Krull and valuative dimension of the Serre conjecture ring $R\langle n\rangle$

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Abstract. In this paper, we deal with the Serre conjecture ring $R\langle n\rangle$. The purpose is to give the Krull dimension and valuative dimension of the ring $R\langle n\rangle$. As a consequence, we characterize when it is Jaffard, and more precisely locally, residually or totally Jaffard.

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Introduction

Throughout this paper R is a commutative ring with a unit element. We denote by $R[n]$ the ring of polynomials in n indeterminates on R (but rather by $R[X]$ the ring in one indeterminate). Letting U be the multiplicative set of monic polynomials in $R[X]$, we denote by $R\langle X\rangle$ the localization $R\langle X\rangle = U^{-1}R[X]$ and we set $R\langle X_1, ..., X_n\rangle = R\langle X_1, ..., X_{n-1}\rangle\langle X_n\rangle$, where $X_1, ..., X_n$ are n indeterminates. We note at once that the order of these indeterminates is in general pertinent in the definition of $R\langle X_1, ..., X_n\rangle$, since, for any two indeterminates X and Y, $R\langle X\rangle\langle Y\rangle$ need not be equal $R\langle Y\rangle\langle X\rangle$ [9, Theorem 10]. Although this order is significant in general, it has no influence throughout this work, so we can denote $R\langle X_1, ..., X_n\rangle$ by $R\langle n\rangle$. We say that $R\langle n\rangle$ is the *Serre conjecture ring* in n indeterminates on R. Letting S be the multiplicative set in $R[n]$ formed by the polynomials whose coefficients generate R, we recall that the localization $R(n) = S^{-1}R[n]$ is called the *Nagata ring* on R with n indeterminates on R. It is clear that $R(n)$ is a localization of $R\langle n\rangle$ and that we always have $R[n] \subseteq R\langle n\rangle \subseteq R(n)$.

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We denote by $\dim R$ the Krull dimension of R and by $\dim_{\mathrm{v}} R$ its valuative dimension, i.e. the limit of the sequence $(\dim R[n] - n)$ (and we emphasize that R need not be a domain with such a definition). In a first section we establish the Krull and valuative dimension of the rings $R\langle n\rangle$ and $R(n)$.

Recall that a finite dimensional ring R is said to be *Jaffard* if $\dim R[n] = \dim R + n$, for all n [1], or equivalently $\dim R = \dim_{\mathrm{v}} R$, *residually* Jaffard if the quotient of R by any prime $\mathfrak{p}$ is Jaffard, *locally* Jaffard if the localization of R at any prime $\mathfrak{p}$ is Jaffard and lastly *totally* Jaffard if any quotient of any localization (equivalently any localization of any quotient) is Jaffard [8] (we may note that these last two definitions make sense if R is only supposed to be locally finite dimensional). In a second section we investigate the transfer of the Jaffard (and more precisely of the locally, residually and totally Jaffard) properties from the Nagata ring $R(n)$ to the Serre conjecture ring $R\langle n\rangle$ and conversely.

Letting $R\langle\infty\rangle$ (resp. $R(\infty)$) be the union $R\langle\infty\rangle = \bigcup_n R\langle n\rangle$ (resp. $R(\infty) = \bigcup_n R(n)$), we say that $R\langle\infty\rangle$ (resp. $R(\infty)$) is the *infinite Serre conjecture ring* (resp. *infinite Nagata ring*) on R. In a third and last section we show the Krull dimension of these rings to be the valuative dimension of R. If $\mathfrak{p}$ is a prime ideal of R, and n is a non negative integer, or $n = \infty$, we denote by $\mathfrak{p}[n]$ the extension of $\mathfrak{p}$ in $R[n]$ (i.e. the set of polynomials with coefficients in $\mathfrak{p}$) and by $\mathfrak{p}(n)$ (resp. $\mathfrak{p}\langle n\rangle$) its localisation in $R(n)$ (resp. in $R\langle n\rangle$). We denote by $\mathrm{ht}\mathfrak{p}$ the height of $\mathfrak{p}$ and as in [4] we let the *valuative height* of $\mathfrak{p}$, denoted by $\mathrm{ht}_{\mathrm{v}}\mathfrak{p}$, be the valuative dimension of the localization $R_{\mathfrak{p}}$. We show that the height of $\mathfrak{p}(\infty)$ and $\mathfrak{p}\langle\infty\rangle$ is the valuative height of $\mathfrak{p}$. Recall that R is said to be a strong S-ring if, for any pair $\mathfrak{p} \subset \mathfrak{q}$ of consecutive primes in R, $\mathfrak{p}[X] \subset \mathfrak{q}[X]$ are consecutive in $R[X]$. If R is a strong S-ring, $R[X]$ need not be so [19]; a ring R such that $R[n]$ is a strong S-ring for any n is said to be a stably strong S-ring. A stably strong S-ring is totally Jaffard and totally Jaffard rings are strong S-rings [8, introduction]. We lastly show that $R\langle\infty\rangle$ and $R(\infty)$ are stably strong S-rings.

