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When Is $D + M$ n-Coherent and an (n, d)-Domain?

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1 INTRODUCTION

All rings considered below are commutative with unit, typically (integral) domains, and all modules and ring homomorphisms are unital. As its title suggests, this article contributes to a program which was begun in [7]. That article determined, i. e., when the classical $D+M$ construction (in which the ambient domain $K+M$ is a valuation domain) produces a coherent domain. In [8], to which the present article may be considered a sequel, [8, Theorem 3.6] treated the more general problem of characterizing n-coherence for the classical $D+M$ construction, with a complete answer being given in case D has quotient field K. (All relevant definitions, including that of n-coherence, will be recalled three paragraphs hence. For the moment, recall that 1-coherence is equivalent to coherence [8, page 270].)

It was noted in [3] that many of the themes and techniques in [7] carry over to more general $D+M$ contexts (in which $K+M$ need not be a valuation domain). In this spirit, our main result, Theorem 2.1, studies the transfer of n-coherence between a general $D+M$ construction and the associated ring D, with best results in case the ambient $K+M$ is a Bézout domain. In Theorem 2.8, we return to the classical $D+M$ context, to study the possible transfer of the strong n-coherence property between $D+M$ and D. Moreover, one upshot of Proposition 3.3 is that for coherent domains, strong n-coherence is equivalent to n-coherence.

There is also a markedly homological aspect to this article. For instance, Theorem 3.4 establishes that for a (context more general than a) general $D+M$ construction, $K+M$ is a flat module over $D+M$ if and only if D has quotient field K. (The proofs of many of our results, including Theorem 3.4, depend on resolutions, specifically, finding that the kernels of certain homomorphisms on $(D+M)^{(n)}$ are canonically isomorphic to $M^{(n-1)}$. While this observation is prominent in the homological considerations in [7, proof of Theorem 3], our first use of it occurred in the first-named author's proof of Proposition 4.5 (ii) in "On going-down for simple overrings II", Comm. Algebra 1 (1974), 439–458. This occurrence predates by two years its oft-cited occurrence in [3, Theorem 3].) Theorem 3.4 may be viewed as a companion for the results in [7, Theorem 7 and Corollary 8] on flatness of ideals in the classical $D+M$ construction.

In addition to pursuing resolution-theoretic themes from [7], the homological aspect of this work owes much to the classification of non-Noetherian rings initiated by Costa in [4]. Specifically, in addition to the n-coherence results described above, Theorem 2.1 also studies the transfer of the weak (n,d)-domain property between $D+M$ and D, Theorem 2.8 also studies the transfer of the (n,d)-domain property between $D+M$ and D, and Corollary 3.2 establishes that for coherent domains, the (n,d)- and the weak (n,d)-properties are equivalent.

This paragraph collects background from [8], [4] and [5] on the concepts mentioned above. Following [4] and [8], if n is a nonnegative integer, we say that an R-module E is n-presented if there is an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

of R-modules in which each F_i is finitely generated and free; and that $\lambda(E) = \lambda_R(E) = sup\{n : E \text{ is an } n\text{-presented } R\text{-module}\}$. If $n \geq 1$, we say that R is n-coherent if each $(n-1)$-presented ideal of R is n-presented; and that R is strong n-coherent if each n-presented R-module is $(n+1)$-presented. (For other inequivalent usages of "n-coherent", see [8, page 270].) Given nonnegative integers n and d, we say that a ring R is an (n,d)-ring if $pd_R(E) \leq d$ for each n-presented R-module E (as usual, pd denotes projective dimension); and that R is a weak (n,d)-ring if $pd_R(I) \leq d-1$ for each $(n-1)$-presented ideal I of R. Since, in case R is a domain, every finitely generated torsionfree R-module can be embedded in a finitely generated free R-module, R is an (n,d)-domain if and only if $pd_R(E) \leq d-1$ for each $(n-1)$-presented torsionfree R-module E. We note, for motivation, that the (n,d)- and weak (n,d)-ring concepts are relevant to a sequel to [8] because, i.a., Prüfer domains are the $(1,1)$-domains and the (possibly weak) $(2,1)$-domains of $D+M$ type are tractable: cf. [4, Theorem 5.1], [5].

It is convenient to use "local" to refer to (not necessarily Noetherian) rings with a unique maximal ideal. Also, unadorned tensor products $\otimes$ are generally taken over the implicit base ring, not necessarily Z. Finally, note that the riding assumptions and notations for Section 2 are announced at its outset.

2 n-COHERENCE AND THE (n,d)-PROPERTY

Throughout this section, we adopt the following riding assumptions and notations: T is a domain of the form $T = K+M$, where K is a field and M is a nonzero maximal ideal of T; D is a subring of K; the quotient field of D is $k = qf(D) \subseteq K$; $R = D+M$; and $T_0 = k+M$.

