Class semigroups and t-class semigroups of integral domains

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Abstract The class (resp., t-class) semigroup of an integral domain is the semigroup of the isomorphy classes of the nonzero fractional ideals (resp., t-ideals) with the operation induced by ideal (t-) multiplication. This paper surveys recent literature which studies ring-theoretic conditions that reflect reciprocally in the Clifford property of the class (resp., t-class) semigroup. Precisely, it examines integral domains with Clifford class (resp., t-class) semigroup and describes their idempotent elements and the structure of their associated constituent groups.

1 Introduction

All rings considered in this paper are integral domains. The notion of ideal class group of a domain is classical in commutative algebra and is also one of major objects of investigation in algebraic number theory. Let R be a domain. The ideal class group $\mathcal{C}(R)$ (also called Picard group) of R consists of the isomorphy classes of the invertible ideals of R, that is, the factor group $\mathcal{I}(R)/\mathcal{P}(R)$, where $\mathcal{I}(R)$ is the group of invertible fractional ideals and $\mathcal{P}(R)$ is the subgroup of nonzero principal fractional ideals of R. A famous result by Claiborne states that every Abelian group can be regarded as the ideal class group of a Dedekind domain.

If R is Dedekind, then $\mathcal{I}(R)$ coincides with the semigroup $\mathcal{F}(R)$ of nonzero fractional ideals of R. Thus, a natural generalization of the ideal class group is the semigroup $\mathcal{F}(R)/\mathcal{P}(R)$ of the isomorphy classes of nonzero fractional ideals of R. The factor semigroup $\mathcal{F}(R)/\mathcal{P}(R)$ is denoted by $\mathcal{S}(R)$ and called the *class*

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semigroup of R. The class semigroup of an order in an algebraic number field was first investigated by Dade, Taussky and Zassenhaus [18] and later by Zanardo and Zannier [59]. Halter-Koch [34] considered the case of the class semigroup of lattices over Dedekind domains.

The investigation of the structure of a semigroup is not as attractive as the study of a group. This is the reason why it is convenient to restrict attention to the case of a particular type of semigroups, namely, the Clifford semigroups. A commutative semigroup S (with 1) is said to be Clifford if every element x of S is (von Neumann) regular, i.e., there exists $a \in S$ such that $x^2a = x$. The importance of a Clifford semigroup S resides in its ability to stand as a disjoint union of groups G_e, each one associated to an idempotent element e of the semigroup and connected by *bonding homomorphisms* induced by multiplications by idempotent elements [16]. The semigroup S is said to be Boolean if for each $x \in S$, $x = x^2$.

Let R be a domain with quotient field K. For a nonzero fractional ideal I of R, let $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$. The v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \bigcup J_v$ where J ranges over the set of finitely generated subideals of I. The ideal I is said to be divisorial or a v-ideal if $I_v = I$, and I is said to be a t-ideal if $I_t = I$. Under the ideal t-multiplication $(I, J) \mapsto (IJ)_t$, the set $\mathcal{F}_t(R)$ of fractional t-ideals of R is a semigroup with unit R. An invertible element for this operation is called a t-invertible t-ideal of R.

The t-operation in integral domains is considered as one of the keystones of multiplicative ideal theory. It originated in Jaffard's 1960 book “Les Systèmes d'Idéaux” [37] and was investigated by many authors in the 1980s. From the t-operation stemmed the notion of (t-)class group of an arbitrary domain, extending both notions of divisor class group (in Krull domains) and ideal class group (in Prüfer domains). Class groups were introduced and developed by Bouvier and Zafrullah [12, 13], and have been extensively studied in the literature. The (t-)class group of R, denoted $\mathrm{Cl}(R)$, is the group under t-multiplication of fractional t-invertible t-ideals modulo its subgroup of nonzero principal fractional ideals. The *t-class semigroup* of R, denoted $\mathcal{S}_t(R)$, is the semigroup under t-multiplication of fractional t-ideals modulo its subsemigroup of nonzero principal fractional ideals. One may view $\mathcal{S}_t(R)$ as the t-analogue of $\mathcal{S}(R)$, similarly as the (t-)class group $\mathrm{Cl}(R)$ is the t-analogue of the ideal class group $\mathcal{C}(R)$. We have the set-theoretic inclusions

$$\mathcal{C}(R) \subseteq \mathrm{Cl}(R) \subseteq \mathcal{S}_t(R) \subseteq \mathcal{S}(R).$$

