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EXAMPLES OF JAFFARD DOMAINS*

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This article presents new examples of Jaffard domains (domains whose valuative dimension defined more accurately coincides with the Krull dimension) which can help to determine more clearly their relationships with other classes of domains, in particular the '$D+M$', the Kaplansky strong S-domains and the non-noetherian U.F.D. (unique factorization domains).

Introduction

Let A be a commutative domain of finite Krull dimension. It is already known [9] that the dimension of the polynomial ring with n indeterminates and with coefficients in A satisfies the double inequality

$$n+\dim A \leq \dim A[X_1,\ldots,X_n] \leq n+(n+1)\dim A,$$

and that $\dim A[X_1,\ldots,X_n]=n+\dim A$ when A is noetherian [15] or Prüferian [17]. In order to specify these explicit results on the dimension of polynomial rings, Jaffard [11] introduced the notion of valuative dimension of a ring (Section 1), denoted $\dim_v A$. For every ring A, it satisfies

$$\dim A \leq \dim_v A \quad \text{and} \quad \dim_v A[X_1,\ldots,X_n]=n+\dim_v A.$$

Moreover, when $\dim_v A$ is finite, Jaffard proved that $\dim A[X_1,\ldots,X_n]=\dim_v A[X_1,\ldots,X_n]$ as soon as $n\geq \dim_v A$ and Arnold [2] established that this is also true if $n\geq \dim_v A-1$.

So it became natural to study those rings A whose valuative dimension coincides with the Krull dimension. That was done by Jaffard in [11]. For that reason, we call *Jaffard domain* any finite Krull dimensional domain A, such that $\dim_v A=\dim A$. The importance of Jaffard domains in relation to catenarity questions [15] must be recalled here as any universally catenarian domain is a Jaffard domain [4]. One will find in [1] most of what is now known on Jaffard domains.

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In this article, we present new examples of Jaffard domains in order to point out the connection between Jaffard domains, Kaplansky strong S-domains [13], Malik and Mott universally strong S-domains [14] and UFD. To this end, we shall first determine under which conditions, $D+M$ rings, $(D+M)[X_1,\dots,X_n]$, $D+XD_S[X]$ and $D+XK[X]$ are Jaffard domains. Then, we will show how to build UFD or non-UFD Jaffard domains with a finite dimension and belonging to no already known families.

1. Terminology and notations

The terminology and notations which are not established in this paragraph are those of [8, 13, 15].

The considered rings are integral, commutative domains. Let A be a ring. Then its Krull dimension is denoted $\dim A \in \mathbb{N} \cup \{\infty\}$. We admit that $n<\infty$ for each $n\in\mathbb{N}$ and that $n+\infty=\infty+\infty=\infty$. Let us recall the following definition [9]:

Definition. Let A be an integer ring, $K=\mathrm{Frac}(A)$ its quotient field, L an algebraic extension of K and n an integer. Then A is said to be of *finite valuative dimension* n and we write $\dim_v A=n$ if the following equivalent properties are satisfied:

(i) Every valuation overring of A in L has Krull dimension $\leq n$ and there exists a valuation overring of A in L of dimension n;

(ii) Every valuation overring of A in K has Krull dimension $\leq n$ and there exists a valuation overring of A in K of dimension n;

(iii) Every overring of A in L has dimension $\leq n$ and there exists an overring of A in L of dimension n;

(iv) Every overring of A in K has dimension $\leq n$ and there exists an overring of A in K of dimension n;

(v) $\dim A[X_1,\dots,X_n]=2n$;

(vi) $\dim A[X_1,\dots,X_s]=n+s$ for every $s\geq n$.

If such an integer n does not exist, we shall say that A has infinite valuative dimension and we write $\dim_v A=\infty$.

It is clear that $\dim A\leq \dim_v A$ and we know, thanks to Jaffard, that the difference $\dim_v A-\dim A$ can be arbitrarily large. For example, with Proposition 2.1 it becomes obvious that if $A=\mathbb{Q}+Y\mathbb{Q}(X_1,\dots,X_n)[[Y]]$ and $B=\mathbb{Q}+Y\mathbb{R}[[Y]]$, then $\dim A=\dim B=1$ whereas $\dim_v A=n+1$ and $\dim_v B=\infty$.