THEOREM 2.1 *Let T, T_0 and R be as above. Then:*

1) *R is n-coherent $\Longrightarrow$ D is n-coherent;*

R is a weak (n,d)-domain $\Longrightarrow$ D is a weak (n,d)-domain.

2) *Suppose that T is a Bézout domain and $[K:k] = \infty$. Then:*

a) T_0 is a weak $(2,1)$-domain but not coherent. In particular, T_0 is n-coherent $\forall n \geq 2$.

b) R is not coherent. Moreover, $\forall n \geq 2$ and $\forall d \geq 1$, we have:

R is n-coherent $\Longleftrightarrow$ D is n-coherent;

R is a weak (n,d)-domain $\Longleftrightarrow$ D is a weak (n,d)-domain.

3) *Suppose that T is a Bézout domain, with $1 \neq [K:k] < \infty$, and M is not a principal ideal of T. Then:*

a) T_0 is a weak $(2,1)$-domain but not coherent. In particular, T_0 is n-coherent $\forall n \geq 2$.

b) R is not coherent. Moreover, $\forall n \geq 2$ and $\forall d \geq 1$, we have:

R is n-coherent $\Longleftrightarrow$ D is n-coherent;

R is a weak (n,d)-domain $\Longleftrightarrow$ D is a weak (n,d)-domain.

4) *Suppose that T is a Bézout domain, with $1 \neq [K:k] < \infty$, and that D is a local $(n,1)$-domain, for some $n \geq 2$. Then R is a weak $(n,1)$-domain; in particular, R is m-coherent, $\forall m \geq n$.*

5) *Suppose that T is a Bézout domain and $k = K$. Then:*

R is n-coherent $\Longleftrightarrow$ D is n-coherent;

R is a weak (n,d)-domain $\Longleftrightarrow$ D is a weak (n,d)-domain.

6) *Suppose that T is a local weak $(n,1)$-domain for some $n \geq 1$ and $K = k$. Then:*

R is n-coherent $\Longleftrightarrow$ D is n-coherent;

R is a weak (n,d)-domain $\Longleftrightarrow$ D is a weak (n,d)-domain.

Before proving Theorem 2.1, we establish the following six Lemmas.

LEMMA 2.2 *Suppose that T is a Bézout domain. If I is a finitely generated ideal of R, then $I = Wa + Ma$, for some $a \in IT$ and some D-submodule W of K.*

Proof: Let I be a finitely generated ideal of R; without loss of generality, $I \neq 0$. Since T is a Bézout domain and IT is a nonzero finitely generated ideal of T, we have that $IT = Ta$, for some nonzero element $a \in IT$. As $aM = aTM = ITM = IM \subseteq I$, it follows that $M \subseteq (1/a)I$. Also, $(1/a)I \subseteq (1/a)IT = T$, and so $M \subseteq (1/a)I \subseteq T$. Put $W = (1/a)I \bigcap K$; evidently, W is a D-submodule of K. Moreover, $(1/a)I \bigcap M = M$, since $M \subseteq (1/a)I$. Hence $(1/a)I = (1/a)I \bigcap T = ((1/a)I \bigcap K) + ((1/a)I \bigcap M) = W + M$, and so $I = Wa + Ma$, as asserted.

LEMMA 2.3 *Let $A \to B$ be an injective flat ring homomorphism and let Q be an ideal of A such that $QB = Q$. Let E be an A-module such that $E \otimes_A B$ is B-flat. Then:*

1) $\lambda_A(E) \geq n \Longleftrightarrow \lambda_B(E \otimes B) \geq n$ *and* $\lambda_{A/Q}(E \otimes A/Q) \geq n$.

2) $pd_A(E) \leq d \Longleftrightarrow pd_B(E \otimes B) \leq d$ *and* $pd_{A/Q}(E \otimes A/Q) \leq d$.

Proof: 1) The assertion for the case $n = 0$ is a well-known result concerning finitely generated modules: cf. [11, Theorem 5.1.1(3)], [9].

Now, using induction on n, suppose the assertion holds for some $n \geq 0$ and let E be an $(n+1)$-presented A-module such that $E \otimes_A B$ is B-flat. We have an exact sequence $0 \to K \to A^m \to E \to 0$, where $\lambda_A(K) \geq n$ and m is some nonnegative integer. By hypothesis, B is a flat A-module; moreover, as in [5, proof of Lemma 1], $Tor_A^1(E, A/Q) = 0$. (Indeed, the exact sequence $0 \to Q \to A \to A/Q \to 0$ yields the exact sequence $0 \to Tor_A^1(E, A/Q) \to E \otimes Q \to E \to E/QE \to 0$. Since we still have $E \otimes_A Q \cong (E \otimes_A B) \otimes_B Q$, $E \otimes_A B \cong (E \otimes_A B) \otimes_B B$, $Q \subseteq B$ and $E \otimes_A B$ B-flat, it follows from [5, diagram (3)] that $Tor_A^1(E, A/Q) = 0$ as claimed). So tensoring over A with B and A/Q respectively, we get the following exact sequences :