The properties of the class group or class semigroup of a domain can be translated into ideal-theoretic information on the domain and conversely. If R is a Prüfer domain, $\mathcal{C}(R) = \mathrm{Cl}(R)$ and $\mathcal{S}_t(R) = \mathcal{S}(R)$; and then R is a Bézout domain if and only if $\mathrm{Cl}(R) = 0$. If R is a Krull domain, $\mathrm{Cl}(R) = \mathcal{S}_t(R)$ equals its usual divisor class group, and then R is a UFD if and only if $\mathrm{Cl}(R) = 0$ (so that R is a UFD if and only if every t-ideal of R is principal). Trivially, Dedekind domains (resp., PIDs) have Clifford (resp., Boolean) class semigroup. In 1994, Zanardo and Zannier proved that all orders in quadratic fields have Clifford class semigroup, whereas the ring of all entire functions in the complex plane (which is Bézout) fails to have this property [59].

Thus, the natural question arising is to characterize the domains with Clifford class (resp., t-class) semigroup and, moreover, to describe their idempotent elements and the structure of their associated constituent groups.

2 Class semigroups of integral domains

A domain is said to be *Clifford regular* if its class semigroup is a Clifford semigroup. The first significant example of a Clifford regular domain is a valuation domain. In fact, in [9], Salce and the first named author proved that the class semigroup of any valuation domain is a Clifford semigroup whose constituent groups are either trivial or groups associated to the idempotent prime ideals of R. Next, the investigation was carried over for the class of Prüfer domains of finite character, that is, the Prüfer domains such that every nonzero ideal is contained in only finitely many maximal ideals. In [5], the first named author proved that if R is a Prüfer domain of finite character, then R is a Clifford regular domain and moreover, in [6] and [7] a description of the idempotent elements of $\mathcal{S}(R)$ and of their associated groups was given.

A complete characterization of the class of integrally closed Clifford regular domains was achieved in [8] where it is proved that it coincides with the class of the Prüfer domains of finite character. Moreover, [8] explores the relation between Clifford regularity, stability and finite stability. Recall that an ideal of a commutative ring is said to be *stable* if it is projective over its endomorphism ring and a ring R is said to be stable if every ideal of R is stable. The notion of stability was first introduced in the Noetherian case with various different definitions which turned out to be equivalent in the case of a local Noetherian ring (cf. [51]). Olberding has described the structural properties of an arbitrary stable domain. In [51] and [50] he proves that a domain is stable if and only if it is of finite character and locally stable. Rush, in [52] considered the class of *finitely stable* rings, that is, rings with the property that every finitely generated ideal is stable and proved that the integral closure of such rings is a Prüfer ring.

In [8], it is shown that the class of Clifford regular domains is properly intermediate between the class of finitely stable domains and the class of stable domains. In particular, the integral closure of a Clifford regular domain is a Prüfer domain. Moreover, this implies that a Noetherian domain is Clifford regular if and only if it is a stable domain. Thus, [8] provides for a characterization of the class of Clifford regular domains in the classical cases of Noetherian and of integrally closed domains. In the general case, the question of determining whether Clifford regularity always implies finite character is still open.

In [8], was also outlined a relation between Clifford regularity and the *local invertibility property*. A domain is said to have the local invertibility property if every locally invertible ideal is invertible. In [5] and again in [8] the question of deciding if a Prüfer domain with the local invertibility property is necessarily of finite character was proposed as a conjecture. The question was of a interest on its own independently of Clifford regularity and it attracted the interest of many authors. Recently the validity of the conjecture has been proved by Holland, Martinez, McGovern and Tesemma [36]. They translated the problem into a statement on the lattice ordered group of the invertible fractional ideals of a Prüfer domain and then used classical results by Conrad [17] on lattice ordered groups.