We notice with that example that B has a non-finite valuative dimension although all the rings $B[X_1,\dots,X_n]$ have a finite Krull dimension. Note also that rings A and B are integrally closed but are not completely integrally closed [9] and therefore are non-noetherian.

Further, recall [11]: $\dim_v A[X_1,\dots,X_n]=n+\dim_v A$ for every $n\geq 1$ and for every ring A, if $A\subset B\subset \mathrm{Frac}(A)$, then $\dim_v B\leq \dim_v A$.

Definition. Any domain A of finite Krull dimension is called a *Jaffard domain* if it satisfies the two following equivalent properties:

(i) $\dim_v A = \dim A$;

(ii) $\dim A[X_1, \ldots, X_n] = n + \dim A$ for every $n \geq 1$.

It is known that the finite noetherian Krull dimensional rings or Prüfer rings [11] or universally catenarian rings [4], or the universally strong S-domains [14] or the polynomial rings $A[X_1, \ldots, X_n]$ with A Jaffard [11] are Jaffard domains. Moreover, if A is a finite-valuative-dimensional ring, then for every $n \geq \dim_v A - 1$, the ring $A[X_1, \ldots, X_n]$ is a Jaffard domain [2]. It becomes obvious with this last example that the quotient of a Jaffard domain by a prime ideal is not necessarily a Jaffard domain, which gives an answer to a question in [11].

Notice also that if $A \hookrightarrow B$ is an integral extension of integral domains, then A is Jaffard if and only if B is Jaffard.

Notations. Let D be a subring of a field K, we write $\deg \operatorname{tr}_D K \in \mathbb{N} \cup \{\infty\}$ the transcendence degree of K on D and D' the integral closure of D (in Frac(D)).

Let p be a prime ideal of a ring A, we write $p^* = p[X] = pA[X]$ for its extension in $A[X]$. If $P \in \operatorname{Spec}(A[X])$ is such that $P \cap A = p$ and if $p[X] \subsetneq P$, then its canonical image in $\operatorname{Frac}(A/p)[X]$ is a principal ideal $\varphi_p \operatorname{Frac}(A/p)[X]$ with $\varphi_p \in (A/p)[X]$ irreducible in $\operatorname{Frac}(A/p)[X]$ and we write $P = \langle p, \varphi \rangle$ where $\varphi \in A[X]$ is an antecedent of $\varphi_p \in (A/p)[X]$ in $A[X]$.

The considered '$D+M$' rings will, as in [9], be subrings of a valuation ring of the form $V = K + M$ where M is the maximal ideal of V and K its residual fields; the ring D is supposed to be a subring of the field K.

A ring A is called a *strong S-domain* if for any pair $P_1 \subset P_2$ of adjacent primes of A their extensions $P_1[X] \subset P_2[X]$ are adjacent primes of $A[X]$.

We say that A is a *universally strong S-domain* [12, 14] if $A[X_1, \ldots, X_n]$ is a strong S-domain for every non-negative integer n.

2. Results

We shall first study under which conditions the 'D + M' rings are Jaffard domains; we shall then give the consequences on the polynomial ring extensions of the '$D+M$' and on strong S-domains.

Proposition 2.1. *Let V be a valuation ring of the form $V = K + M$, where M is its maximal ideal, K its residual field, D a subring of K with quotient field k and $R = D + M$. Then*

(a) $\dim_v R = \dim_v D + \dim V = \deg \operatorname{tr}_D K$;

(b) *R is a Jaffard ring if and only if D is Jaffard and K is algebraic over D.*

Proof. Let $A \subset K$ be a ring such that $\dim A = \dim_v D$ and $B = A + M$. Then, on the one hand, we have [9]:

$$\dim B = \dim A + \dim V = \dim_v D + \dim V$$

and on the other hand, as B is an overring of R, $\dim B \leq \dim_v R$, so

$$\dim_v D + \dim V \leq \dim_v R. \tag{$*$}$$

If the degree of transcendence of K on k is non-finite, the ring R is included in the overring $B_1 = k[X_1, \ldots, X_n, \ldots] + M$ where the $x_i \in K$ are algebraically independent over K. Therefore B_1 has infinite Krull dimension, R has non-finite valuative dimension and (a) is trivially proved. From now on, we shall suppose that K has a finite degree of transcendence on k and that $B_1 = k[X_1, \ldots, X_d] + M$ where the $x_i \in K$ are algebraically independent over k. Then