(*) $\quad 0 \to B \otimes K \to B \otimes A^m (\cong B^m) \to B \otimes E \to 0$ and

$\quad 0 \to A/Q \otimes K \to A/Q \otimes A^m (\cong (A/Q)^m) \to A/Q \otimes E \to 0$

of B- and A/Q-modules, respectively. On the other hand, since $\lambda_A(K) \geq n$ and

$K \otimes_A B$ is B-flat (using (*), since $E \otimes_A B$ is B-flat), the induction assumption applies to the A-module K; thus, $\lambda_B(B \otimes K) \geq n$ and $\lambda_{A/Q}(A/Q \otimes K) \geq n$. Therefore, the exact sequences (*) and [8, Lemma 2.2(b)] allow us to conclude that $\lambda_B(B \otimes E) \geq n+1$ and $\lambda_{A/Q}(A/Q \otimes E) \geq n+1$.

Conversely, let E be any A-module such that $\lambda_B(B \otimes E) \geq n+1$, $\lambda_{A/Q}(A/Q \otimes E) \geq n+1$, and $E \otimes_A B$ is B-flat. For some $m \geq 0$, we have an exact sequence $0 \to K \to A^m \to E \to 0$ of A-modules. The exact sequences (*), in conjunction with [8, Lemma 2.2(c)], yield that $\lambda_B(B \otimes K) \geq n$, $\lambda_{A/Q}(A/Q \otimes K) \geq n$, and $K \otimes_A B$ is B flat (since $E \otimes_A B$ is B-flat). By the induction assumption, it follows that $\lambda_A(K) \geq n$; and the exact sequence $0 \to K \to A^m \to E \to 0$, together with [2, Lemma 2.2(b)], shows that $\lambda_A(E) \geq n+1$.

2) We induct on d. The case $d = 0$ is well known: cf. [11, Theorem 5.1.1(1)], [14]; and the case $d = 1$ follows from the proof of [5, Lemma 1]. Let $d > 1$ and assume that 2) is true for any integer $d' < d$. Let E be an A-module such that $E \otimes_A B$ is B-flat. Suppose that $pd_A(E) \leq d$. Since B is A-flat, we have that $pd_B(E \otimes_A B) \leq d$. Choose an exact sequence of A-modules $0 \to K \to F \to E \to 0$ in which F is free. Hence, $pd_A(K) \leq d-1$. By the induction assumption, $pd_{A/Q}(K \otimes_A A/Q) \leq d-1$. Hence, by (*), $pd_{A/Q}(E \otimes_A A/Q) \leq d$.

Conversely, suppose that $pd_B(E \otimes_A B) \leq d$ and $pd_{A/Q}(E \otimes_A A/Q) \leq d$. As above, $Tor_1^A(E, A/Q) = 0$, so that $pd_{A/Q}(K \otimes_A A/Q) \leq d-1$, where K is the kernel of an epimorphism from a free A-module to E. Reasoning as above, $K \otimes_A B$ is B-flat and $pd_B(K \otimes_A B) \leq d-1$. Then, by the induction assumption, $pd_A(K) \leq d-1$, so that $pd_A(E) \leq d$, and this completes the proof.

LEMMA 2.4 *Consider the pullback*

$$\begin{array}{ccc} A & \hookrightarrow & B(= S^{-1}A) \\ \downarrow & & \downarrow \\ A/Q & \longrightarrow & B/Q \end{array}$$

where $B = S^{-1}A$ for some multiplicative subset S of A, $A \to B$ is an injective flat ring homomorphism, and Q is an ideal of both A and B. Then:

1) *Assume that B is a local weak $(n,1)$-domain. Let I be any nonzero $(n-1)$-presented ideal of A. Then there exists $0 \neq x \in B$ and an ideal $I' \supseteq Q$ of A such that $I \otimes A/Q \cong I'/Q$ as A/Q-modules and $I = xI' \cong I'$ as A-modules.*

2) *Assume that B is a local weak $(n,1)$-domain. Then:*

A/Q is n-coherent $\Longrightarrow$ A is n-coherent;

A/Q is a weak (n,d)-domain $\Longrightarrow$ A is a weak (n,d)-domain.

3) *Assume that B is a valuation domain and Q, the maximal ideal of B, is a finitely generated ideal of B. Then:*

A is n-coherent $\Longleftrightarrow$ A/Q is n-coherent;

A is a weak (n,d)-domain $\Longleftrightarrow$ A/Q is a weak (n,d)-domain.

