Dekker Lect. Notes Pure Appl. Math. 205 (1999) 307-321.

Polynomial Closure in Essential Domains and Pullbacks

Marco Fontana *, Università degli Studi "Roma Tre", Italy

Lahoucine Izelgue, Université de Marrakech, Morocco

Salah-Eddine Kabbaj, King Fahd University, Saudi Arabia

Francesca Tartarone, Università degli Studi "La Sapienza", Italy

ABSTRACT. Let D be a domain with quotient field K. Let $E \subseteq K$ be a subset; the ring of *D-integer-valued polynomials* over E is $\text{Int}(E, D) := \{f \in K[X]; f(E) \subseteq D\}$. The *polynomial closure* in D of a subset $E \subseteq K$ is the largest subset $F \subseteq K$ containing E such that $\text{Int}(E, D) = \text{Int}(F, D)$, and it is denoted by $cl_D(E)$. We study the polynomial closure of ideals in several classes of domains, including essential domains and domains of strong Krull-type, and we relate it with the t-closure. For domains of Krull-type we also compute the Krull dimension of $\text{Int}(D)$.

INTRODUCTION

Let D be any integral domain with quotient field K. For each subset $E \subseteq K$, $\text{Int}(E, D) := \{f \in K[X]; f(E) \subseteq D\}$ is called the *ring of D-integer-valued polynomials over E*. As usual, when $E = D$, we set $\text{Int}(D) := \text{Int}(D, D)$. When E is "large enough", it may happen that $\text{Int}(E, D) = D$ (for instance, $\text{Int}(S^{-1}\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ for each nontrivial multiplicative subset S of $\mathbb{Z}$, [CC2, Corollary I.1.10]). This does not happen if E is a *D-fractional subset* of K, i.e. if there exists $d \in D\backslash(0)$ such that $dE \subseteq D$. Indeed, in this case, $dX \in \text{Int}(E, D)$. It is well known that

* Partially supported by a NATO Collaborative Research Grant No. 970140.

$\mathrm{Int}(D) \subseteq \mathrm{Int}(E, D)$ if and only if $E \subseteq D$ [CC2, Corollary I.1.7]. Two subsets E and F of K may be distinct while $\mathrm{Int}(E, D) = \mathrm{Int}(F, D)$. When this happens, we say that E and F are *polynomially D-equivalent*. For instance, $\mathbb{N}$ and $\mathbb{Z}$ are polynomially $\mathbb{Z}$-equivalent [CC2, Corollary I.1.2]. In particular, if $\mathrm{Int}(E, D) = \mathrm{Int}(D)$ we say that E is a *polynomially dense* subset of D (so, $\mathbb{N}$ is a polynomially dense subset of $\mathbb{Z}$). In [G1] Gilmer characterized the polynomially dense subsets of $\mathbb{Z}$ and, in [C2] and [C4], Cahen studied the polynomial density, with special emphasis to the Noetherian domains case. McQuillan, pursuing Gilmer's work, investigated the polynomially D-equivalent subsets of a Dedekind domain D [Mc]. Among other results, he proved that two fractional ideals I and J of a Dedekind domain D are polynomially equivalent if and only if $I = J$. After noticing that, Cahen introduced the notion of *polynomial closure* (in D) of a subset E of K as follows:

$$cl_D(E) := \{x \in K; f(x) \in D, \text{ for each } f \in \mathrm{Int}(E, D)\},$$

that is, $cl_D(E)$ is the largest subset $F \subseteq K$ such that $\mathrm{Int}(E, D) = \mathrm{Int}(F, D)$. Obviously, E is said to be *polynomially dense* in D if $cl_D(E) = D$ and *polynomially D-closed* if $cl_D(E) = E$.

In the first section of this paper, we consider *essential domains*, that is, domains D such that

$$D = \cap_{P \in \mathcal{P}} D_P, \tag{1}$$

where $\mathcal{P}$ is a subset of $\mathrm{Spec}(D)$ and D_P is a valuation domain. In particular, among these domains, we will focus our attention on the *strong Krull-type domains*, that is the essential domains D such that the intersection (1) is locally finite (i.e., each nonzero of D belongs to finitely many prime ideals $P \in \mathcal{P}$) and the valuation rings D_P are pairwise independent. Examples of strong Krull-type domains are *Krull domains* and *generalized Krull domains* [G1, p. 524]. We prove that if E is a fractional subset of a strong Krull-type domain D, then $cl_D(E) = \cap_{P \in \mathcal{P}} cl_{D_P}(E_P)$, where $cl_{D_P}(E_P)$ denotes the polynomial D_P-closure of $E_P := \{e/s; e \in E, s \in D \setminus P\}$. This yields a generalization of a result proved by Cahen for Krull domains, [C3] or [C4]. Moreover, we study the polynomial closure as a star-operation and we relate it to the t-operation. We find that if D is an essential domain, then $I_t \subseteq cl_D(I)$, for each fractional ideal I of D and, for some distinguished classes of domains, as Krull domains, Prüfer domains in which each nonzero ideal is divisorial, $I_t = cl_D(I)$.

In the second section we compute the Krull dimension of the ring of $\mathrm{Int}(D)$, when D is a domain with a locally finite representation. By using this result we show that if D is a domain of Krull-type then $\dim(\mathrm{Int}(D)) = \dim(D[X])$, obtaining further evidence for the validity of the conjecture about the Krull dimension of $\mathrm{Int}(D)$ stating that $\dim(\mathrm{Int}(D)) \leq \dim(D[X])$ for each integral domain D (cf. [Ch], [C1], and [FIKT]).

In Section 3, we study the quotient of the polynomial closure of a subset modulo a divided prime ideal, and we apply this result to some classes of domains defined by making use of pullback constructions.

1. POLYNOMIAL CLOSURE IN STRONG KRULL-TYPE AND NOETHERIAN DOMAINS

We start this section by studying the polynomial closure of the ideals in a *strong Krull-type domain*, that is a domain D having the following representation:

$$D = \cap_{P \in \mathcal{P}} D_P, \tag{1.0.I}$$

where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, the intersection (1.0.I) is locally finite and the rings D_P are pairwise independent valuation domains.

We prove for this class of domains some results already proved by Cahen in case of Noetherian and Krull domains [C3, Proposition 1.3], [C4, Proposition 3.2, 3.5, 3.6, Corollaries 3.7 and 3.8].

THEOREM 1.1. *With the notation above, let D be a strong Krull-type domain with quotient field K and let E be any D-fractional subset of K. Then:*
(1) $\mathrm{Int}(E, D)_P = \mathrm{Int}(E, D_P)$, *for each* $P \in \mathcal{P}$;
(2) $cl_D(E) = \cap_{P \in \mathcal{P}} cl_{D_P}(E_P)$;
(3) *if $E \subseteq D$, then E is polynomially dense in D if and only if E is polynomially dense in D_P for each $P \in \mathcal{P}$.*

Proof. (1) The argument of the proof runs parallel with the one used in [C4, Proposition 3.2] (cf. also [CC2, Proposition I.2.8 and IV.2.9]). We wish to prove that $\mathrm{Int}(E, D_{\overline{P}}) \subseteq \mathrm{Int}(E, D)_{\overline{P}}$, for each fixed ideal $\overline{P} \in \mathcal{P}$ (the opposite inclusion holds in general, [CC2, Lemma I.2.4]).