$$d + \dim V = \dim B_1 \leq \dim_v R$$

and

$$\deg \operatorname{tr}_k K + \dim V \leq \dim_v R. \tag{$**$}$$

(1) The above inequalities ($*$) and ($**$) prove that if $\dim_v R$ is finite, the same happens for $\dim_v D$ and $\dim V$. Let $n = \dim_v R$; with [10], we have,

$$\begin{aligned} 2n &= \dim R[X_1, \ldots, X_n] \\ &= \dim V + \dim D[X_1, \ldots, X_n] + \inf(n, \deg \operatorname{tr}_k K) \\ &= \dim V + n + \dim_v D + \inf(n, \deg \operatorname{tr}_k K). \end{aligned}$$

Since in ($**$) we have $n = \dim_v R \geq \deg \operatorname{tr}_k K$, we come to write $\dim_v R = n = \dim V + \dim_v D + \deg \operatorname{tr}_k K$.

(2) Now assume that $\dim_v D$ and $\dim V$ are finite, then R is necessarily of finite valuative dimension. Indeed, consider a valuation ring W such that $R \subset W \subset \operatorname{Frac}(V)$. The ring $k \cap W$ is a valuation ring of k; the ring $A = (k \cap W) + M$ contains R, is included in $V \cap W$ and has as quotient field $\operatorname{Frac}(V) = L$.

A is a valuation ring of L. As a matter of fact, let x be in L. Then, as $x \in V$ or $x^{-1} \in V$, we suppose that $x \in V$. We write $x = \alpha + M$ with $\alpha \in k$ and $m \in M$. If $\alpha \in W \cap k$, we have $x \in A$. We now suppose that $\alpha \notin W \cap k$. Then, $\alpha^{-1} \in k \cap W$ and $x^{-1} = \alpha^{-1} + m' \in A$ where $m' \in M$.

Note that $\dim A = \dim(W \cap k) + \dim V$. As $W \cap k$ is an overring of D on k, we have $\dim A \leq \dim_n D + \dim V < \infty$. As W is an overring of the valuation ring A on L, we can deduce $\dim W \leq \dim_n D + \dim V < \infty$. That establishes that the valuative dimension of R is finite. □

Corollary 2.2. *Let V be a valuation ring of the form $K + M$ of finite dimension $d \geq 1$; let $D \subsetneq K$ be an overring of K of finite valuative dimension s and with quotient field k. If $e = \deg \operatorname{tr}_k K$ is finite, then*

(a) *if* $n \geq d+s+e-1$, *the ring* $(D+M)[X_1, \ldots, X_n]$ *is a Jaffard ring of dimension* $n+d+s+e$;

(b) *if* $\dim_v D - \dim D + e > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $(D+M)[X_1, \ldots, X_n]$ *is not a strong S-domain for every* $n \geq n_0$.

Proof. Let $R = D+M$. From Proposition 2.1, we have $\dim_v R = d+s+e$ and then, for $n \geq d+s+e-1$

$$\dim R[X_1, \ldots, X_n] = \dim_v R[X_1, \ldots, X_n] = n+d+s+e.$$

Still from Proposition 2.1, if $\dim_v D - \dim D + e > 0$, the ring R is not Jaffard, and therefore not a universal strong S-domain. Hence, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, the ring $R[X_1, \ldots, X_n]$ is not a strong S-domain. □

Example 2.3. Jaffard domains A_n with dimension $n+3$ which are not strong S-domains for every $n \geq 0$. The ring $B = \mathbb{Z} + V\mathbb{Q}(U)[V]_{(V)}$ has Krull dimension equal to 2 and has a valuative dimension equal to 3. It is therefore not Jaffard. For $n \geq 2$, the ring $A_n = B[X_1, \ldots, X_n]$ is Jaffard with dimension $n+3$. For $n = 1$, we have $\dim B[X_1] = 4 = \dim_v B[X_1]$. The ring $B[X_1]$ is not a strong S-domain, otherwise B would also be a strong S-domain and we would have $\dim B[X_1, X_2] = 1 + \dim B[X_1] = 2 + \dim B = 4$, which is nonsense. So, none of the rings $A_n = B[X_1, \ldots, X_n]$ is a strong S-domain for $n \geq 0$.