Terminology is standard as in [17]. We use "$\subset$" to denote proper containment. If $\mathfrak{P}$ is a prime ideal of $R\langle n\rangle$, $R[n]$, $R(n)$, $R\langle\infty\rangle$ or $R(\infty)$ and $\mathfrak{p} = \mathfrak{P} \cap R$, we say that $\mathfrak{P}$ is *above* $\mathfrak{p}$. By convention, we let $R[0]$, $R(0)$ and $R\langle 0\rangle$ be the ring R.

1 Krull and valuative dimensions

It is clear that every prime ideal upper to a maximal ideal in $R[X]$ contains a monic polynomial [17, theorem 28]. Hence every maximal ideal of $R\langle X\rangle$ is either the extension $\mathfrak{m}\langle X\rangle$ of a maximal ideal $\mathfrak{m}$ of R, or the localisation of a prime ideal $\mathfrak{P}$ of $R[X]$ which is an upper to a non maximal prime ideal $\mathfrak{p}$ of R. We thus get immediately the following, as already shown in [5, lemma1] and [18, Th.2.1].

Lemma 1.1 *For any ring R,* $\dim R\langle X\rangle = \dim R[X] - 1$.

We generalize the result of lemma 1.1 as follows:

Proposition 1.2 *Let R be a ring and n, r two non negative integers, then*

$$\dim R(n)[r] = \dim R\langle n\rangle[r] = \dim R[n+r] - n.$$

Proof. Since $R(n)$ is a localization of $R\langle n\rangle$, we have

$$\dim R(n)[r] \leq \dim R\langle n\rangle[r] \tag{1}$$

We next prove that

$$\dim R[n+r] - n \leq dim R(n)[r] \tag{2}$$

Letting $\mathfrak{m}$ be a maximal ideal of R such that $\dim R[n+r] = \mathrm{ht}\mathfrak{m}[n+r] + (n+r)$, then $\dim R(n)[r] \geq \mathrm{ht}\mathfrak{m}(n)[r] + r = \mathrm{ht}\mathfrak{m}[n][r] + r = \mathrm{ht}\mathfrak{m}[n+r] + r$, thus $\dim R(n)[r] \geq \dim R[n+r] - n$.
Lastly we prove, by induction on $n \geq 1$, that

$$\dim R\langle n\rangle[r] \leq \dim R[n+r] - n \tag{3}$$

Case $n = 1$. From the special chain theorem [6, theorem 1], we have,

$$\dim R\langle 1\rangle[r] = Sup\{\mathrm{ht}\mathfrak{M}[r] + r\} \tag{4}$$

where $\mathfrak{M}$ runs among the maximal ideals of $R\langle 1\rangle$. As noticed above, two cases may occur.
a) $\mathfrak{M}$ is the extension $\mathfrak{m}\langle 1\rangle$ of a maximal ideal $\mathfrak{m}$ of R. In this first case, $ht\mathfrak{m}\langle 1\rangle = \mathrm{ht}\mathfrak{m}[1]$, and

$$\mathrm{ht}\mathfrak{M}[r] = \mathrm{ht}\mathfrak{m}[1][r] = \mathrm{ht}\mathfrak{m}[1+r] \tag{5}$$

b) $\mathfrak{M}$ is the localisation of a prime ideal $\mathfrak{P}$ of $R[1]$, which is an upper to a non maximal prime ideal $\mathfrak{p}$ of R. Hence there is a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subset \mathfrak{m}$. Therefore, from [6, lemma 1], we get

$$\mathrm{ht}\mathfrak{M}[r] = \mathrm{ht}\mathfrak{P} = \mathrm{ht}\mathfrak{p}[1][r] + 1 = \mathrm{ht}\mathfrak{p}[1+r] + 1 \leq \mathrm{ht}\mathfrak{m}[1+r]. \tag{6}$$