Since E is a fractional D-subset of K, with a standard argument we can easily assume, without loss of generality, that $E \subseteq D$. Let $f \in \mathrm{Int}(E, D_{\overline{P}})$, $f \neq 0$. It is obvious that there exists $d \in D$, $d \neq 0$, such that $df \in D[X]$. Set

$$\mathcal{P}(d) := \{Q \in \mathcal{P}; d \in Q \text{ and } Q \not\subseteq \overline{P}\}.$$

Since the given representation of D is locally finite, then $\mathcal{P}(d)$ is a finite set.

We claim that there exists $a \in D$ such that $a \in dD_Q \setminus \overline{P}$, for each $Q \in \mathcal{P}(d)$, that is, $\mathbf{v}_Q(a) \geq \mathbf{v}_Q(d)$, for each $Q \in \mathcal{P}(d)$, and $\mathbf{v}_{\overline{P}}(a) = 0$, where $\mathbf{v}_P$ is the valuation associated to the ring D_P for each $P \in \mathcal{P}$.

The Approximation Theorem for valuations [B, Ch. VI § 7 n. 2, Corollaire 1, p. 135] states that there exists an element $b \in K$ such that $\mathbf{v}_{\overline{P}}(b) = 0$ and $\mathbf{v}_Q(b) \geq \mathbf{v}_Q(d)$, for each $Q \in \mathcal{P}(d)$. Now, there exist $a, c \in D$, $a \notin \overline{P}$, such that $b = a/c$. For each $P \in \mathcal{P}$, we have that $\mathbf{v}_P(a) = \mathbf{v}_P(b) + \mathbf{v}_P(c)$. Thus, if $Q \in \mathcal{P}(d)$, then $\mathbf{v}_Q(c) \geq 0$ (since $c \in D \subseteq D_Q$) and $\mathbf{v}_Q(a) \geq \mathbf{v}_Q(b) \geq \mathbf{v}_Q(d)$. Moreover, since $\mathbf{v}_{\overline{P}}(b) = \mathbf{v}_{\overline{P}}(c) = 0$, then also $\mathbf{v}_{\overline{P}}(a) = 0$.

Therefore $af \in D_P[X]$ for each $P \in \mathcal{P}(d)$ and $P \in \mathcal{P}$ such that $d \notin P$. As a matter of fact, for each $P \in \mathcal{P} \setminus \mathcal{P}(d)$, with $d \notin P$, we have that d is a unit in D_P. Thus, since $df \in D[X]$, we deduce that $f \in D_P[X]$ and $af \in D_P[X]$. If $Q \in \mathcal{P}(d)$, then $ad^{-1} \in D_Q$ and $af = (ad^{-1})(df) \in D_Q[X]$. In these cases, since $E \subseteq D$ and $af \in D_P[X]$, then $af(E) \subseteq D_P$, that is, $af \in \mathrm{Int}(E, D_P)$. On the other hand, $f \in \mathrm{Int}(E, D_{\overline{P}})$ and $a \in D$, whence $af(E) \subseteq D_{\overline{P}}$. If $d \in P$ and $P \subseteq \overline{P}$, then $D_P \supseteq D_{\overline{P}}$. Thus $f(E) \subseteq D_{\overline{P}} \subseteq D_P$ and $f \in \mathrm{Int}(E, D_P)$.

We conclude that $af(E) \subseteq \cap_{P \in \mathcal{P}} D_P = D$, that is $af \in \mathrm{Int}(E, D)$. Hence $f \in \mathrm{Int}(E, D)_{\overline{P}}$, because $a \in D \setminus \overline{P}$.

(2) follows from (1) and from [C4, Proposition 3.5].

(3) is a straightforward consequence of (2). □

In order to deepen the study of the polynomial closure of fractional ideals, we recall some properties about star-operations.

Let D be an integral domain with quotient field K, let $\mathfrak{F}(D)$ denote the set of nonzero fractional ideals of D and let $\mathfrak{F}_{fg}(D)$ denote the subset of $\mathfrak{F}(D)$ of nonzero finitely generated fractional ideals of D. A mapping $I \mapsto I^*$ of $\mathfrak{F}(D)$ into $\mathfrak{F}(D)$ is called a *star-operation on* D if the following conditions hold for all $a \in K\setminus\{0\}$ and $I, J \in \mathfrak{F}(D)$:

$(*1)$ $(aD)^* = aD$;

$(*2)$ $(aI)^* = aI^*$;

$(*3)$ $I \subseteq I^*$;

$(*4)$ $I \subseteq J \Rightarrow I^* \subseteq J^*$;

$(*5)$ $I^{**} = I^*$.

A fractional ideal $I \in \mathfrak{F}(D)$ is called a *star-ideal* if $I = I^*$. A *star-operation* $*$ on D is said to be *of finite character* if, for each $I \in \mathfrak{F}(D)$,

$$I^* = \cup\{J^*; J \subseteq I \text{ and } J \in \mathfrak{F}_{fg}(D)\}.$$

Given a star-operation $*$, then the function $*_s$ defined as follows:

$$I \mapsto I^{*_s} = \cup\{J^*; J \subseteq I \text{ and } J \in \mathfrak{F}_{fg}(D)\}$$

is a star-operation of finite character. The star-operation $*_s$ is called the *star-operation of finite character associated to* $*$. It is obvious that:

$$J^* = J^{*_s}, \text{ for each } J \in \mathfrak{F}_{fg}(D),$$

$$I^{*_s} \subseteq I^*, \text{ for each } I \in \mathfrak{F}(D).$$

The *v-operation*

$$I \mapsto I_v := (D: (D: I))$$

is a star-operation. The *t-operation*

$$I \mapsto I_t := \cup\{J_v; J \subseteq I \text{ and } J \in \mathfrak{F}_{fg}(D)\}$$

is the star-operation of finite character associated to the v-operation (cf. [G2, Sections 32 and 34]).

The following result is implicitly proved by Cahen [C4, Lemma 1.2].

LEMMA 1.2. *Let D be an integral domain, then the polynomial closure*

$$cl_D : \mathfrak{F}(D) \to \mathfrak{F}(D), \quad I \mapsto cl_D(I),$$

is a star-operation. □

COROLLARY 1.3. *Let D be an integral domain. Then, for all I, J in $\mathfrak{F}(D)$ and for each subset $\{I_\alpha; \alpha \in A\}$ of $\mathfrak{F}(D)$, we have:*

(1) $cl_D(\sum_\alpha I_\alpha) = cl_D(\sum_\alpha cl_D(I_\alpha))$, *if* $\sum_\alpha I_\alpha \in \mathfrak{F}(D)$;

(2) $\cap_\alpha cl_D(I_\alpha) = cl_D(\cap_\alpha cl_D(I_\alpha))$, *if* $\cap_\alpha I_\alpha \neq (0)$;

(3) $cl_D(IJ) = cl_D(I cl_D(J)) = cl_D(cl_D(I)J) = cl_D(cl_D(I) cl_D(J))$;

(4) $cl_D(I) \subseteq I_v$;

(5) $cl_D(I_v) = I_v$.