Proof: 1) Let $I = \sum_{i=1}^{m} a_i A$ be any nonzero $(n-1)$-presented ideal of A. We have $I \otimes A/Q \cong I/IQ$. Since $IB(\cong I \otimes_A B)$ is an $(n-1)$-presented B-module and B is a local weak $(n,1)$-domain, IB is a nonzero projective, hence principal ideal of B. Hence there exists $0 \neq x \in B$ such that $IB = xB$; then $IQ = IQB = Q(IB) = xBQ = xQ$. Also, by replacing x with a suitable x', we may assume without loss of generality that $I = xI'$, where I' is an ideal of A. (In detail: $\forall i = 1,\ldots,m$, we have $a_i \in I \subseteq IB = xB$, then $\exists b_i \in A$ and $\exists s_i \in S$ such that $a_i = x(b_i/s_i)$. Thus, for $x' = x/\prod_{j=1}^{m} s_j \in B$, we have $a_i = x'b'_i$, where $b'_i = (\prod_{j=1, j\neq i}^{m} s_j)b_i \in A$ and $I' = \sum_{i=1}^{m} Ab'_i$. Then $I = x'I'$; and $IB = xB = x'B$ since elements of S are units in B.) Therefore, $IQ = xQ$ and $I = xI' \cong I'$ as A-modules, where I' is an ideal of A, so that we have: $I \otimes A/Q \cong I/IQ = xI'/xQ \cong I'/Q$ as A/Q-modules.

2) A/Q **is** n**-coherent** $\Longrightarrow$ A **is** n**-coherent:** Let I be any nonzero $(n-1)$-presented ideal of A. Since $I \otimes_A B \cong IB$ is a nonzero projective, hence principal ideal of B, we have that $I \otimes_A B$ is B-flat. Then Lemma 2.3, 1) may be applied to the given pullback and the A-module $E = I$, giving $\lambda_{A/Q}(I \otimes A/Q) \geq n-1$. Express I via x and I' as in 1). Observe that $I \otimes_A B \cong IB = xB \cong B$ which, in particular, is an n-presented B-module. Now, $I \otimes A/Q \cong I'/Q$ is an $(n-1)$-presented ideal of the n-coherent ring A/Q, so $\lambda_{A/Q}(I \otimes A/Q) = \lambda_{A/Q}(I'/Q) \geq n$. Thus from Lemma 2.3, 1), $\lambda_A(I) \geq n$; and so A is n-coherent.

A/Q **is a weak** (n,d)**-domain** $\Longrightarrow$ A **is a weak** (n,d)**-domain:** Argue as above, using both Lemma 2.3, 1) and Lemma 2.3, 2).

3) A **is** n**-coherent** $\Longleftrightarrow$ A/Q **is** n**-coherent:** Since any valuation domain is a local weak $(n,1)$-domain $\forall n \geq 1$, then A/Q is n-coherent implies that A is n-coherent by 2). Conversely, let $J = I/Q$ be any nonzero $(n-1)$-presented ideal of A/Q, where I is an ideal of A such that $Q \subset I$. Moreover, IB is a finitely generated ideal of B since Q is. Then $I \otimes_A B \cong IB = xB \cong B$ for some $x \in B$, since B is a valuation domain and IB is a finitely generated ideal of B. In particular, $I \otimes_A B$ is B-flat. We apply Lemma 2.3 to the above pullback and the A-module $E = I$. We have $I \otimes_A B \cong B$ which, in particular, is an $(n-1)$-presented B-module. Moreover, $I \otimes A/Q \cong I/IQ = I/Q(=: J)$. Indeed, since $Q \subset I$, then $\exists b \in I \setminus Q$; then b is a unit in B, whence $Q = bQ \subseteq IQ \subseteq Q$ and $Q = IQ$. As J is an $(n-1)$-presented A/Q-module, we now see from Lemma 2.3, 1) that $\lambda_A(I) \geq n-1$. But A is assumed to be n-coherent, and so $\lambda_A(I) \geq n$. Thus from Lemma 2.3, 1), $\lambda_{A/Q}(J) = \lambda_{A/Q}(I \otimes A/Q) \geq n$, and so A/Q is n-coherent.

A **is a weak** (n,d)**-domain** $\Longleftrightarrow$ A/Q **is a weak** (n,d)**-domain:** Argue as above, using both Lemma 2.3, 1) and Lemma 2.3, 2). The proof is complete.

LEMMA 2.5 *Suppose that T is a Bézout domain (but not a field). Then each nonzero 1-presented ideal of T_0 is isomorphic to T_0. Consequently, T_0 is a weak $(2,1)$-domain and n-coherent $\forall n \geq 2$, in each of the following cases:*

1) $[K:k] = \infty$,

2) $1 \neq [K:k] < \infty$ *and M is not a principal ideal of T.*

Proof: We first claim that M is not a finitely generated ideal of T_0. Indeed, [3, Lemma 1] shows that if $[K:k] = \infty$, then M is not a finitely generated ideal of T_0. On the other hand, if $1 \neq [K:k] < \infty$ and M is not a principal ideal of T, then M is not a finitely generated ideal of T_0. (Otherwise, M would be finitely generated, hence principal, over T, since T is a Bézout domain.) Thus, the claim has been established.