Remark 2.4. We show that a universally strong S-domain with finite dimension is a Jaffard ring by establishing that in such a ring A, for every prime ideal p on A and for every $n \geq 1$, we have

$$\operatorname{ht} p[X_1, \ldots, X_n] = \operatorname{ht} p.$$

On the other hand, a Jaffard domain does not necessarily satisfy this property as shown in [4, Example 5]. Moreover, the Jaffard ring A of that example has a multiplicative part S such that the ring $S^{-1}A$ is not Jaffard.

We nevertheless have the following result:

Proposition 2.5. (a) *For a finite-dimensional ring* A, *the following assertions are equivalent*:

(i) *The ring* $S^{-1}A$ *is Jaffard for every multiplicative set* S *of* A;

(ii) *The ring* A_p *is Jaffard for every prime ideal* p *of* A.

(b) *Let* A *be an equicodimensional* (*resp. equicodimensional and catenarian*) *Jaffard ring with finite dimension*; *for every maximal ideal* m *of* A (*resp. for every prime ideal* p *of* A) *the ring* A_m (*resp.* A_p) *is Jaffard.*

Proof. (a) We have to establish (ii) ⇒ (i). Let S be a multiplicative set of A and suppose that A satisfies (ii). If $S^{-1}A$ is not Jaffard there exists a valuation ring V such that $S^{-1}A \subset V \subset \operatorname{Frac}(A)$ with $\dim S^{-1}A < \dim V$. Then $S^{-1}A \cap M$ is a prime ideal

$S^{-1}p$ where $p \in \operatorname{Spec}(A)$. Therefore, we have $\dim A_p = \operatorname{ht} p \leq \dim S^{-1}A < \dim V$, which is nonsense.

(b) Let m be a maximal ideal of A and let $n = \dim A$. We just have to notice that A being equicodimensional, we have:

$$n + \dim A_m = 2n \leq \dim A_m[X_1, \ldots, X_n] \leq \dim A[X_1, \ldots, X_n] = 2n.$$

Let p be a prime ideal of A; we know [11] that $\dim_v A_p + \dim_v A/p \leq \dim_v A$. As A is equicodimensionally catenarian, we have $0 = \dim_v A - \dim A \geq \dim_v A_p + \dim_v A/p - \dim A_p - \dim A/p \geq 0$. Therefore, $\dim_v A_p = \dim A_p$ (and also $\dim_v A/p = \dim A/p$). □

Let us now recall the characterizations of Jaffard domains of dimension 1 given in [4].

Lemma 2.6. *For a 1-dimensional domain, the following assertions are equivalent:*
(i) *A is a Jaffard domain;*
(ii) *$A[X]$ is catenarian;*
(iii) *A is universally catenarian;*
(iv) *A is a strong S-domain;*
(v) *A is an S-domain;*
(vi) $\dim A[X] = 2$;
(vii) *A' is Prüferian;*
(viii) *$Q \subset p[X]$ implies $Q = (Q \cap A)[X]$ for every $p \in \operatorname{Spec}(A)$;*
(ix) *$Q \subset p[X_1, \ldots, X_n]$ implies $Q = (Q \cap A)[X_1, \ldots, X_n]$ for every $p \in \operatorname{Spec}(A)$.* □

In dimension 2, we can state:

Proposition 2.7. *Let A be a 2-dimensional equicodimensional Jaffard ring. Then*
(a) *A is a strong S-domain;*
(b) *If $p \in \operatorname{Spec}(A)$ has height 1, then A_p is universally catenarian and A'_p is Prüfer;*
(c) *For every multiplicative set S, the ring $S^{-1}A$ is Jaffard;*
(d) *Let $p \in \operatorname{Spec}(A)$ and $Q \subset p[X_1, \ldots, X_n]$. If $Q \supsetneq (Q \cap A)[X_1, \ldots, X_n]$, then $Q \cap A = (0)$ and $\operatorname{ht} p = 2$.*

Proof. (a) Under these hypotheses, in A, every saturated maximal chain of prime ideals is of the form $(0) \subsetneq p \subsetneq m$ with $\operatorname{ht} p = 1$ and $\operatorname{ht} m = 2$. As $A[X]$ is 3-dimensional and as m^* is not maximal, there exists a maximal saturated chain $(0) \subsetneq p^* \subsetneq m^* \subset M$ so, $\operatorname{ht} p^* = 1$ and $\operatorname{ht}(m^*/p^*) = 1$.