Proof. It is a straightforward consequence of Lemma 1.2 and of [G2, Proposition 32.2 and Theorem 34.1(4)]. □

Note that, from Corollary 1.3(1) and (3), we recover for the fractional ideals some results proved by Cahen for subsets [C4, Lemma 2.4], in particular we obtain that $cl_D(I) + cl_D(J) \subseteq cl_D(I+J)$, $cl_D(I)cl_D(J) \subseteq cl_D(IJ)$.

We will need the following result, that is a consequence of [C4, Proposition 3.5(2)], in order to deepen the relation between the polynomial closure and the star-operations.

LEMMA 1.4. *Let D be an integral domain and let $\mathcal{P}$ be a subset of* Spec(D) *such that $D = \cap_{P\in\mathcal{P}} D_P$. For each $I \in \mathfrak{F}(D)$, we have:*

$$\cap_{P\in\mathcal{P}} cl_{D_P}(ID_P) \subseteq cl_D(I). \quad \square$$

It is well known from the theory of star-operations that, if $\{D_\alpha; \alpha \in A\}$ is a collection of overrings of an integral domain D such that $D = \cap_{\alpha\in A} D_\alpha$, and if $*_\alpha$ is a star-operation on D_α, for each $\alpha \in A$, then the mapping

$$I \mapsto I^{*_A} := \cap\{(ID_\alpha)^{*_\alpha}; \alpha \in A\}$$

is a star-operation on D and $(I^{*_A}D_\alpha)^{*_\alpha} = (ID_\alpha)^{*_\alpha}$, for each $\alpha \in A$ [A, Theorem 2]. If $D = \cap_{P\in\mathcal{P}} D_P$, for some subset $\mathcal{P} \subseteq$ Spec(D), we call the *$\mathcal{P}$-polynomial closure of* $I \in \mathfrak{F}(D)$, the following fractional ideal of D:

$$\mathcal{P}\text{-}cl_D(I) := \cap_{P\in\mathcal{P}} cl_{D_P}(ID_P).$$

PROPOSITION 1.5. *Let D be an integral domain such that $D = \cap_{P\in\mathcal{P}} D_P$, for some subset $\mathcal{P}$ of* Spec(D).

(1) *The mapping:*

$$I \mapsto \mathcal{P}\text{-}cl_D(I)$$

defines a star-operation on D, with $\mathcal{P}$-$cl_D(I) \subseteq cl_D(I)$ for each $I \in \mathfrak{F}(D)$.

(2) *Let $I \in \mathfrak{F}(D)$. If $ID_P \neq D_P$ for finitely many $P \in \mathcal{P}$, then*

$$\mathcal{P}\text{-}cl_D(I)D_P = cl_{D_P}(ID_P).$$

(3) *If $cl_{D_P}(F \cap G) = cl_{D_P}(F) \cap cl_{D_P}(G)$, for each $P \in \mathcal{P}$ and for all $F, G \in \mathfrak{F}(D_P)$, then*

$$\mathcal{P}\text{-}cl_D(I \cap J) = \mathcal{P}\text{-}cl_D(I) \cap \mathcal{P}\text{-}cl_D(J), \text{ for all } I, J \in \mathfrak{F}(D).$$

(4) *If* $cl_{D_P}((F:_{D_P} G)) = (cl_{D_P}(F):_{D_P} cl_{D_P}(G))$, *for each* $P \in \mathcal{P}$, $F \in \mathfrak{F}(D_P)$ *and* $G \in \mathfrak{F}_{fg}(D_P)$, *then*

$$\mathcal{P}\text{-}cl_D((I:_D J)) = (\mathcal{P}\text{-}cl_D(I):_D \mathcal{P}\text{-}cl_D(J)),$$

for each $I \in \mathfrak{F}(D)$ *and* $J \in \mathfrak{F}_{fg}(D)$.

(5) *If* $D = \cap_{P \in \mathcal{P}} D_P$ *is locally finite and if, for each* $P \in \mathcal{P}$, cl_{D_P} *is a star-operation on* D_P *with finite character, then* $\mathcal{P}\text{-}cl_D$ *is a star-operation on* D *with finite character.*

Proof. These results are a straightforward consequence of Lemma 1.4, of the definition of the $\mathcal{P}$-polynomial closure and of [A, Theorem 2]. □

COROLLARY 1.6. *If* $D = \cap_{P \in \mathcal{P}} D_P$ *is a strong Krull-type or a Noetherian domain, then, for each* $I \in \mathfrak{F}(D)$, *we have that*

$$\mathcal{P}\text{-}cl_D(I) = cl_D(I).$$

Proof. If D is strong Krull-type, the thesis is a consequence of Theorem 1.1(2) and the definition of $\mathcal{P}$-polynomial closure. More precisely, if $\text{Int}(ID_P, D_P) = \text{Int}(I, D)_P$, then $cl_D(I) \subseteq cl_{D_P}(ID_P)$ [C4, Proposition 3.5(1)] and hence $cl_D(I) \subseteq \mathcal{P}\text{-}cl_D(I)$. Therefore $cl_D(I) = \mathcal{P}\text{-}cl_D(I)$ by Lemma 1.4.

If D is Noetherian, then $S^{-1}\text{Int}(I, D) = \text{Int}(S^{-1}I, S^{-1}D)$, for each multiplicative set S of D [CC2, Proposition I.2.7]. As shown above for strong Krull-type domains, also in the Noetherian domain case, for $\mathcal{P} = \text{Max}(D)$, we have that:

$$cl_D(I) = \cap_{P \in \mathcal{P}} cl_{D_P}(ID_P),$$

for each $I \in \mathfrak{F}(D)$, that is, $cl_D(I) = \mathcal{P}\text{-}cl_D(I)$, by definition of $\mathcal{P}$-polynomial closure. □

COROLLARY 1.7. *Let* D *be a Noetherian domain.*

(1) *For each* $M \in \text{Max}(D)$ *and for each* $I \in \mathfrak{F}(D)$, *we have:*

$$cl_D(I)D_M = cl_{D_M}(ID_M).$$

(2) *If* S *is a multiplicative subset of* D, *for each* $I \in \mathfrak{F}(D)$ *we have:*

$$S^{-1}cl_D(I) = cl_{S^{-1}D}(S^{-1}I).$$

In particular, if I *is polynomially closed in* D *then* $S^{-1}I$ *is polynomially closed in* $S^{-1}D$.

(3) *For each* $M \in \text{Max}(D)$,

$$cl_D(M) = (MD_M)_v \cap D = M_v.$$

In particular, M *is polynomially closed (respectively, polynomially dense) in* D *if and only if* $M = M_v$ *(respectively,* $M_v = D$*) or, equivalently, if and only if* MD_M *is polynomially closed (respectively, polynomially dense) in* D_M.