We shall prove that each nonzero 1-presented ideal I of T_0 is projective, in fact principal, over T_0. Use Lemma 2.2 to write $I = Wa + Ma$, where W is a k-submodule of K and $a \in IT$. Since M is not a finitely generated ideal of T_0, we have $W \neq 0$. Now, $I \otimes k \cong I \otimes T_0/M \cong I/IM = (Wa+Ma)/M(Wa+Ma) = (Wa+Ma)/Ma \cong Wa \cong W$ is a finite dimensional k-vector space, since I is a finitely generated ideal of T_0. Thus, there exists a nonnegative integer p such that $W \cong k^p$. We claim that $p = 1$.

Indeed, if $p \geq 2$, let $e_1, \ldots, e_p$ be a k-vector space basis of W, and consider the surjective T_0-module homomorphism

$u : (k+M)^p \longrightarrow W + M(\cong I)$, given by

$(d_1 + m_1, \ldots, d_p + m_p) \mapsto \sum_{i=1}^{p}(d_i + m_i)e_i.$

(To verify that u is surjective, it suffices to show that $im(u)$ contains each nonzero element $m \in M$. Consider a nonzero element $\alpha = \sum \delta_i e_i \in W$, with each $\delta_i \in k$. A straightforward calculation shows that $u(\delta_1\alpha^{-1}m, \ldots, \delta_p\alpha^{-1}m) = m$.) Since I is a 1-presented T_0-module, $ker(u)$ is a finitely generated T_0-module. On the other hand, we have

$ker(u) = \{(d_1+m_1,\ldots,d_p+m_p) : \sum_{i=1}^{p}(d_i+m_i)e_i = 0\} =$ [since $K \cap M = 0$ and $\{e_i\}$ is linearly independent over k] $= \{(m_1,\ldots,m_p) : \sum_{i=1}^{p} m_i e_i = 0\}$.

Since $Me_i = M$ for each i, it follows that $ker(u) \cong M^{p-1}$. As $p-1 \geq 1$ and $ker(u)$ is finitely generated over T_0, so is M, a contradiction. This proves the claim that $p = 1$, and so $W \cong k$. Hence $I = Wa + Ma \cong W + M \cong k + M = T_0$ as a T_0-module. In fact, we have also proved that each nonzero $(n-1)$-presented ideal of T_0 is isomorphic to T_0, hence infinitely presented, $\forall n \geq 2$, that is T_0 is n-coherent $\forall n \geq 2$, to complete the proof.

LEMMA 2.6 *Suppose that T is a Bézout domain (but not a field) such that $1 \neq [K : k] < \infty$. Suppose also that D is a local $(n, 1)$-domain (but not a field) for some $n \geq 2$. Then each nonzero $(n-1)$-presented ideal of R is isomorphic to R. Consequently, R is a weak $(n, 1)$-domain and is m-coherent $\forall m \geq n$.*

Proof: Let I be any nonzero $(n-1)$-presented ideal of R. Use Lemma 2.2 to write $I = Wa + Ma$, where W is a D-submodule of K and $a \in IT$. Now, $M(\cong Ma)$ is not a finitely generated ideal of R (by [3, Lemma 1] since D is not a field), and so $W \neq 0$. Since R is D- flat, we have $\lambda_R(W \otimes_D R) = \lambda_R(WR) = \lambda_R(W(D+M)) = \lambda_R(W + M) = \lambda_R(I) \geq n-1$; therefore, $\lambda_D(W) \geq n-1$ since R is a faithfully flat D-module. On the other hand, since $W \subseteq K \cong k^r$ where $r = [K : k] < \infty$, there exists $0 \neq \delta \in D$ such that $W \cong \delta W \subseteq D^r$ since $k = qf(D)$. So, there exists a nonnegative integer m such that $W \cong D^m$ as a D-module (since D is a local $(n, 1)$-domain). We claim that $m = 1$.

Indeed, if $m \geq 2$, let $e_1, \ldots, e_m$ be a basis for W as a D-module. Consider the R-module homomorphism

$$v : (D+M)^m \to W + M(\cong I), \text{ given by}$$

$$(d_1 + m_1, \ldots, d_m + m_m) \mapsto \sum_{i=1}^{m}(d_i + m_i)e_i.$$

As in the proof of Lemma 2.5, v is surjective and $ker(v) \cong M^{m-1}$. Hence, since I is an $(n-1)$-presented R-module, $ker(v)$ is an $(n-2)$-presented R-module; in particular, $ker(v) \cong M^{m-1}$ is a finitely generated R-module. Thus, M is a finitely generated ideal of R, a contradiction. This proves the claim that $m = 1$. Therefore, $W \cong D$ and $I = Wa + Ma \cong W + M \cong D + M = R$ as R-modules, to complete the proof.

LEMMA 2.7 *Let n, d be nonnegative integers. If R is a weak (n, d)-domain, then so is D.*

Proof: Mimic the end of the proof of [5, Lemma 2], with Lemma 2.3 replacing the role of [5, Lemma 1].

Proof of Theorem 2.1: 1) Since R is faithfully flat over D, the first assertion follows from [8, Theorem 2.12, page 274]; and the second assertion is the conclusion of Lemma 2.7.