2.1 Preliminaries and notations

Let $\mathcal{S}$ be a commutative multiplicative semigroup. The subsemigroup $\mathcal{E}$ of the idempotent elements of $\mathcal{S}$ has a natural partial order defined by $e \leq f$ if and only if $ef = e$, for every $e, f \in \mathcal{E}$. Clearly, $e \wedge f = ef$ and thus $\mathcal{E}$ is a $\wedge$-semilattice under this order. An element a of a semigroup $\mathcal{S}$ is *von Neumann regular* if $a = a^2x$ for some $x \in \mathcal{S}$.

Definition 2.1. A commutative semigroup $\mathcal{S}$ is a *Clifford semigroup* if every element of $\mathcal{S}$ is regular.

By [16] a Clifford semigroup $\mathcal{S}$ is the disjoint union of the family of groups $\{G_e \mid e \in \mathcal{E}\}$, where G_e is the largest subgroup of $\mathcal{S}$ containing the idempotent element e, that is:

$$G_e = \{ae \mid abe = e \quad \text{for some} \quad b \in \mathcal{S}\}.$$

In fact, if $a \in \mathcal{S}$ and $a = a^2x, x \in \mathcal{S}$, then $e = ax$ is the unique idempotent element such that $a \in G_e$. We say that $e = ax$ is the idempotent associated to a. The groups G_e are called the constituent groups of $\mathcal{S}$. If $e \leq f$ are idempotent elements, that is $fe = e$, the multiplication by e induces a group homomorphism $\phi_e^f : G_f \to G_e$ called the bonding homomorphism between G_f and G_e. Moreover, the set $\mathcal{S}^*$ of the regular elements of a commutative semigroup $\mathcal{S}$ is a Clifford subsemigroup of $\mathcal{S}$. In fact, if $a^2x = a$ and $e = ax$, then also $a^2xe = a$ and xe is a regular element of $\mathcal{S}$, since $(xe)^2a = xe$.

Throughout this section R will denote a domain and Q its field of quotients. For R-submodules A and B of Q, $(A : B)$ is defined as follows:

$$(A : B) = \{q \in Q \mid qB \subseteq A\}.$$

A fractional ideal F of R is an R-submodule of Q such that $(R : F) \neq 0$. By an overring of R is meant any ring between R and Q. We say that a domain R is of finite character if every nonzero ideal of R is contained only in a finite number of maximal ideals. If (P) is any property, we say that a fractional ideal F of R satisfies (P) locally if each localization FR_m of F at a maximal ideal m of R satisfies (P).

Let $\mathcal{F}(R)$ be the semigroup of the nonzero fractional ideals of R and let $\mathcal{P}(R)$ be the subsemigroup of the nonzero principal fractional ideals of the domain R. The factor semigroup $\mathcal{F}(R)/\mathcal{P}(R)$ is denoted by $\mathcal{S}(R)$ and called the *class semigroup of* R. For every nonzero ideal I of R, $[I]$ will denote the isomorphism class of I.

Definition 2.2. A domain R is said to be Clifford regular if the class semigroup $\mathcal{S}(R)$ of R is a Clifford semigroup.

2.2 Basic properties of regular elements of $\mathcal{S}(R)$ and of Clifford regular domains

If R is a domain and I is a nonzero ideal of R, $[I]$ is a regular element of $\mathcal{S}(R)$ if and only if $I = I^2X$ for some fractional ideal X of R. Let $E(I) = (I : I)$ be the endomorphism ring of the ideal I of R. The homomorphisms from I to $E(I)$ are multiplication by elements of $(E(I) \colon I) = (I \colon I^2)$. The trace ideal of I in $E(I)$ is the sum of the images of the homomorphisms of I into $E(I)$, namely $I(I : I^2)$. Thus, we have the following basic properties of regular elements of $\mathcal{S}(R)$.