(4) *For each nonzero ideal* I *of* D *there exists a prime ideal* P *of* D *such that* $I \subseteq P = cl_D(P)$.

(5) *If* $\dim(D) = 1$, *then every nonzero prime ideal of* D *is polynomially closed.*

Proof. (1) is an easy consequence of Proposition 1.5(2) and Corollary 1.6, since D is Noetherian and $D = \cap_{M \in \text{Max}(D)} D_M$.

(2) Note that $S^{-1}D = \cap\{D_M; M \in \text{Max}(D) \text{ and } M \cap S = \emptyset\}$ is locally finite. Hence the conclusion follows from (1) and from Corollary 1.6, since

$$S^{-1}cl_D(I) = \cap\{cl_D(I)D_M; M \in \text{Max}(D) \text{ and } M \cap S = \emptyset\} =$$

$$= \cap\{cl_{D_M}(ID_M); M \in \text{Max}(D) \text{ and } M \cap S = \emptyset\} =$$

$$= cl_{S^{-1}D}(S^{-1}I).$$

(3) By Corollary 1.6 we have

$$cl_D(M) = \cap_{N \in \text{Max}(D)} cl_{D_N}(MD_N) =$$

$$= cl_{D_M}(MD_M) \cap (\cap\{D_N; N \in \text{Max}(D),\ N \neq M\}) = cl_{D_M}(MD_M) \cap D.$$

Moreover, by the proof of [C4, Proposition 2.3], we know that if $(R, \mathbf{m})$ is a local Noetherian domain then $cl_R(\mathbf{m}) = \mathbf{m}_v$. Finally, since D is Noetherian, by [G2, Theorem 4.4(4)], we have that $(MD_M)_v = M_vD_M$ and by [G2, Theorem 4.10(3)] we have

$$M_v = (MD_M)_v \cap D.$$

The conclusion is straightforward.

(4) Since D is Noetherian, cl_D is a star-operation on D with finite character (Corollary 1.6). It is well known, in this situation, that each proper star-ideal of D is contained in a maximal proper star-ideal of D and that a maximal proper star-ideal of D is a prime ideal (cf. for example [J]).

(5) follows immediately from (4). □

Note that Corollary 1.7(2) gives a positive answer to Question 3.10 in [C4] and Corollary 1.7(3) generalizes to the nonlocal case [C4, Proposition 2.3]. Note also that Cahen [C4, Example 3.9] has given an example of an ideal I of an integrally closed (non Noetherian) domain D such that $I = cl_D(I)$ and $S^{-1}I \neq cl_{S^{-1}D}(S^{-1}I)$, for some multiplicative set S of D.

The equality in Corollary 1.7(3) does not hold for the nonmaximal ideals, i.e. the inclusion $cl_D(I) \subseteq I_v$ may be a proper inclusion even in the Noetherian local case. In fact, it is enough to consider a local, Noetherian, one-dimensional, analitically irreducible domain D with finite residue field and a nonzero nonmaximal ideal I of D (cf. [C4, Corollary 4.8] or [CC2, Theorem IV.1.15]). For instance, let k be a finite field, $D := k[[X^3, X^4, X^5]]$ and $I := (X^3, X^4)D$. In this case $(D:I) = k[[X]]$, hence $I_v = (X^3, X^4, X^5)D$; but $I = cl_D(I)$ [C4, Corollary 4.8].

We recall some definitions. An *essential domain* is an integral domain D such that $D = \cap_{P \in \mathcal{P}} D_P$, where D_P is a valuation domain for P belonging to a subset $\mathcal{P}$ of $\text{Spec}(D)$. If D is an essential domain with the valuation rings D_P pairwise independent and $D = \cap_{P \in \mathcal{P}} D_P$ is locally finite (i.e. each nonzero element of D belongs to finitely many prime ideals P of $\mathcal{P}$) then D is a domain of strong Krull-type. Obviously, each Prüfer domain is an essential domain and each Krull domain is a domain of strong Krull-type.

A relevant case is when $\mathcal{P}$ is the set $t_m(D)$ of all *t-maximal ideals of D* (i.e. the maximal elements among the integral t-ideals of D). It is well known that each maximal t-ideal is a prime ideal and, for each ideal I of D, $I = \cap_{P \in t_m(D)} ID_P$; in particular $D = \cap_{P \in t_m(D)} D_P$ [Gr, Proposition 4]. A *Prüfer v-multiplication domain* D is an integral domain such that D_P is a valuation domain for each $P \in t_m(D)$. This class of domains was introduced by Griffin [Gr].

In order to study the polynomial closure of fractional ideals in an essential domain, we start by considering the local case, i.e. when $D = V$ is a valuation domain.

PROPOSITION 1.8. *Let V be a valuation domain with maximal ideal M.*

(1) *If M is principal, then, for each nonzero fractional ideal I of V, $I = cl_V(I) = I_v$.*

(2) *If M is not principal, then:*

a) $cl_V(M) = M_v = V$;

b) *for each nonzero ideal I of V, $cl_V(I) = I_v$; moreover, if $I \neq I_v$, then $cl_V(I) = I_v$ is a principal ideal of V.*

Proof. We recall that, in general, for each integral domain D and for each $I \in \mathfrak{F}(D)$ we have the following inclusions: $I \subseteq cl_D(I) \subseteq I_v$ (Corollary 1.3).

(1) If M is principal, then each nonzero fractional ideal of V is divisorial [G2, Exercise 12, p. 431]. The conclusion follows immediately from the previous tower of inclusions.

(2) If M is not principal, then $\{aM; a \in V, a \neq 0\}$ is the set of all nonzero nondivisorial (integral) ideals of V [G2, Exercise 12, p. 431]. Therefore, in this case, $M \neq M_v$, hence $M_v = V$.

(a) In order to prove that $cl_V(M) = V$, we will show that $\mathrm{Int}(V) = \mathrm{Int}(V, V) = \mathrm{Int}(M, V)$.

Since M is not principal, $V[X] = \mathrm{Int}(V)$ [CC2, Proposition I.3.16].

Let $f := c_0 + c_1X + \cdots + c_nX^n \in \mathrm{Int}(M, V)$ be a polynomial of degree n. By [CC2, Corollary I.3.3], if $a_0, a_1, \cdots, a_n$ are $n+1$ elements of M and if $d := \prod_{0 \leq i < j \leq n}(a_i - a_j)$, then $df \in V[X]$. Let v be the valuation associated to V. By using the assumption that M is not finitely generated, we can choose the elements a_i's such that $0 < \mathrm{v}(d) < |\mathrm{v}(c_i)|$, for each c_i such that $\mathrm{v}(c_i) \neq 0$. On the other hand, $df \in V[X]$ hence $\mathrm{v}(dc_i) = \mathrm{v}(d) + \mathrm{v}(c_i) \geq 0$, for each $0 \leq i \leq n$. If $f \notin V[X]$, then $\mathrm{v}(c_i) < 0$ for some i with $0 \leq i \leq n$, hence we have a contradiction. Therefore, we can conclude that $V[X] = \mathrm{Int}(M, V)$ and thus a) holds.