In any case, (4), (5) and (6) lead to

$$\mathrm{dim}R\langle 1\rangle[r] \leq Sup\{\mathrm{ht}\mathfrak{m}[1+r] + r\} \leq \mathrm{dim}R[1+r] - 1.$$

Case $n \geq 2$. From the case $n = 1$, we get
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$$\mathrm{dim}R\langle n\rangle[r] = \mathrm{dim}R\langle n-1\rangle\langle 1\rangle[r] \leq \mathrm{dim}R\langle n-1\rangle[1+r] - 1$$

thus, by induction hypothesis,
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$$\mathrm{dim}R\langle n\rangle[r] \leq \mathrm{dim}R[(n-1)+(1+r)] - (n-1) - 1 \leq \mathrm{dim}R[n+r] - n.$$

This proves (3). The result follows, putting (1), (2) and (3) together. $\diamond$

In particular $\mathrm{dim}R\langle n\rangle = \mathrm{dim}R(n) = \mathrm{dim}R[n] - n$. Thus we derive:

Corollary 1.3 *Let R be a ring and n a non negative integer, then*

$$\mathrm{dim_v}R = Sup_n \mathrm{dim}R\langle n\rangle = Sup_n \mathrm{dim}R(n).$$

It results also clearly from proposition 1.2 that, if $T = R(n)$ or $T = R\langle n\rangle$, then $\mathrm{dim}T[r] - r = \mathrm{dim}R[n+r] - n - r$, hence the limit of the sequence $(\mathrm{dim}T[r] - r)$, is the same as the limit of the sequence $(\mathrm{dim}R[m] - m)$. Thus we get the following:

Corollary 1.4 *Let R be a ring and n a non negative integer, then*

$$\mathrm{dim_v}R\langle n\rangle = \mathrm{dim_v}R(n) = \mathrm{dim_v}R.$$

From proposition 1.2 and corollary 1.4, we note that $R(n)$ and $R\langle n\rangle$ have the same Krull dimension and the same valuative dimension.

2 Jaffard properties

It is clear that a finite dimensional ring T is a Jaffard ring if and only if, for each non negative integer k, $\dim T[k] = \dim T + k$. From proposition 1.2 we thus get:

Lemma 2.1 *For any ring T, the following assertions are equivalent*

(i) T is a Jaffard ring,

(ii) for any non negative integer k, $\dim T(k) = \dim T$,

(iii) for any non negative integer k, $\dim T\langle k\rangle = \dim T$.

From the same proposition 1.2 we obtain also the following results for the transfer of the Jaffard (resp. locally Jaffard) property from the Nagata ring $R(n)$ to the Serre conjecture ring $R\langle n\rangle$.

Proposition 2.2 *Let R be a finite dimensional ring and n a non negative integer. Then the following assertions are equivalent:*

(i) $R[n]$ is a Jaffard ring,

(ii) $R\langle n\rangle$ is a Jaffard ring,

(iii) $R(n)$ is a Jaffard ring,

(iv) for any non negative integer k, $\dim R(n) = \dim R(n+k)$,

(v) for any non negative integer k, $\dim R\langle n\rangle = dim R\langle n+k\rangle$.

Proposition 2.3 *Let R be a finite dimensional ring and n a non negative integer. Then the following assertions are equivalent:*

(i) $R[n]$ is a locally Jaffard ring,

(ii) $R\langle n\rangle$ is a locally Jaffard ring,

(iii) $R(n)$ is a locally Jaffard ring.