2) Assume that T is a Bézout domain and $[K : k] = \infty$.

a) T_0 is a weak $(2, 1)$-domain and n-coherent $\forall n \geq 2$ by Lemma 2..5, 1). On the other hand, T_0 is not coherent by [3, Theorem 3] since $[K : k] = \infty$.

b) R is not coherent by [3, Theorem 3]. Let $n \geq 2$ and $d \geq 1$.

R is n-coherent $\Longleftrightarrow$ D is n-coherent: If R is n-coherent, then D is n-coherent by 1). Conversely, assume that D is n-coherent, and let I be any nonzero $(n-1)$-presented ideal of R. By Lemma 2.2, write $I = Wa + Ma$, where W is a D-submodule of K and $a \in IT$. Since $Ma \cong M$ is not a finitely generated ideal of R by [3, Lemma 1], we have $W \neq 0$. Since R is D-flat, $\lambda_R(W \otimes_D R) = \lambda_R(WR) = \lambda_R(W(D + M)) = \lambda_R(W + M) = \lambda_R(I) \geq n-1$; therefore, $\lambda_D(W) \geq n-1$, since R is a faithfully flat D-module. Moreover, since T_0 is R-flat, $I \otimes T_0 \cong IT_0 = kWa + Ma$ is an $(n-1)$-presented ideal of T_0. Thus, by Lemma 2.5, 1), since $n \geq 2$, IT_0 is isomorphic to T_0. Also, by the proof of Lemma 2.5, 1), we can identify $kW(\cong W \otimes_D k) \cong k$. Since W is finitely generated over D, there exists $0 \neq \delta \in D$ such that $W \cong \delta W \subseteq D$. But D is n-coherent, so $\lambda_D(W) = \lambda_D(\delta W) \geq n$ (since δW is an $(n-1)$-presented ideal of D). Therefore, since R is D-flat, $\lambda_R(I) = \lambda_R(W+M) = \lambda_R(W \otimes_D R) \geq n$, and so R is n-coherent.

R is a weak (n, d)-domain $\Longleftrightarrow$ D is a weak (n, d)-domain: If R is a weak (n, d)-domain, then D is a weak (n, d)-domain by 1). Conversely, assume that D is a weak (n, d)-domain, and let J be any nonzero $(n-1)$-presented ideal of R. By Lemma 2.2, write $J = Wa + Ma$, where W is a D-submodule of K and $a \in JT$. Since $Ma \cong M$ is not a finitely generated ideal of R by [3, Lemma 1], $W \neq 0$. As in the above argument, we have $\lambda_D(W) \geq n-1$ and there exists $0 \neq \delta \in D$ such that $W \cong \delta W \subseteq D$. Since D is a weak (n, d)-domain, $pd_D(W) = pd_D(\delta W) \leq d-1$. Therefore, $pd_R(J) = pd_R(W+M) = pd_R(W \otimes_D R) \leq d-1$ (the inequality holding since R is a flat D-module). Thus, R is a weak (n, d)-domain.

3) Argue as for 2).

4) This is a restatement of Lemma 2.6.

5) Assume that T is a Bézout domain and $K = k$.

R is n-coherent $\Longleftrightarrow$ D is n-coherent: By 1), it remains to show that if D is n-coherent, then R is n-coherent. Without loss of generality, $R \neq T$, and so D is not a field. Let I be a nonzero $(n-1)$-presented ideal of R. Write $I = Wa + Ma$, where W is a D-submodule of K and $a \in IT$. Since $Ma \cong M$ is not a finitely generated ideal of R by [3, Lemma 1], $W \neq 0$. We have $I \otimes T \cong IT$ (since $T = (D \setminus \{0\})^{-1}R$ is R-flat) $= Ta \cong T$ is T-flat, and Lemma 2.3 may be applied to $A = R$, $B = T$, $E = I$, and the pullback

$$\begin{array}{ccc} R & \hookrightarrow & T \\ \downarrow & & \downarrow \\ D(= R/M) & \longrightarrow & k(= T/M). \end{array}$$

Now, $I \otimes T \cong Ta$ which is, in particular, an n-presented ideal of T. Also, $I \otimes R/M = I/IM = (Wa + Ma)/(Wa + Ma)M = (Wa + Ma)/Ma \cong Wa \cong W$ is an $(n-1)$-presented D-module by Lemma 2.3, 1) and so there exists $0 \neq \delta \in D$ such that $\delta W \subseteq D$. Then we have $\lambda_{R/M}(I \otimes R/M) = \lambda_D(W) = \lambda_D(\delta W) \geq n$ since D is n-coherent and δW is an $(n-1)$-presented ideal of D. Thus, by Lemma 2.3, 1), $\lambda_R(I) \geq n$, and so R is n-coherent.

R is a weak (n, d)-domain $\Longleftrightarrow$ D is a weak (n, d)-domain: Argue as above, using both Lemma 2.3, 1) and Lemma 2.3, 2).