b) It is obvious that, if $I = I_v$, then $cl_V(I) = cl_V(I_v) = I_v$ (Corollary 1.3(5)). If $I \neq I_v$ and $I \subseteq V$, then $I = aM$ for some nonzero element $a \in V$, hence $cl_V(I) = cl_V(aM) = a\,cl_V(M)$ (Lemma 1.2). By point a), we deduce that $cl_V(I) = aV$ is a principal (hence, divisorial) ideal and $cl_V(I) = I_v$. If $I \neq I_v$ and I is a fractional ideal of V, then $bI \subset V$ and $bI \neq bI_v$, for some nonzero element $b \in V$. The conclusion follows easily from the previous argument. □

THEOREM 1.9. *Let $D = \cap_{P \in \mathcal{P}} D_P$ be an essential domain.*

(1) *For each $J \in \mathfrak{F}_{fg}(D)$, we have:*

$$\mathcal{P}\text{-}cl_D(J) = cl_D(J) = J_v.$$

(2) *For each $I \in \mathfrak{F}(D)$, we have:*

$$I_t \subseteq \mathcal{P}\text{-}cl_D(I) \subseteq cl_D(I) \subseteq I_v.$$

Proof. (1) If J is finitely generated, then

$$J_vD_P \subseteq (JD_P)_v, \text{ for each prime ideal } P \text{ of } D,$$

[B, Ch. I § 2 n. 11 (11), p. 41]. On the other hand, by Proposition 1.8,

$$J_v \subseteq \cap_{P \in \mathcal{P}} J_vD_P \subseteq \cap_{P \in \mathcal{P}} (JD_P)_v =$$
$$= \cap_{P \in \mathcal{P}} cl_D(JD_P) = \mathcal{P}\text{-}cl_D(J).$$

The conclusion follows by recalling that, in general for each $I \in \mathfrak{F}(D)$, we have:

$$\mathcal{P}\text{-}cl_D(I) \subseteq cl_D(I) \subseteq I_v.$$

(2) Since $I_t := \cup\{J_v; J \subseteq I \text{ and } J \in \mathfrak{F}_{fg}(D)\}$ then, by Proposition 1.5(1) and by (1), we have:

$$I_t := \cup\{\mathcal{P}\text{-}cl_D(J); J \subseteq I \text{ and } J \in \mathfrak{F}_{fg}(D)\} \subseteq \mathcal{P}\text{-}cl_D(I). \quad \square$$

COROLLARY 1.10. *Let $D = \cap_{P \in \mathcal{P}} D_P$ be an essential domain.*

(1) *For all $J', J'' \in \mathfrak{F}_{fg}(D)$:*

$$cl_D(J' \cap J'') = cl_D(J') \cap cl_D(J'').$$

(2) *For all $I', I'' \in \mathfrak{F}(D)$, then:*

$$\mathcal{P}\text{-}cl_D(I' \cap I'') = \mathcal{P}\text{-}cl_D(I') \cap \mathcal{P}\text{-}cl_D(I'').$$

Proof. (1) follows from (2) and from Theorem 1.9(1).

(2) Since D_P is a valuation domain, for each $P \in \mathcal{P}$, then either $I'D_P \subseteq I''D_P$ or $I''D_P \subset I'D_P$, hence $cl_{D_P}(I'D_P \cap I''D_P) = cl_{D_P}(I'D_P) \cap cl_{D_P}(I''D_P)$. The conclusion follows from Proposition 1.5(3). □

Let $D = \cap_{P \in t_m(D)} D_P$ and let $I \in \mathfrak{F}(D)$. In this case we set $\mathcal{P} = t_m(D)$ and

$$t\text{-}cl_D(I) := \cap_{P \in t_m(D)} cl_{D_P}(ID_P).$$

COROLLARY 1.11. *Let $D = \cap_{P \in t_m(D)} D_P$ be a Prüfer v-multiplication domain. Assume that, for each maximal t-ideal P of D, PD_P is a principal ideal. Then, for each $I \in \mathfrak{F}(D)$, we have:*

$$I_t = t\text{-}cl_D(I).$$

If, moreover, $D = \cap_{P \in t_m(D)} D_P$ is locally finite and the valuation rings D_P are pairwise independent (i.e. D is an integral domain of strong Krull-type) then, for each $I \in \mathfrak{F}(D)$, we have

$$I_t = t\text{-}cl_D(I) = cl_D(I).$$

Proof. In a Prüfer v-multiplication domain D, for each $I \in \mathfrak{F}(D)$, $I_t = \cap_{P \in t_m(D)} ID_P$ (cf. for instance [A, Theorem 6]).

On the other hand, by Proposition 1.8(1),

$$\cap_{P \in t_m(D)} ID_P \subseteq t\text{-}cl_D(I) = \cap_{P \in t_m(D)} cl_{D_P}(ID_P) = \cap_{P \in t_m(D)} ID_P,$$

hence $I_t = t\text{-}cl_D(I)$. The last statement is a consequence of Corollary 1.6. □

COROLLARY 1.12. *Let $D = \cap_{P\in t_m(D)} D_P$ be an integral domain of strong Krull-type. Assume that, for each $P \in t_m(D)$, there exists a finitely generated ideal J of D such that $J \subseteq P$ and $J^{-1} = P^{-1}$ and that each prime t-ideal of D is contained in a unique maximal t-ideal. Then, for each $I \in \mathfrak{F}(D)$, we have:*

$$I_t = t\text{-}cl_D(I) = cl_D(I) = I_v.$$

Proof. These assumptions characterize the Prüfer v-multiplication domains such that each t-ideal is divisorial [HZ, Theorem 3.1]. The conclusion is a straightforward consequence of Theorem 1.9(2). □

Examples of integral domains satisfying the assumptions of Corollary 1.12 are Krull domains and the Prüfer domains in which each nonzero ideal is divisorial (cf. [H, Theorem 5.1] and [K, 127]).

REMARK 1.13. For each nonzero fractional ideal I of an integral domain D, since I^{-1} is divisorial, $I \subseteq cl_D(I) \subseteq I_v$ and $I^{-1} = I_v^{-1}$, we have:

$$\text{(1.13.I)} \qquad cl_D(I^{-1}) = cl_D(I)^{-1} = I^{-1}.$$

Since $I_v = (I^{-1})^{-1}$, then the previous identity generalizes the fact that $cl_D(I_v) = I_v$ (Corollary 1.3(5)). From (1.13.I), we deduce that if $D \neq I^{-1}$, then I is not polynomially dense in D. In particular, for a maximal ideal M of D, we obtain that $D \neq M^{-1}$ implies that $M = cl_D(M)$. This statement could be obtained also as a consequence of Corollary 1.3(5), since $M = M_v$ if and only if $D \neq M^{-1}$.

2. KRULL DIMENSION OF INT(D) WHEN D IS OF KRULL-TYPE

This section is devoted to the study of the Krull dimension of the ring Int(D) for the integral domains D having a locally finite representation. If a domain D has a representation

$$D = \cap_{P\in\mathcal{P}} D_P$$

where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, the ring D_P are valuation domains for each $P \in \mathcal{P}$ and the intersection is locally finite, then D is called a *Krull-type domain.* Strong Krull-type domains studied in Section 1, are a particular case of Krull-type domains.