Proposition 2.3 ([8, Lemma 1.1, Proposition 1.2]). *Let I be a nonzero ideal of a domain R with endomorphism ring $E = (I : I)$ and let $T = I(E : I)$ be the trace ideal of I in E. Assume that $[I]$ is a regular element of $\mathcal{S}(R)$, that is, $I = I^2X$ for some fractional ideal X of R. The following hold:*

(1) $I = I^2(I : I^2)$.
(2) $IX = T$ *and* $[T]$ *is an idempotent of* $\mathcal{S}(R)$ *associated to* $[I]$.
(3) T *is an idempotent ideal of* E *and* $IT = I$.
(4) $E = (T : T) = (E : T)$

Proof. (1) By assumption $X \subseteq (I : I^2)$ and so $I = I^2X \subseteq I^2(I : I^2) \subseteq I$ implies $I = I^2(I : I^2)$.

(2) and (3). Since $(I \colon I^2) = (E \colon I)$, part (1) implies $IX = I^2(E \colon I)X = I(E \colon I)$, hence $T = IX$ is an idempotent ideal of E and $IT = I$.

(4) We have $E \subseteq (E \colon T) = (I \colon IT) = E$ and $E \subseteq (T \colon T) \subseteq (E \colon T)$.

Recall that a nonzero ideal of a domain is said to be *stable* if it is projective, or equivalently invertible, as an ideal of its endomorphism ring and R is said to be *(finitely) stable* if every nonzero (finitely generated) ideal of R is stable.

An ideal I of a domain R is said to be *L-stable* (here L stands for Lipman) if $R^I := \bigcup_{n\geq 1}(I^n : I^n) = (I : I)$, and R is called L-stable if every nonzero ideal is L-stable. Lipman introduced the notion of stability in the specific setting of one-dimensional commutative semi-local Noetherian rings in order to give a characterization of Arf rings; in this context, L-stability coincides with Boole regularity [46].

The next proposition illustrates the relation between the notions of (finite) stability, L-stability and Clifford regularity. A preliminary key observation is furnished by the following lemma.

Lemma 2.4 ([8, Lemma 2.1]). *Let I be a nonzero finitely generated ideal of a domain R. Then $[I]$ is a regular element of $\mathcal{S}(R)$ if and only if I is a stable ideal.*

Proposition 2.5 ([8, Propositions 2.2 and 2.3, Lemma 2.6]).

(1) *A stable domain is Clifford regular.*
(2) *A Clifford regular domain is finitely stable.*
(3) *A Clifford regular domain is L-stable.*

In order to better understand the situation, it is convenient to recall some properties of finitely stable and stable domains.

Theorem 2.6 ([52, Proposition 2.1] and [51, Theorem 3.3]).

(1) *The integral closure of a finitely stable domain is a Prüfer domain.*
(2) *A domain is stable if and only if it has finite character and every localization at a maximal ideal is a stable domain.*

It is also useful to state properties of Clifford regular domains relative to localization and overrings. To this end we can state:

Lemma 2.7 ([8, Lemmas 2.14 and 2.5]).

(1) *A fractional overring of a Clifford regular domain is Clifford regular.*
(2) *If R is a Clifford regular domain and S is a multiplicatively closed subset of R, then R_S is a Clifford regular domain.*

Recall that an overring T of a domain R is fractional if T is a fractional ideal of R. The next result is useful in reducing the problem of the characterization of a Clifford regular domain to the local case: it states that a domain is Clifford regular if and only if it is locally Clifford regular and the trace of any ideal in its endomorphism ring localizes. In this vein, recall that [58] contributes to the classification of Clifford regular local domains.

Proposition 2.8 ([8, Proposition 2.8]). *Let R be a domain. The following are equivalent:*

(1) *R is a Clifford regular domain;*
(2) *For every maximal ideal m of R, R_m is a Clifford regular domain and for every ideal I of R, $(I(I:I^2))_m = I_m(I_m:I_m^2)$, i.e., the trace of the localization I_m in its endomorphism ring coincides with the localization at m of the trace of I in its endomorphism ring.*

In case the Clifford regular domain R is stable or integrally closed, a better result can be proved.