Proof. It is trivial that (i) implies (ii) and (ii) implies (iii). Conversely, if $R(n)$ is a locally Jaffard ring, then $R_{\mathfrak{p}}(n)$ is a Jaffard ring, for any prime ideal $\mathfrak{p}$ of R, and so is $R_{\mathfrak{p}}[n]$, from the previous proposition. Thus $R[n]$ is a locally Jaffard ring, by [3, lemma 1.11]. Therefore (iii) imples (i). $\diamond$

Remarks 2.4 (i) From propositions 2.2 and 2.3, $R\langle n\rangle$ and $R(n)$ are Jaffard rings (resp. locally Jaffard rings) whenever R is a Jaffard ring (resp. a locally Jaffard ring) or if $n \geq dim_v R - 1$ [8, proposition 1].
(ii) If $R[n]$ is a totally Jaffard ring, then so are clearly $R(n)$ and $R\langle n\rangle$. The converse does not hold: [4, example 5.3] is a dimension 2, quasi-local and totally Jaffard domain such that $R[X]$ is not a strong S-ring. Thus $R[n]$ is not totally Jaffard for $n \geq 1$. According to the previous remark, $R(n)$ and $R\langle n\rangle$ are however dimension 2 locally Jaffard domains, *for all* n, thus even totally Jaffard domains from [8, corollaire 1] (and therefore strong S-rings).

We show next that $R\langle n\rangle$ is totally Jaffard for n large if and only if it is a strong S-ring. First we set a lemma:

Lemma 2.5 *Let R be a finite dimensional ring such that $R\langle n\rangle$ is a strong S-ring for all n, then R is totally Jaffard.*

Proof. If $R\langle n\rangle$ is a strong S-ring for all n, so is $R(n)$ by localisation. For any prime $\mathfrak{p}$ of R, letting $\overline{R} = R/\mathfrak{p}$, $\overline{R}(n)$ is isomorphic to $R(n)/\mathfrak{p}(n)$, hence is also a strong S-ring. For any prime $\mathfrak{q}$ of R containing $\mathfrak{p}$, letting $\overline{\mathfrak{q}} = \mathfrak{q}/\mathfrak{p}$ then $\overline{R}_{\overline{\mathfrak{q}}}(n)$ is isomorphic to the localisation of $\overline{R}(n)$ at the prime $\overline{\mathfrak{q}}(n)$, hence $\overline{R}_{\overline{\mathfrak{q}}}(n)$ is again a strong S-ring. From [16, theorem 2] it results that $\overline{R}_{\overline{\mathfrak{q}}} = R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ is Jaffard. $\diamond$

Since $R\langle n+m\rangle$ is clearly the same as $R\langle n\rangle\langle m\rangle$ and totally Jaffard rings are strong S-rings, we derive immediately the following:

Proposition 2.6 *Let R be finite dimensional and k be a non negative integer. The following assertions are equivalent:*

- *(i) for $n \geq k$, $R\langle n\rangle$ is a strong S-ring*
- *(ii) for $n \geq k$, $R\langle n\rangle$ is totally Jaffard.*

We close this section with some questions and an example:

Question 2.7 Are $R\langle n\rangle$ and $R(n)$ residually Jaffard rings, when $R[n]$ is?

We note that conversely, $R\langle n\rangle$ and $R(n)$ may be residually (even totally) Jaffard rings, whereas $R[n]$ is not: indeed, if R is a domain such that $\dim R = 1$ and $\dim_v R = 2$. From remark 2.4 (i), $R\langle n\rangle$ and $R(n)$ are thus dimension 2 locally Jaffard domains, for $n \geq 1$, hence totally Jaffard domains [8, corollaire 1]. But R is not Jaffard, thus $R[n]$ is not residually Jaffard, for any n.

Question 2.8 Is $R\langle n\rangle$ a residually Jaffard (resp. a totally Jaffard ring, resp. a strong S-ring), if and only if $R(n)$ is?

Question 2.9 Is $R\langle n\rangle$ a totally Jaffard ring (or equivalently a strong S-ring) for $n \geq \dim_v R$ or at least for n large?

Lastly the following example presents a totally Jaffard domain R such that $R[X]$, $R\langle X\rangle$ and $R(X)$ are not residually Jaffard domains.