6) Since $K = k$, we have that $T = S^{-1}R$, with $S = D \setminus \{0\}$. The assertions now follow by combining 1) and Lemma 2.4, 2).

THEOREM 2.8 *Suppose that T is a valuation domain. Suppose also that one of the following three conditions holds:*
(a) $[K : k] = \infty$; (b) $1 < [K : k] < \infty$ *and* $M = M^2$; (c) $K = k$.
Let n and d be nonnegative integers such that $n \geq 2$. Then:
1) *R is an (n, d)-domain $\Longleftrightarrow$ D is an (n, d)-domain.*
2) *R is strong n-coherent $\Longleftrightarrow$ D is strong n-coherent.*

We need the following lemma before proving Theorem 2.8:

LEMMA 2.9 *Suppose that T is an $(n_0, 1)$-domain for some $n_0 \geq 1$ and that $k = K$. Let n and d be nonnegative integers such that $n \geq n_0$. Then:*
1) *R is an (n, d)-domain $\Longleftrightarrow$ D is an (n, d)-domain.*
2) *R is strong n-coherent $\Longleftrightarrow$ D is strong n-coherent.*

Proof: 1) $\Longleftarrow$) Using a criterion mentioned in the introduction, it suffices to show that if E is an $(n-1)$-presented torsionfree R-module, then $pd_R(E) \leq d-1$. Since $k = K$, T is R-flat. Thus, $E \otimes_R T$ is an $(n-1)$-presented torsionfree T-module (since R is a domain, E embeds in some free R-module F, hence $E \otimes_R T$ embeds in

the free T-module $F \otimes_R T$); hence, it is T-projective since T is an $(n_0, 1)$-domain and $n \geq n_0$. Now $E \otimes_R D$ is $(n-1)$-presented by Lemma 2.3, 1) and is a torsionfree D-module as in the proof of [5, Lemma 2]. Since D is an (n, d)-domain, $pd_D(E \otimes_R D) \leq d-1$. It follows from Lemma 2.3, 2) that $pd_R(E) \leq d-1$.

$\Longrightarrow$) Assume that R is an (n, d)-domain. Let E be an $(n-1)$-presented torsionfree D-module. Replacing [5, Lemma 1] by our Lemma 2.3, we may mimic the end of the proof of [5, Lemma 2] to show that $pd_D(E) \leq d-1$.

2) Argue as above, using Lemma 2.3, 1). This achieves the proof.

Proof of Theorem 2.8: If *c*) holds, Lemma 2.9 gives the conclusion. Next, suppose that *a*) or *b*) hold. Then [4, Corollary 5.2] shows that R is a $(2, 1)$-domain. Replacing T by T_0, Lemma 2.9 once more gives the result.

3 FURTHER RESULTS

In [8, page 277], the question was raised whether strong n-coherence is equivalent to n-coherence for $n \geq 2$. An affirmative answer is given in Proposition 3.3 for rings satisfying certain properties P_n, Q_n (defined below). It is shown in Proposition 3.1 that the P_n, Q_n conditions also imply the equivalence of the (n, d)-domain and the weak (n, d)-domain conditions. Since coherence implies P_n, Proposition 3.1 may be viewed as a companion of the result of [5, Proposition 2] that for a coherent ring, one has equivalence of the $(1, d)$-ring and the weak (n, d)-ring conditions. Finally, the section concludes, in the spirit of [7], by characterizing when $K + M$ is $(D + M)$-flat.

We next focus the setting for (3.1) - (3.3). Let R be a domain with quotient field Q, and M be a torsionfree R-module. As usual, *rank*(M) denotes the Q-vector space dimension of $Q \otimes_R M$. An R-submodule M' of M is said to be *pure* (in M) if M/M' is a torsionfree R-module.

Let n be a positive integer. We say that R satisfies P_n if, for every $(n-1)$-presented torsionfree R-module M, there exists $f \in Hom(M, R^{rank(M)-1})$ such that $f(M)$ is n-presented. This is equivalent to saying that every nonzero $(n-1)$-presented torsionfree R-module M has a proper $(n-1)$-presented pure submodule. Observe that if R is a coherent domain, then R satisfies P_n, $\forall n$.

We say that R satisfies Q_n if, for every $(n-1)$-presented torsionfree R-module M, there exists a projective submodule M' of $R^{rank(M)-1}$ such that $M + M'$ is a projective R-module.

PROPOSITION 3.1 *Let n, d be positive integers. Let R be a domain which satisfies P_n or Q_n. Then the following conditions are equivalent:*

a) R *is a weak* (n,d)-*domain;*

b) R *is an* (n,d)-*domain.*

Proof: The implication b) $\Rightarrow$ a) holds even without the hypothesis of P_n or Q_n. Conversely, assume a), and let M be a nonzero $(n-1)$-presented torsionfree R-module. We have to show that $pd_R(M) \leq d-1$. We proceed by induction on $p = rank(M)$. If $p = 1$, then M is finitely generated over R and embeds canonically in the quotient field of R, whence M is isomorphic to an ideal of R and the assertion follows from a). We now proceed to the induction step, with $p > 1$.