As a consequence of our main result we prove that, for each domain of Krull-type D, $\dim(\mathrm{Int}(D)) = \dim(D[X])$. This improves the knowledge of the Krull dimension of the ring of integer-valued polynomials giving further evidence for the conjecture stating that $\dim(\mathrm{Int}(D)) \leq \dim(D[X])$, for each integral domain D.

THEOREM 2.1. *Let D be an integral domain and $\mathcal{P}$ a subset of* Spec(D). *Assume that $D = \cap_{P\in\mathcal{P}} D_P$ is a locally finite representation of D. Set $\mathcal{P}_0 := \{P \in \mathcal{P} \cap \mathrm{Max}(D);\ \mathrm{Card}(D/P) < \infty\}$ and $\mathcal{M} := \mathrm{Max}(D) \setminus \mathcal{P}_0$. Then*

$$\dim(\mathrm{Int}(D)) = \mathrm{Max}(\{\dim(D_M[X]);\ M \in \mathcal{M}\}, \{\dim(\mathrm{Int}(D_P));\ P \in \mathcal{P}_0\}).$$

Proof. We note that, for each maximal ideal M of D,

$$\text{(2.1.I)} \quad D_M = \cap_{P\in\mathcal{P}} (D_P)_{(D\setminus M)} = (\cap_{P\in\mathcal{P},\, P\subseteq M} D_P) \cap (\cap_{P\in\mathcal{P},\, P\not\subseteq M} (D_P)_{(D\setminus M)})$$

is a locally finite representation of D_M [G2, Proposition 43.5]. Since $D = \cap_{M\in\mathrm{Max}(D)} D_M$, then, by [CC1, Corollaire 3, p. 303],

$$\mathrm{Int}(D) = \cap_{M\in\mathrm{Max}(D)} \mathrm{Int}(D_M).$$

Since $\mathrm{Int}(D) \subseteq \mathrm{Int}(D)_M \subseteq \mathrm{Int}(D_M)$, for each $M \in \mathrm{Max}(D)$, it follows that

$$\text{(2.1.II)} \qquad \mathrm{Int}(D) = \cap_{M\in\mathrm{Max}(D)} \mathrm{Int}(D)_M.$$

Now, we will show that for each $M \in \mathcal{M}$, $\mathrm{Int}(D_M) = \mathrm{Int}(D)_M = D_M[X]$. As a matter of fact, if $M \in \mathrm{Max}(D) \setminus \mathcal{P}$, for each $P \in \mathcal{P}$, $P \subset M$, D/P is infinite and then $\mathrm{Int}(D_P) = D_P[X]$. If $P \in \mathcal{P}$ with $P \not\subseteq M$, then clearly the maximal ideals of $(D_P)_{(D\setminus M)}$ contract to nonmaximal prime ideals of D. Therefore, $(D_P)_{(D\setminus M)}$ has infinite residue fields and whence $\mathrm{Int}((D_P)_{(D\setminus M)}) = (D_P)_{(D\setminus M)}[X]$ [CC2, Corollary I.3.7]. From (2.1.I) and [CC1, Corollaire 3, p. 303], we deduce that, if $M \in \mathrm{Max}(D) \setminus \mathcal{P}$,

$$\mathrm{Int}(D_M) = \cap_{P\in\mathcal{P}} \mathrm{Int}((D_P)_{(D\setminus M)}) =$$
$$= (\cap_{P\in\mathcal{P},\, P\subseteq M} \mathrm{Int}(D_P)) \cap (\cap_{P\in\mathcal{P},\, P\not\subseteq M} \mathrm{Int}((D_P)_{(D\setminus M)})) =$$
$$= (\cap_{P\in\mathcal{P},\, P\subseteq M} D_P[X]) \cap (\cap_{P\in\mathcal{P},\, P\not\subseteq M} (D_P)_{(D\setminus M)}[X]) = D_M[X].$$

It is obvious that if $M \in (\mathrm{Max}(D) \cap \mathcal{P}) \setminus \mathcal{P}_0$ then $\mathrm{Int}(D_M) = D_M[X]$, since D/M is infinite. Since $D_M[X] \subseteq \mathrm{Int}(D)_M \subseteq \mathrm{Int}(D_M)$, we have that $D_M[X] = \mathrm{Int}(D)_M = \mathrm{Int}(D_M)$.
From the previous claim and from (2.1.II), we deduce that:

$$\text{(2.1.III)} \qquad \mathrm{Int}(D) = (\cap_{M\in\mathcal{M}} D_M[X]) \cap (\cap_{M\in\mathcal{P}_0} \mathrm{Int}(D)_M).$$

We know that there exists $N \in \mathrm{Max}(\mathrm{Int}(D))$ such that $\mathrm{ht}(N) = \dim(\mathrm{Int}(D))$ and $N \cap D$ maximal. (In fact, if $N \cap D = P$ is a nonmaximal prime ideal of D, then $\mathrm{Int}(D)_P = \mathrm{Int}(D_P) = D_P[X]$ and hence $\mathrm{ht}(N) = \dim(D_P[X]) \leq \dim(D[X]) - 1 \leq \dim(\mathrm{Int}(D))$ [CC2, Proposition V.1.6]. Therefore $\dim(\mathrm{Int}(D)) = \dim(D[X]) - 1$. Arguing as in the proof of [CC2, Proposition V.1.6], we can find a chain $\mathcal{C}$ of prime ideals of $D_P[X]$ of length $n = \dim(D_P[X]) = \dim(\mathrm{Int}(D))$,

$$\mathcal{C}: \qquad (0) \subset Q_1 \subset \cdots \subset Q_{n-1} \subset Q_n,$$

such that $(Q_i \cap D_P)[X] \in \mathcal{C}$, for each $i = 1, \cdots, n$. Therefore, $Q_{n-1} = PD_P[X]$. When we contract $\mathcal{C}$ to Int(D) we get a chain

$$\mathcal{C}': \qquad (0) \subset Q'_1 \subset \cdots \subset Q'_{n-1} \subset Q'_n,$$

where $Q'_{n-1} = PD_P[X] \cap \mathrm{Int}(D)$. If M is a maximal ideal of D containing P and $a \in D$, then it is easy to see that $Q'_{n-1} \subset P_a \subseteq M_a$, where $Q_a := \{f \in \mathrm{Int}(D); f(a) \in Q\}$ for $Q \in \{P, M\}$ and M_a is a maximal ideal of Int(D) above M [CC2, Lemma V.1.3]. Therefore we reach a contradiction: $\dim(\mathrm{Int}(D)) \geq n + 1$.

Let $M := (N \cap D) \in \mathrm{Max}(D)$. Since $N \cap (D \setminus M) = \emptyset$, then $N\mathrm{Int}(D)_M$ is a maximal ideal of $\mathrm{Int}(D)_M$. The conclusion follows immediately by examining the two possible cases:

Case 1. $M \in \mathcal{M}$. In this case, $\mathrm{Int}(D)_M = \mathrm{Int}(D_M) = D_M[X]$, hence $\mathrm{ht}(N) = \dim(D_M[X])$.