Example 2.10 As in [8, example 8], we let k be a field, u, v, w indeterminates and S the multiplicative subset complement of the union $\mathfrak{m}_1 \cup \mathfrak{n}_1$ of the prime ideals $\mathfrak{m}_1 = (u-1)$ and $\mathfrak{n}_1 = (u, v, w)$ of $k[u, v, w]$. The localisation $B = S^{-1}k[u, v, w]$ is a three dimensional semi-local domain, with two maximal ideals $\mathfrak{m} = S^{-1}\mathfrak{m}_1$ and $\mathfrak{n} = S^{-1}\mathfrak{n}_1$ such that $\mathrm{ht}\mathfrak{m} = 1$ and $\mathrm{ht}\mathfrak{n} = 3$. Finally, let $I = \mathfrak{m} \cap \mathfrak{n}$ and $R = k + I$. Then R is a 3 dimensional quasi-local totally Jaffard domain such that $R[X]$ is not a residually Jaffard domain. More precisely, it has been established in [8, example 8] that there exists a prime ideal $\mathfrak{P}$ in $R[X]$ such that $\mathfrak{P} \subset I[X]$ are consecutive in $R[X]$, whereas $\mathfrak{P}[Y] \subset I[X, Y]$ are not in $R[X, Y]$. This prime $\mathfrak{P}$ lifts as a prime $\mathfrak{P}'$ of $R(X)$ (resp. $R\langle X\rangle$). Clearly $\dim R(X)/\mathfrak{P}' = htI[X]/\mathfrak{P} = 1$ (resp. $\dim R\langle X\rangle/\mathfrak{P}' = 1$). On the other hand $\dim(R(X)/\mathfrak{P}')[Y] \geq htI[X, Y]/\mathfrak{P}[Y] + 1 \geq 3$ (resp. $\dim(R\langle X\rangle/\mathfrak{P}')[Y] \geq 3$). Therefore $R(X)/\mathfrak{P}'$ (resp. $R\langle X\rangle/\mathfrak{P}'$) is not a Jaffard ring.

3 Infinitely many indeterminates

We first give the Krull dimension of $R(\infty)$ and $R\langle\infty\rangle$, as already done by D.E. Dobbs at al. in [9, corollary 2.5] for the infinite Nagata ring in the particular case of a domain. We also give the height of the extended primes:

Proposition 3.1 *For any ring R,*

(i) $\dim_{\mathrm{v}} R = \dim R(\infty) = \dim R\langle\infty\rangle$,

(ii) for any prime $\mathfrak{p}$ of R, $\mathrm{ht}_{\mathrm{v}}\mathfrak{p} = \mathrm{ht}\mathfrak{p}(\infty) = \mathrm{ht}\mathfrak{p}\langle\infty\rangle$.

Proof. Since $R\langle\infty\rangle$ (resp. $R(\infty)$) is the union of the rings $R\langle n\rangle$ (resp. $R(n)$, by [9, lemma 2.1] we have the inequality $\dim R\langle\infty\rangle \leq Sup_n\{\dim R\langle n\rangle\}$ (resp. $\dim R(\infty) \leq Sup_n\{\dim R(n)\}$. Moreover, any chain of primes in $R\langle n\rangle$ (resp. in $R(n)$) lifts in $R\langle\infty\rangle$ (resp. in $R(\infty)$), hence the reverse inequality proving

(i) from corollary 1.3. For any prime $\mathfrak{p}$ of R, $\mathrm{ht_v}\mathfrak{p} = \dim_{\mathrm{v}} Rp = \dim R_{\mathfrak{p}}(\infty)$. But $\dim R_{\mathfrak{p}}(\infty) = \mathrm{ht}\mathfrak{p}(\infty)$, since $R_{\mathfrak{p}}(\infty)$ is the localization of $R(\infty)$ with respect to the prime $\mathfrak{p}(\infty)$. Thus $\mathrm{ht_v}\mathfrak{p} = \mathrm{ht}\mathfrak{p}(\infty)$. On the other hand, $\mathrm{ht}\mathfrak{p}(\infty) = \mathrm{ht}\mathfrak{p}\langle\infty\rangle$, since $\mathfrak{p}(\infty)$ is a localisation of $\mathfrak{p}\langle\infty\rangle$. This proves (ii). $\diamond$

