[C2] P.-J. CAHEN, *Construction B, I, D et anneaux de Jaffard*, Arch. Math. **54** (1990), 125–141.

[C3] P.-J. CAHEN, *Dimension de l'anneau des polynômes à valeurs entières*, Manuscripta Math. **67** (1990), 333–343.

[C4] P.-J. CAHEN, *Integer-valued polynomials on a subset*, Proc. Amer. Math. Soc. **117** (1993), 919–929.

[CC] P.-J. CAHEN - J.-L. CHABERT, *Coefficients et valeurs d'un polynôme*, Bull. Sc. Math. **95** (1971), 295–304.

[CH] P.-J. CAHEN - Y. HAOUAT, *Polynômes à valeurs entières sur un anneau de pseudovaluation*, Manuscripta Math. **61** (1988), 23–31.

[Ch1] J.-L. CHABERT, *Anneaux de polynômes à valeurs entières et anneaux de Fatou*, Bull. Sc. Math. France **99** (1971), 273–283.

[Ch2] J.-L. CHABERT, *Les idéaux premiers de l'anneau des polynômes à valeurs entières*, J. Reine Angew. Math. **293/294** (1977), 275–283.

[Ch3] J.-L. CHABERT, *Un anneau de Prüfer*, J. Algebra **107** (1987), 1–16.

[DF1] D.E. DOBBS - M. FONTANA, *Universally incomparable ring-homomorphisms*, Bull. Austral. Math. Soc. **9** (1984), 289–302.

[DF2] D.E. DOBBS - M. FONTANA, *Sur les suites dimensionnelles et une classe d'anneaux distingués qui les déterminent*, C.R. Acad. Sci. Paris **306** (1988), 11–16.

[DFK] D.E. DOBBS - M. FONTANA - S. KABBAJ, *Direct limits of Jaffard domains and S-domains*, Comm. Math. Univ. St. Pauli **39** (1990), 143–155.

[F1] M. FONTANA, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl. **123** (1980), 331–345.

[F2] M. FONTANA, *Carrés cartésiens et anneaux de pseudo-valuation*, Publ. Dépt. Math. Univ. Lyon **17** (1980), 57–95.

[F3] M. FONTANA, *Sur quelques classes d'anneaux divisés*, Rend. Sem. Mat. Fis. Milano **51** (1981), 179–200.

[FIK1] M. FONTANA - L. IZELGUE - S. KABBAJ, *Krull and valuative dimension of the rings of the form $A + XB[X]$*, Proc. Fès Conference "Commuative Ring Theory", Lecture Notes Pure Appl. Math., Marcel Dekker **153** (1993), 111–130.

[FIK2] M. FONTANA - L. IZELGUE - S. KABBAJ, *Quelques propriétés des chaînes d'idéaux premiers dans les anneaux $A + XB[X]$*, Comm. Algebra **22** (1994), 9–27.

[FIK3] M. FONTANA - L. IZELGUE - S. KABBAJ, *Les sous-anneaux de la forme $D + I$ d'un anneau intègre*, Comm. Algebra **23** (1995), 4180–4210.

[FK] M. FONTANA - S. KABBAJ, *On the Krull and valuative dimension of $D+(X_1,\cdots,X_r)D_S[X_1,X_2,\cdots,X_r]$ domains*, J. Pure Appl. Algebra **63** (1990), 231–245.

[G] R. GILMER, Multiplicative ideal theory, Dekker, New York (1972).

[J] P. JAFFARD, *Théorie de la dimension dans les anneaux de polynômes*, Mem. Sc. Math., Gauthier-Villars, Paris **146** (1960).

[HH] J.R. HEDSTROM - E.G. HOUSTON, *Pseudo-valuation domains II*, Houston J. Math. **4** (1978), 199–207.

[Mc] D. McQUILLAN, *On Prüfer domains of polynomials*, J. Reine Angew. Math. **385** (1985), 162–178.

[R] F. RICHMAN, *Generalized quotient rings*, Proc. Amer. Math. Soc. **16** (1965), 794–799.

[T] F. TARTARONE, *On the Krull dimension of Int(D) when D is a pullback*, Proc. Fès Conference "Commutative Ring Theory II", Lecture Notes Pure Appl. Math., Marcel Dekker **185** (1997), 457–476.

[V] S. VISWESWARAN, *Subrings of $\dot{K}[Y_1,\cdots,Y_t]$ of the type $D+I$*, J. Algebra **117** (1988), 374–389.

[S1] A. SEIDENBERG, *A note on the dimension theory of rings*, Pac. J. Math. **3** (1952), 305–312.

[S2] A. SEIDENBERG, *On the dimension theory of rings II*, Pac. J. Math. **4** (1954), 603–614.

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