Lemma 2.9. *Let R be a stable or an integrally closed Clifford regular domain. If I is any ideal of R and m is any maximal ideal of R, then the following hold:*

(1) $(I:I)_m = (I_m:I_m)$.
(2) $(I:I^2)_m = (I_m:I_m^2)$.

The connection between Clifford regularity and stability stated by Proposition 2.5 is better illustrated by the concepts of local stability and local invertibility in the way that we are going to indicate.

Definition 2.10. A domain R is said to have the *local invertibility property* (resp., *local stability property*) if every locally invertible (resp., locally stable) ideal is invertible (resp., stable).

The next result is a consequence of Proposition 2.8 and the fact that a locally invertible ideal of a domain is cancellative.

Proposition 2.11 ([8, Lemmas 4.2 and 5.7]). *A Clifford regular domain has the local invertibility property and the local stability property.*

The preceding result together with the observation that stable domains are of finite character, prompts one to ask if a Clifford regular domain is necessarily of finite type. The question has a positive answer if the Clifford regular domain is Noetherian or integrally closed as we are going to show in the next two sections.

2.3 The Noetherian case

From Proposition 2.5, the characterization of the Clifford regular Noetherian domains is immediate.

Theorem 2.12 ([8, Theorem 3.1]). *A Noetherian domain is Clifford regular if and only if it is stable.*

The Noetherian stable rings have been extensively studied by Sally and Vasconcelos in the two papers [53] and [54]. We list some of their results.

(a) A stable Noetherian ring has Krull dimension at most 1.

(b) If every ideal of a domain R is two-generated (i.e., generated by at most two elements), then R is stable.

(c) If R is a Noetherian domain and the integral closure $\bar{R}$ of R is a finitely generated R-module, then R is stable if and only if every ideal of R is two-generated.

(d) Ferrand and Raynaud [24, Proposition 3.1] constructed an example of a local Noetherian stable domain admitting non two-generated ideals. This domain is not Gorenstein.

(e) A local Noetherian Gorenstein domain is Clifford regular if and only if every ideal is two-generated. ([8, Theorem 3.2])

It is not difficult to describe the idempotent elements of the class semigroup of a Noetherian domain and the groups associated to them.

Proposition 2.13 ([8, Proposition 3.4 and Corollary 3.5]). *Let R be a Noetherian domain. The following hold:*

(1) *The idempotent elements of $\mathcal{S}(R)$ are the isomorphy classes of the fractional overrings of R and the groups associated to them are the ideal class groups of the fractional overrings of R.*
(2) *If R is also a Clifford regular domain, then the class semigroup $\mathcal{S}(R)$ of R is the disjoint union of the ideal class groups of the fractional overrings of R and the bonding homomorphisms between the groups are induced by extending ideals to overrings.*

2.4 The integrally closed case

The starting point for the study of integrally closed Clifford regular domains is the following fact.

Proposition 2.14 ([59, Proposition 3]). *An integrally closed Clifford regular domain is a Prüfer domain.*

In [9], it was proved that any valuation domain is Clifford regular and in [5] the result was extended by proving that a Prüfer domain of finite character is a Clifford regular domain. Finally, in [8] it was proved that an integrally closed Clifford regular domain is of finite character.

While trying to prove the finite character property for a Clifford regular Prüfer domain, a more general problem arose and in the papers [7] and [8] the following conjecture was posed. Its interest goes beyond the Clifford regularity of Prüfer domains.

Conjecture. If R is a Prüfer domain with the local invertibility property, then R is of finite character.

In [8], the conjecture was established in the affirmative for the class of Prüfer domains satisfying a particular condition. To state the condition we need to recall a notion on prime ideals: a prime ideal P of a Prüfer domain is *branched* if there exists a prime ideal Q properly contained in P and such that there are no other prime ideals properly between Q and P.

Theorem 2.15 ([8, Theorem 4.4]). *Let R be a Prüfer domain with the local invertibility property. If the endomorphism ring of every branched prime ideal of R satisfies the local invertibility property, then R is of finite character.*

Theorem 2.15 together with Proposition 2.11 and the fact that every fractional overring of a Clifford regular domain is again Clifford regular, imply the characterization of integrally closed Clifford regular domains.