(b) Let $p \in \operatorname{Spec}(A)$ be a prime ideal of height 1. We have $2 \leq \dim A_p[X] \leq 3$. If $\dim A_p[X] = 3$, there exists in $A_p[X]$ a saturated chain of prime ideals $(0) \subsetneq Q \subsetneq (pA_p)^* \subsetneq M$, hence in $A[X]$ a saturated chain $(0) \subsetneq Q_1 \subsetneq p^*$, which contradicts (a). Therefore $\dim A_p[X] = 2$ and it follows from Lemma 2.6 that A_p is universally catenarian and A'_p is Prüferian.

(c) If we take Proposition 2.5 into account, we just have to show that A_p is a Jaffard ring for every $p \in \mathrm{Spec}(A)$. If p has height 1, from (a) A_p is universally catenarian and also Jaffard. Let us suppose p has height 2; then $4 \le \dim A_p[X, Y] \le \dim A[X, Y] = 4$. Therefore, $\dim A_p[X, Y] = 4$ and therefore, $\dim_v A_p = 2 = \dim A_p$.

(d) Let $q = Q \cap A$; suppose that $q \neq (0)$. As $(0) \subsetneq q[X_1, \dots, X_n] \subsetneq Q \subset p[X_1, \dots, X_n]$, we then have the strictly increasing chain

$$(0) \subsetneq q[X_1, \dots, X_n] \subsetneq Q \subsetneq p[X_1, \dots, X_n] \subsetneq (p, X_1) \subsetneq \cdots \subsetneq (p, X_1, \dots, X_n),$$

which implies that $\dim A[X_1, \dots, X_n] = n + 2$; from which one can conclude that there is a contradiction and that consequently $q = 0$. If we had $\mathrm{ht}\, p = 1$, from (b) the ring A_p would be Jaffard and $A_p[X_1, \dots, X_n]$ would be $n+1$ dimensional. That is nonsense as there also exists a strictly increasing series of prime ideals in $A_p[X_1, \dots, X_n]$:

$$(0) \subsetneq Q_p \subsetneq pA_p[X_1, \dots, X_n] \subsetneq (p, X_1)_p \subsetneq \cdots \subsetneq (p, X_1, \dots, X_n)_p.$$

Therefore necessarily $\mathrm{ht}\, p = 2$. □

We remark that under the hypotheses of Proposition 2.7, if m is a maximal ideal of A, the ring A_m is not necessarily universally catenarian: to see that, we need only choose A as a 2-dimensional local noetherian and not universally catenarian ring. See [15] or [16].

We can show that for a 2-dimensional equicodimensional ring, the following statements are equivalent:

(i) A is a strong S-domain;

(ii) $\dim A[X] = 3$.

On the other hand, such a ring – e.g. $\mathbb{Z} + X\mathbb{R}[[X]]$ – is not necessarily Jaffard.

Notice also that in the case of a 2-dimensional equicodimensional Jaffard ring A, the ring $A[X]$ is not necessarily catenarian [16].

Corollary 2.8. *Let A be a 2-dimensional equicodimensional Jaffard ring B such that $Q \subsetneq P$ and $Q \cap A = (0)$ implies* $\mathrm{ht}(P/Q) \ge \mathrm{ht}(P \cap A)$. *Then*

(a) *$A[X]$ is catenarian and*

(b) *$Q \subset p[X]$ implies $Q = (Q \cap A)[X]$.*

Proof. (b) Let $p \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(A[X])$ be such that $(0) \neq Q \subsetneq p[X]$. Let us prove, ad absurdum, that $Q = (Q \cap A)[X]$. On the other hand, from Proposition 2.7 we have $Q \cap A = (0)$ and $\mathrm{ht}\, p = 2$. Thus, $\mathrm{ht}(p[X]/Q) \ge \mathrm{ht}\, p = 2$. As $\mathrm{ht}\, Q = 1$ and as $p[X]$ is non-maximal, we should have $\dim A[X] \ge 4$: nonsense.