Case 2. $M \in \mathcal{P}_0$. In this case, $\mathrm{Int}(D)_M = \mathrm{Int}(D_M)$ and $\mathrm{ht}(N) = \dim(\mathrm{Int}(D_M))$. □

COROLLARY 2.2. *With the same notation and hypotheses of* Theorem 2.1, *if* $\dim(\mathrm{Int}(D_P)) \leq \dim(D_P[X])$ *for each* $P \in \mathcal{P}_0$, *then* $\dim(\mathrm{Int}(D)) \leq \dim(D[X])$. *In particular, the previous inequality holds in the following cases:*

(a) $D_P[X]$ *is a Jaffard domain, for each* $P \in \mathcal{P}_0$ *(e.g. when D is a locally Jaffard domain* [ABDFK]);

(b) D_P *is a* P^n*VD, with* $n \geq 0$, *for each* $P \in \mathcal{P}_0$ *(e.g. when D is a locally PVD domain,* [F] *or* [DF]).

Proof. The first inequality is a straightforward consequence of Theorem 2.1. As concerns the particular cases, we proceed as follows.

(a) We note that, for each $P \in \mathcal{P}_0$, we have:

$$\dim(\mathrm{Int}(D_P)) \leq \dim_v(\mathrm{Int}(D_P)) \leq \dim_v(D_P[X]) = \dim(D_P[X]).$$

(b) follows from [FIKT, Lemma 3.1]. $\square$

COROLLARY 2.3. *If D is a domain of Krull-type (e.g. a generalized Krull domain* [G2, p.524]), *then* $\dim(\mathrm{Int}(D)) = \dim(D[X])$.

Proof. In this case, $\mathcal{P} = t_m(D)$, $\mathcal{P}_0 = \{P \in \mathcal{P} \cap \mathrm{Max}(D); \mathrm{Card}(D/P) < \infty\}$ and D_P is a valuation domain, for each $P \in \mathcal{P}$. In particular $\dim(\mathrm{Int}(D_P)) = \dim(D_P)+1 = \dim(D_P[X])$ for each $P \in \mathcal{P}_0$. The conclusion follows from Theorem 2.1. $\square$

3. POLYNOMIAL CLOSURE IN PULLBACK DOMAINS

Let D be any domain and let P be a prime ideal of D with $\mathrm{Card}(D/P) = \infty$. Let I be an ideal of D such that $P \subset I$, thus $I_P = ID_P = D_P$. From [C4, Lemma 3.4] we have

$$\mathrm{Int}(I, D_P) = \mathrm{Int}(I_P, D_P) = \mathrm{Int}(D_P) = D_P[X],$$

hence $\mathrm{Int}(I, D) \subseteq D_P[X]$. Therefore, we can consider the canonical map

$$\varphi : \mathrm{Int}(I, D) \to \mathrm{Int}(I/P, D/P), \quad f \mapsto \overline{f} := f + PD_P[X],$$

where it is easily seen that $\overline{f} \in \mathrm{Int}(I/P, D/P)$.

We start this section with an observation about $\mathrm{Int}(I, D)$ and $\mathrm{Int}(I/P, D/P)$ when P is a divided prime ideal, i.e. $P = PD_P$.

LEMMA 3.1. *Let D be an integral domain, P a divided prime ideal of D with* $\mathrm{Card}(D/P) = \infty$ *and I an ideal of D with* $P \subseteq I$. *Then,*

(1) *the canonical map* $\varphi : \mathrm{Int}(I, D) \to \mathrm{Int}(I/P, D/P)$ *is a surjective homomorphism;*

(2) $ker(\varphi) = P[X]$;

(3) $\mathrm{Int}(I, D)/P[X] \cong \mathrm{Int}(I/P, D/P)$.

Proof. (1) Let $g \in (D_P/PD_P)[X]$ such that $g(I/P) \subseteq D/P$. Then it is easy to see that $g = G + PD_P[X] = G + P[X]$, where $G \in D_P[X]$ and $G(i) + P \in D/P$, for each $i \in I$. Therefore $G \in \mathrm{Int}(I, D)$.

(2) It is obvious that $ker(\varphi) = PD_P[X] \cap \mathrm{Int}(I, D) = P[X]$.

(3) is a straightforward consequence of (1) and (2). $\square$

PROPOSITION 3.2. *Let D, P and I as in* Lemma 3.1. *Then, the canonical homomorphism* φ *defines the following isomorphism:*

$$cl_D(I)/P \cong cl_{D/P}(I/P).$$

Proof. Let $x \in cl_D(I)$, then $f(x) \in D$ for each $f \in \mathrm{Int}(I, D)$. Hence, $\overline{f}(\overline{x}) = \overline{f(x)} \in D/P$ for each $f \in \mathrm{Int}(I, D)$. Since the map $\varphi : f \mapsto \overline{f}$ is surjective (Lemma 3.1(1)), then $g(\overline{x}) \in D/P$ for each $g \in \mathrm{Int}(I/P, D/P)$, i.e. $\overline{x} \in cl_{D/P}(I/P)$. Therefore $cl_D(I)/P \subseteq cl_{D/P}(I/P)$. Conversely, if $y = x + P \in cl_{D/P}(I/P)$ then for each $g \in \mathrm{Int}(I/P, D/P)$, $g(y) \in D/P$. Since φ is surjective, $g = \overline{f} = f + P[X]$ for some $f \in \mathrm{Int}(I, D)$. By the fact that $g(y) \in D/P$, for each $g \in \mathrm{Int}(I/P, D/P)$, we deduce that $f(x) \in D$, for each $f \in \mathrm{Int}(I, D)$, i.e. $x \in cl_D(I)$. $\square$

COROLLARY 3.3. *Let D be a domain with a divided prime ideal P. Suppose that D/P is a valuation domain V with nonzero principal maximal ideal. Then, each ideal of D containing P is polynomially closed.*

Proof. If I is an ideal of D and $P \subset I$, then from Propositon 3.2 $cl_D(I)/P \cong cl_{D/P}(I/P)$. But $D/P = V$ is a valuation domain with principal maximal ideal and, by Proposition 1.8, $cl_{D/P}(I/P) = I/P$. Therefore $cl_D(I)/P = I/P$ and $cl_D(I) = I$, since they both contain P. $\square$

Relevant examples of divided domains are the pseudo-valuation domains (PVD) or, more generally, the pseudo-valuation domains of type n (P^nVD). We recall that a PVD, D, is defined by a pullback of the following type:

(3.I)
$$\begin{tikzcd} D := \alpha^{-1}(k) \arrow[r] \arrow[d] & k \arrow[d] \\ V \arrow[r, "\alpha"] & V/M \end{tikzcd}$$

where (V, M) is a valuation domain (called the *valuation overring associated to* D), $\alpha : V \twoheadrightarrow V/M$ is the canonical projection and k is a subfield of the residue field of V. From [HH, Theorem 2.13], every prime ideal of D is divisorial, hence it is polynomially closed. Moreover, if I is any nonprincipal integral ideal of D, then $I_v = IV$ [HH, Corollary 2.14]. It follows immediately that $cl_D(I) \subseteq IV$. Moreover, for a nonvaluation PVD, the t-operation and the v-operation coincide [HZ, Proposition 4.3] so that $cl_D(I) \subseteq I_t$. If $D = V$ is a valuation domain, then it is known that the t-operation and the v-operation coincide if and only if the maximal ideal M of V is principal [HZ, Remark 1.5]; in fact, in this situation, every nonzero ideal of V is divisorial.