Suppose first that R satisfies P_n, so that M has a proper $(n-1)$-presented pure submodule M'. As $rank(M')$, $rank(M/M') \leq rank(M) - 1 = p - 1$, it follows from the induction assumption that $pd_R(M')$, $pd_R(M/M') \leq d-1$, whence $pd_R(M) \leq d-1$ as desired.

Suppose next that R satisfies Q_n, so that $R^{rank(M)-1}$ has a projective submodule M' such that $M+M'$ is projective. Then $pd_R(M) \leq pd_R(M \cap M')$. Note that if $M \cap M' \neq 0$, then $M \cap M'$ satisfies the induction assumption. Thus, in all cases, $pd_R(M \cap M') \leq d-1$, whence $pd_R(M) \leq d-1$, completing the proof.

COROLLARY 3.2 *Let R be a coherent domain, and let n, d be positive integers. Then the following conditions are equivalent:*

a) R *is a weak* (n,d)-*domain;*

b) R *is an* (n,d)-*domain.*

Proof: Since coherence implies the P_n-property, Proposition 3.1 applies.

By reasoning as in the proof of Proposition 3.1, one can prove the following result (cf. also [5, Proposition 2]).

PROPOSITION 3.3 *Let n be a positive integer. Let R be a domain which satisfies P_n or Q_n (for instance, let R be coherent). Then the following conditions are equivalent:*

a) R *is* n-*coherent;*

b) R *is strong* n-*coherent.*

Finally, we turn to questions involving flatness in the $D+M$ construction. In view of a result [7, Theorem 7] for the classical $D+M$ context in which T is a valuation domain, one might well conjecture that if T is a domain and $k \neq K$, then T is not T_0-flat. This assertion is included in Corollary 3.5 below. First, we show that T is R-flat if and only if $k = K$.

THEOREM 3.4 *Let T be a domain of the form $K+M$, where K is a field and M is a nonzero ideal of T. Let $R = D+M$, where D is a subring of K. Then T is R-flat if and only if $qf(D) = K$.*

Proof: If $qf(D) = K$, then $T = S^{-1}R$ is R-flat, where $S = D \setminus \{0\}$. Conversely, assume that T is R-flat. Let $T_0 = k+M$, where $k = qf(D)$. Since T is R-flat, then $T \otimes_R T_0$ is T_0-flat. Now, $T \otimes_R T_0 = T \otimes_R S^{-1}R \cong S^{-1}T = T$, and so T is T_0- flat. Our aim is to show that $K = k$. Assume, on the contrary, that $K \neq k$. Choose a k-vector space basis $\{e_i : i \in L\}$ of K; well-order $L = \{1, 2, \ldots\}$. Consider the surjective T_0-module homomorphism

$u : F(= T_0^{(L)}) \longrightarrow T(= K+M)$, given by

$(t_i)_i \mapsto \sum_i t_i e_i$. Put $E = ker(u)$. Then

$$E = \{(a_i + m_i)_i \in F : \sum_i (a_i + m_i)e_i = 0\} = [\text{since } K \cap M = 0]$$

$$= \{(m_i)_i \in F : \sum_i m_i e_i = 0\} \subseteq M^{(L)}.$$

Since $T \cong F/E$ is T_0-flat, we have from [13, Theorem 3.55, page 88] that $EI = E \cap FI$ for each ideal I of T_0. Consider $I = Ta$, where $0 \neq a \in M$. We have $EI \subseteq M^{(L)}I = (MI)^{(L)} = (MTa)^{(L)} = (Ma)^{(L)} = (M)^{(L)}a$. On the other hand, $FI = T_0^{(L)}I = (T_0 I)^{(L)} = (T_0 Ta)^{(L)} = (Ta)^{(L)} = (I)^{(L)}$. Let $m_1 = a$ and $m_2 = -(e_1/e_2)m_1 = -(e_1/e_2)a$. Set $f = (m_1, m_2, 0, 0, \ldots)$. Since $m_1 e_1 + m_2 e_2 = 0$, we have $f \in E$, and so $f \in E \cap FI$. However, $f \notin EI \subseteq M^{(L)}a$, since $m_1 = a \notin Ma$. This contradiction shows that $K = k$, thus completing the proof.

COROLLARY 3.5 *Under the hypothesis of Theorem 3.4, put $k = qf(D)$ and $T_0 = k + M$. Then T is T_0-flat if and only if $k = K$.*

Proof: This is the conclusion of Theorem 3.4 for the special case in which D is a field (for then $D = k$ and $T_0 = R$).

We close by noting that [3, Theorem 5] and [14, Theorem 1.1] lead to a direct proof of the special case of Corollary 3.5 in which T is assumed to be a Prüfer domain.

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