(a) If $A[X]$ is non-catenarian, there exist two chains of prime ideals $(0) \subset P_1 \subset P_2 \subset M$ and $(0) \subset Q \subset M$ such that

$$\mathrm{ht}(M/P_2) = \mathrm{ht}(P_2/P_1) = \mathrm{ht}(P_1) = \mathrm{ht}(M/Q) = \mathrm{ht}(Q) = 1.$$

Let $m = M \cap A$; from Proposition 2.5, A_m is a two-dimensional Jaffard ring, therefore, $\dim A_m[X]=3$ and ht $m=2$. Thus, $\mathrm{ht}(M/Q)=1<\mathrm{ht}\, m$. From the hypotheses, this implies that $Q\cap A=q\neq 0$. As Q has height 1, necessarily $Q=q[X]\subsetneq m[X]\subsetneq M$, which contradicts $\mathrm{ht}(M/Q)=1$. □

In order to 'generalize' the '$D+M$', some authors, as in [7], have introduced the following construction: given a multiplicative set S of a ring D, we assume $T^{(S)}=D+XD_S[X]$ where $D_S=S^{-1}D$. This ring is formed with $f(X)\in D_S[X]$ such that $f(0)\in D$. Particular case: $S=D-\{0\}$ and $T=D+Xk[X]$ where $k=\mathrm{Frac}(D)$.

In general, the Krull dimension of the ring $T^{(S)}$ is not known; we can however establish [7] that

$$1+\dim D\leq \dim T^{(S)}\leq \dim D[X] \tag{1}$$

so that

Lemma 2.9. *If D is a Jaffard ring, then*

$$\dim T^{(S)}=1+\dim D. \qquad \square$$

The following result shows that the valuative dimension of $T^{(S)}$ can now be made clearer:

Proposition 2.10. *Let S be a multiplicative set of a ring D and $T^{(S)}=D+XD_S[X]$; then*

(a) $\dim_v T^{(S)}=1+\dim_v D$.

(b) *The following statements are equivalent*:

(i) *D is a Jaffard ring*;

(ii) *$T^{(S)}$ is a Jaffard ring and* $\dim T^{(S)}=1+\dim D$.

Proof. (a) If $\dim_v D$ is finite, $\dim_v T^{(S)}$ is also finite. As a matter of fact, set $k=\mathrm{Frac}(D)$. As $T^{(S)}\subset k(X)$ is an overring over $D[X]$, we have: $\dim_v T^{(S)}\leq \dim_v D[X]=1+\dim_v D<+\infty$.

The converse is a consequence of the inequality $\dim_v D\leq \dim_v T^{(S)}$ which we will now establish. Given $A\subset k$, a ring containing D such that $\dim A=\dim_v D$; the ring $B=A+Xk[X]$ is an overring of $T^{(S)}$ which has the same quotient field. Consequently $\dim A\leq \dim B\leq \dim_v T^{(S)}$, whence $\dim_v D\leq \dim_v T^{(S)}$. Now suppose that $\dim_v D$ and $\dim_v T^{(S)}$ are finite and that $n=\dim_v T^{(S)}\geq \dim_v D$. As

$$\begin{aligned}T^{(S)}[X_1,\dots,X_n]&=D[X_1,\dots,X_n]+XD_S[X_1,\dots,X_n,X]\\&=D[X_1,\dots,X_n]+X(D[X_1,\dots,X_n])_S[X],\end{aligned}$$

it follows from (1) that

$$1+\dim D[X_1,\dots,X_n]\leq \dim T^{(S)}[X_1,\dots,X_n]\leq \dim D[X_1,\dots,X_n][X]$$

whence, the choice of n being taken into account:

$$1+n+\dim_v D \le 2n \le n+1+\dim_v D$$

and finally, $n = 1 + \dim_v D$.

(b) Immediate consequence of (1), Lemma 2.9 and (a). □

If we use Proposition 2.10 for the particular case $S = D - \{0\}$, we obtain the following:

Corollary 2.11. *Let K be the quotient field of an integral ring D.*

(a) $\dim_v (D + XK[X]) = 1 + \dim_v D$;

(b) $D + XK[X]$ *is a Jaffard ring if and only if D is a Jaffard ring.* □

Let $n \ge 1$ be an integer; this corollary enables us to build a Jaffard ring A_n, which is $n+1$-dimensional, which is neither noetherian, nor Prüferian, nor a polynomial ring extension of a Jaffard ring. We just have to set

$$A_n = \mathbb{Z} + X_1 Q[X_1] + \cdots + X_{n-1} Q(X_1, \ldots, X_{n-2})[X_{n-1}] + X_n Q(X_1, \ldots, X_{n-1})[X_n].$$

The fact that every noetherian ring and every finite-dimensional Prüferian ring is a Jaffard ring leads us to wonder whether it is the same for every Krull ring and more simply for every factorial ring, whence the following:

Problem 1. Does there exist a finite-dimensional factorial ring which is not a Jaffard ring?