In [C4, § 4] the author establishes some relations between the polynomial closure of a fractional subset and its $\mathfrak{A}$-adic closure, where $\mathfrak{A}$ is an ideal of a *Zariski domain* D (i.e. a Noetherian domain, equipped with the $\mathfrak{A}$-adic topology, in which every ideal is $\mathfrak{A}$-adic closed). Next goal is to obtain a link between the polynomial closure and the adic closure for a special class of PVD's.

PROPOSITION 3.4. *Let D be a PVD. Assume that D possesses a height-one prime ideal P such that* $P \neq P^2$. *Then, the polynomial closure of each D-fractional subset E of K contains the P-adic closure of E.*

Proof. We start by proving that all ideals of D are closed in the P-adic topology. If I is any ideal of D, then its P-adic closure is given by $\overline{I} := \cap_{n \geq 0}(I + P^n)$. If

$I \supseteq P$, it is obvious that $\overline{I} = I$. If $I \subset P$, then $P^2 \subsetneq \sqrt{I} = P$ and I contains a power of P by [HH, Corollary 2.5]. Therefore, $\overline{I} = I$. Since each ideal of D is closed in the P-adic topology, we can use the same argument of [C4, Theorem 4.1] in order to conclude. □

Recall that a P^{n}VD, D, is defined by induction on n in the following way. A P^{0}VD is a PVD and a P^{n}VD is obtained by a pullback diagram of the following type:

$$\begin{tikzcd} D := \alpha^{-1}(A_{n-1}) \arrow[r] \arrow[d] & A_{n-1} \arrow[d] \\ W_{n-1} \arrow[r, "\alpha"] & F = W_{n-1}/\mathcal{M} \end{tikzcd}$$

where W_{n-1} is a P^{n-1}VD with maximal ideal $\mathcal{M}$, F is its residue field, $\alpha : W_{n-1} \to F$ is the canonical projection and A_{n-1} is a PVD with quotient field F. For details about P^{n}VD the reader is referred to [F].

In the next proposition, we will show that also in a P^{n}VD all prime ideals are polynomially closed.

PROPOSITION 3.5. *Let D be a P^{n}VD, then all nonzero prime ideals of D are polynomially closed.*

Proof. Since in a P^{n}VD every prime ideal is divided [F, Theorem 1.9], then if Q is a prime ideal of D, then D_Q is a P^{m}VD, with $m \leq n$, and $Q = (D{:}\,D_Q)$, since $QD_Q = Q$. Then, if $Q \neq 0$, Q is a divisorial ideal, whence it is polynomially closed. □

References

[A] D.F. Anderson, *Star-operations induced by overrings*, Comm. Algebra **16**, (1988), 2535–2553.

[ABDFK] D.F. Anderson - A. Bouvier - D.E. Dobbs - M. Fontana - S. Kabbaj, *On Jaffard domains*, Expo. Math. **6** (1988), 145–175.

[B] N. Bourbaki, *Algèbre Commutative*, Hermann, Paris (1961-1965).

[CC1] P.-J. Cahen - J.L. Chabert, *Coefficients et valeurs d'un polynôme*, Bull. Sci. Math. **95** (1971), 295–304.

[CC2] P.-J. Cahen - J.L. Chabert, *Integer-Valued Polynomials*, Math. Surveys and Monographs (AMS) **48** (1997).

[C1] P.-J. Cahen, *Dimension de l'anneau des polynômes à valeurs entières*, Manuscripta Math. **67** (1990), 333–343.

[C2] P.-J. Cahen, *Parties pleines d'un anneau noethérien*, J. Algebra **157** (1993), 192–212.

[C3] P.-J. Cahen, *Integer-valued polynomials on a subset*, Proc. Amer. Math. Soc. **117** (1993), 919–929.

[C4] P.-J. Cahen, *Polynomial closure*, J. Number Theory **61** (1996), 226–247.

[Ch] J.L. Chabert, *Les idéaux premiers de l'anneau des polynômes à valeurs entières*, J. reine angew. Math. **293/294** (1977), 275–283.

[DF] D.E. Dobbs - M. Fontana, *Locally pseudo-valuation domains*, Ann. Mat. Pura Appl. **134** (1983), 147–168.

[F] M. Fontana, *Sur quelques classes d'anneaux divisés*, Rend. Sem. Mat. Fis. Milano **51** (1981), 179–200.

[FIKT] M. Fontana - L. Izelgue - S.E. Kabbaj - F. Tartarone, *On the Krull dimension of domains of integer-valued polynomials*, Expo. Math. **15** (1997), 433–465.

[G1] R. Gilmer, *Sets that determine integer-valued polynomials*, J. Number Theory **33** (1989), 95–100.

[G2] R. Gilmer, *Multiplicative Ideal Theory*, Marcel-Dekker, New York (1972).

[Gr] M. Griffin, *Some Results on v-multiplication Rings*, Canad. J. Math. **10** (1967), 710–722.

[HH] J.R. Hedstrom - E.G. Houston, *Pseudo-valuation domains*, Pac. J. Math. **75** (1978), 137–147.

[H] W. Heinzer, *Integral domains in which each nonzero ideal is divisorial*, Mathematika **15** (1968), 164–170.

[HZ] E.G. Houston - M. Zafrullah, *Integral domains in which each t-ideal is divisorial*, Michigan Math. J. **35** (1988), 291–300.

[J] P. Jaffard, *Les Systèmes d'Idéaux*, Dunod, Paris (1960).

[K] W. Krull, *Idealtheorie*, Springer-Verlag, Berlin (1935).

[Mc] D.L. McQuillan, *On a Theorem of R. Gilmer*, J. Number Theory **39** (1991), 245–250.

Dipartimento di Matematica, Università degli Studi "Roma Tre", 00146 Roma, Italy, e-mail: fontana@mat.uniroma3.it;

Département de Mathématiques, Faculté des Sciences "Semlalia", Université de Marrakech, Marrakech, Morocco, e-mail: fssm.math@cybernet.net.ma;

Department of Mathematical Sciences KFUPM, P.O.Box 849 Dhahran 31261, Saudi Arabia, e-mail: kabbaj@kfupm.edu.sa;

Dipartimento di Matematica, Università "La Sapienza", 00185 Roma, Italy, e-mail: tartaron@mat.uniroma1.it.

Faculté des Sciences de Saint-Jérôme , Université d'Aix-Marseille III, 13397 Marseille, France, e-mail: francesca.tartarone@vmesa12.u-3mrs.fr.