We may note, as D.E. Dobbs et al. for the infinite Nagata ring, in the special case of a domain [9, corollary 2.5], that it results easily from this proposition that $R(\infty)$ and $R\langle\infty\rangle$ are Jaffard rings (if their dimension are finite). We will show that they are in fact stably strong S-rings. First, we set the following:

Lemma 3.2 *Let $\mathfrak{P} \subset \mathfrak{Q}$ be consecutive primes of finite height in $R[\infty]$; then $\mathfrak{P}[1] \subset \mathfrak{Q}[1]$ are consecutive in $R[\infty][1]$.*

Proof. We note first that there is an integer k such that $\mathfrak{P}$ is the extension of a prime ideal of $R[k]$. Indeed, letting $\mathfrak{P}_n$ be the intersection $\mathfrak{P}_n = \mathfrak{P} \cap R[n]$, if the extension $\mathfrak{P}_n[1]$ of $\mathfrak{P}_n$ to $R[n+1] = R[n][1]$ is such that $\mathfrak{P}_n[1] \subset \mathfrak{P}_{n+1}$, then $\mathrm{ht}\mathfrak{P}_{n+1} > \mathrm{ht}\mathfrak{P}_n$, since any chain of $R[n]$ lifts in $R[n+1]$ (taking the extension of each prime of the chain). If the set of integers such that $\mathfrak{P}_n[1] \subset \mathfrak{P}_{n+1}$ were infinite, so would be $\mathrm{ht}\mathfrak{P}$, contrary to the hypothesis. Therefore, there is an integer k such that $\mathfrak{P}_k[n] = \mathfrak{P}_{k+n}$, for all n, thus $\mathfrak{P} = \bigcup_n \mathfrak{P}_{k+n} = \bigcup_n \mathfrak{P}_k[n] = \mathfrak{P}_k[\infty]$. For the same reason, there is an integer k such that both $\mathfrak{P}$ and $\mathfrak{Q}$ are extensions of primes of $R[k]$ to $R[\infty]$. Replacing R by $R[k]$, since $R[\infty]$ and $R[\infty][k]$ are clearly isomorphic, we may thus consider that $\mathfrak{P} = \mathfrak{p}[\infty]$ and $\mathfrak{Q} = \mathfrak{q}[\infty]$, where $\mathfrak{P}$ and $\mathfrak{Q}$ are respectively above the primes $\mathfrak{p}$ and $\mathfrak{q}$ of R. The infinite polynomial ring $R[\infty]$ is the set theoretic union of the rings $R[n]$ and $R[\infty][1]$ the set theoretic union of the rings $R[n][1]$. Since $R[n][1]$ is isomorphic to $R[n+1]$, $R[\infty][1]$ is thus isomorphic to $R[\infty]$. Similarly $\mathfrak{P} = \mathfrak{p}[\infty]$ and $\mathfrak{Q} = \mathfrak{q}[\infty]$ are respectively the union of the primes $\mathfrak{p}[n]$ and $\mathfrak{q}[n]$, whereas $\mathfrak{P}[1]$ and $\mathfrak{Q}[1]$ are respectively the union of the primes $\mathfrak{p}[n][1]$ and $\mathfrak{q}[n][1]$, thus $\mathfrak{P}[1]$ and $\mathfrak{Q}[1]$ correspond to the primes $\mathfrak{P}$ and $\mathfrak{Q}$ under the isomorphism of $R[\infty][1]$ with $R[\infty]$ $\diamond$

Since $R(\infty)[m]$ (resp. $R\langle\infty\rangle[m]$) is a localisation of $R[\infty][m]$, which is isomorphic to $R[\infty]$, consecutive primes of $R(\infty)[m]$ (resp. $R\langle\infty\rangle[m]$) correspond to consecutive primes of $R[\infty]$. Thus we get:

Theorem 3.3 *If R is a ring such that $\dim_v R$ is finite, then $R(\infty)$ and $R\langle\infty\rangle$ are stably strong S-rings.*

Corollary 3.4 *If R is a ring such that $\dim_v R$ is finite, then $R(\infty)$ and $R\langle\infty\rangle$ are totally Jaffard rings.*

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