We can now observe that

(1) Every factorial ring is trivially an S-domain;

(2) A factorial ring is not necessarily a strong S-domain [14].

As the examples given in [14] are factorial rings with non-finite Krull dimension, we can set the following:

Problem 2. Does there exist a finite-dimensional factorial ring which is not a strong S-domain?

We conjecture that the answer to these two problems is positive and, more precisely, for Problem 1, that for every integer $n \ge 1$, there exists a factorial ring A (necessarily non-noetherian) such that $\dim_v A - \dim A = n$.

Note that non-noetherian factorial rings built in [5, 8, 10] are all Jaffard rings. Being unable to give an answer to the two preceding problems, we can however establish the following:

Proposition 2.12. (a) *For every integer $n \ge 2$, there exists a factorial ring which is Jaffard, non-noetherian, n-dimensional and with characteristic $p \ge 0$.*

(b) *For every integer $n \geq 1$, there exists a local factorial ring whch is Jaffard, non-noetherian, n-dimensional and with characteristic* 0.

Proof. (a) We use the construction of [10]. Set P the Pontriagin group, a group without torsion and with rank 2 such that every subgroup of rank 1 is cyclic. Let $G_n = P \oplus \mathbb{Z}^{n-2}$; this group has no torsion and has rank n, and in [10] it is proved that if k is a field, the group ring $A = k[G_n]$ is factorial, non-noetherian, n-dimensional. Then we also need only establish that A is Jaffard.

There exists in G_n a free subgroup F, with rank n such that G_n/F is torsion; consequently, the ring $k[G_n]$ is integer on the noetherian ring - therefore Jaffard ring - $k[F]$. This implies that $k[G_n]$ is Jaffard.

(b) We use the construction of [7]. Let $r = n-1$. There exists a group with no torsion L, with rank r which has a free subgroup F with rank r, such that L/F is torsion, the group ring $\mathbb{Z}[L]$ is factorial (non-noetherian), equicodimensional, n-dimensional and has a maximal ideal M such that $\mathbb{Z}[L]_M$ is non-noetherian. Since $\mathbb{Z}[L]$ is integer on the Jaffard ring $\mathbb{Z}[F]$, this ring is Jaffard and the same applies for $\mathbb{Z}[L]_M$ with Proposition 2.5(b). □

A 2-dimensional factorial ring, even noetherian like $\mathbb{Z}_{(2)}[X]$ is not necessarily equicodimensional and a prime ideal with height 1 can be maximal. The last point cannot happen in dimension >3; more precisely:

Remark 2.13. Let A be a factorial ring with dimension ≥ 2 and M a maximal ideal of $A[X]$; the height of M is higher than or equal to 2. We prove it ad absurdum. Suppose M is a maximal ideal of $A[X]$ with height 1. Thus M is principal generated by $\alpha(X) \in A[X]$. Since M is maximal, it is not an extension; hence $\alpha \notin A$ and $s = \deg \alpha \geq 1$. Write $\alpha = a_0 + a_1 X + \cdots + a_s X^s$. Since A has dimension >1, there exists in A an infinity of prime ideals with height 1, thus there also exists in A a prime ideal p with height 1 such that $a_i \neq 0 \bmod p$ for $0 \leq i \leq s$. Therefore the canonical image $\bar{\alpha}$ of α in $(A/p)[X]$ has degree $s \geq 1$ and can also be factorized in a product $\bar{\varphi}_1 \cdots \bar{\varphi}_t$ of irreducible polynomials so that there exist $i = 1, \ldots, t$ such that $\alpha \in \langle p, \varphi_i \rangle$ and $M = \alpha A[X] \subset \langle p, \varphi_i \rangle$. Since $\langle p, \varphi_i \rangle$ has height 2, this inclusion is strict and contradicts the maximality of M.

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