Theorem 2.16 ([8, Theorem 4.5]). *An integrally closed domain is Clifford regular if and only if it is a Prüfer domain of finite character.*

We wish to talk a little about the conjecture mentioned above. It attracted the interest of many authors and its validity has been proved recently. In [36], Holland, Martinez, McGovern, and Tesemma proved that the conjecture is true by translating the problem into a statement on lattice ordered groups. In fact, as shown by Brewer and Klingler in [14], the group G of invertible fractional ideals of a Prüfer domain endowed with the reverse inclusion, is a latticed ordered group and the four authors noticed that both the property of finite character and the local invertibility property of a Prüfer domain can be translated into statements on prime subgroups of the group G and filters on the positive cone of G.

Then, they used a crucial result by Conrad [17] on lattice ordered groups with finite basis to prove that the two statements translating the finite character and the local invertibility property are equivalent, so that the validity of the conjecture follows.

Subsequently, McGovern [47] has provided a ring theoretic proof of the conjecture by translating from the language of lattice ordered groups to the language of ring theory the techniques used in [36]. At one point it was necessary to introduce a suitable localization of the domain in order to translate the notion of the kernel of a lattice homomorphism on the lattice ordered group.

Independently, almost at the same time, Halter-Koch [35] proved the validity of the conjecture by using the language of ideal systems on cancellative commutative monoids and he proved that an r-Prüfer monoid with the local invertibility property is a monoid of Krull type (see [33, Theorem 22.4]).

2.5 The structure of the class semigroup of an integrally closed Clifford regular domain

In order to understand the structure of the class semigroup $\mathcal{S}(R)$ of a Clifford regular domain it is necessary to describe the idempotent elements, the constituent groups associated to them and the bonding homomorphisms between those groups. Complete information is available for the case of integrally closed Clifford regular domains, that is, the class of Prüfer domains of finite character.

In [9], Salce and the first named author proved that the class semigroup of a valuation domain R is a Clifford semigroup with idempotent elements of two types: they are represented either by fractional overrings of R, that is, localizations R_P at prime ideals P, or by nonzero idempotent prime ideals. The groups corresponding to localizations are trivial and the group associated to a nonzero idempotent prime ideal P is described as a quotient of the form $\overline{\Gamma}/\Gamma$, where Γ is the value group of the localization R_P and $\overline{\Gamma}$ is the completion of Γ in the order topology. This group is also called the *archimedean group* of the localizations R_P and denoted by $\operatorname{Arch} R_P$.

If I is a nonzero ideal of R, $[I]$ belongs to $\operatorname{Arch} R_P$ if and only if R_P is the endomorphism ring of I and I is not principal as an R_P-ideal. Note that the endomorphism ring of an ideal I of a valuation domain R is the localization of R at the prime ideal P associated to I defined by $P = \{r \in R \mid rI \subsetneq I\}$ (cf. [29, II p. 69]).

The idempotent elements, the constituent groups and the bonding homomorphisms of the class semigroup of a Prüfer domain of finite character have been characterized by the first named author in [6] and [7].

If $\mathcal{S}(R)$ is a Clifford semigroup and I is a nonzero ideal of R, then by Proposition 2.3, the unique idempotent of $\mathcal{S}(R)$ associated to $[I]$ is the trace ideal T of I in its endomorphism ring, that is, $T = I(I\colon I^2)$. Moreover, every idempotent of $\mathcal{S}(R)$ is of this form. The next two propositions describe the subsemigroup $\mathcal{E}(R)$ of the idempotent elements of $\mathcal{S}(R)$

Proposition 2.17 ([6, Theorem 3.1 and Proposition 3.2]). *Assume that R is a Prüfer domain of finite character. Let I be a nonzero ideal of R such that $[I]$ is* *an idempotent element of $\mathcal{S}(R)$. Then there exists a unique nonzero idempotent fractional ideal L isomorphic to I such that*

$$L = P_1 \cdot P_2 \cdot \cdots \cdot P_n D \qquad n \geq 0$$

with uniquely determined factors satisfying the following conditions:

(1) *$D = (L\colon L)$ is a fractional overring of R;*
(2) *The P_i are pairwise incomparable idempotent prime ideals of R;*
(3) *Each P_iD is a maximal ideal of D;*
(4) *$D \supseteq \mathrm{End}(P_i)$.*

The preceding result shows that the semigroup $\mathcal{E}(R)$ of the idempotent elements of $\mathcal{S}(R)$ is generated by the classes $[P]$ and $[D]$ where P vary among the nonzero idempotent prime ideals of R and D are arbitrary overrings of R. Moreover, every element of $\mathcal{E}(R)$ has a unique representation as a finite product of these classes provided they satisfy the conditions of Proposition 2.17.

For each nonzero idempotent fractional ideal L, denote by G_L the constituent group of $\mathcal{S}(R)$ associated to the idempotent element $[L]$ of $\mathcal{E}(R)$, as defined in Section 2.1. The properties and the structure of the groups G_L have been investigated in [7].

We recall some useful information on ideals of a Prüfer domain of finite character.

Lemma 2.18 ([7, Lemma 3.1]). *Let I and J be locally isomorphic ideals of a Prüfer domain of finite character R. Then there exists a finitely generated fractional ideal B of $D = \mathrm{End}(I)$ such that $I = BJ$. In particular, if R is also a Bézout domain, then $I \cong J$.*

A key observation in order to describe the constituent groups of the class semigroup of a Prüfer domain of finite character R is to note that, for each nonzero idempotent prime ideal P of R, there is a relation between G_P and the archimedean group $\mathrm{Arch}\, R_P$ of the valuation domain R_P (cf. [7, Proposition 3.3]). In fact, the correspondence

$$[I] \mapsto [IR_P], \quad [I] \in G_P$$

induces an epimorphism of Abelian groups

$$\psi\colon G_P \to \mathrm{Arch}\, R_P$$

such that $\mathrm{Ker}\,\psi = \{[CP] \mid \text{C is a finitely generated ideal of } \mathrm{End}(P)\}$. In particular, $\mathrm{Ker}\,\psi \cong \mathcal{C}(\mathrm{End}(P))$ and ψ is injective if and only if $\mathrm{End}(P)$ is a Bézout domain.

The preceding remark can be extended to each group G_L in the class semigroup $\mathcal{S}(R)$.

Theorem 2.19 ([7, Theorem 3.5]). *Assume that R is a Prüfer domain of finite character. Let $L = P_1 \cdot P_2 \cdot \cdots \cdot P_n D$ be a nonzero idempotent fractional ideal of R satisfying the conditions of Proposition 2.17. For every nonzero ideal I of R such that $[I] \in G_L$,* *consider the diagonal map* $\pi([I]) = ([IR_{P_1}], \dots, [IR_{P_n}])$. *Then the group* G_L *fits in the short exact sequence:*

$$1 \to \mathcal{C}(D) \to G_L \xrightarrow{\pi} \operatorname{Arch} R_{P_1} \times \cdots \times \operatorname{Arch} R_{P_n} \to 1.$$

If R is a Bézout domain, then so is every overring D of R, hence the ideal class groups $\mathcal{C}(D)$ are all trivial. The constituent groups are then built up by means of the groups associated to the idempotent prime ideals of R and the structure of the class semigroup $\mathcal{S}(R)$ is simpler, precisely we can state the following:

Proposition 2.20 ([7, Proposition 4.4]). *If R is a Bézout domain of finite character, then the constituent groups associated to every idempotent element of $\mathcal{S}(R)$ are isomorphic to a finite direct product of archimedean groups* $\operatorname{Arch} R_P$ *of the valuation domain R_P, where P is a nonzero idempotent prime ideal of R.*

It remains to describe the partial order on the semigroup $\mathcal{E}(R)$ of the idempotent elements and the bonding homomorphisms between the constituent groups of $\mathcal{S}(R)$.

Recalling that if P and Q are two idempotent prime ideals of a domain R, $P \subseteq Q$ if and only if $PQ = P$ and if D and S are overrings of R, then $S \subseteq D$ if and only if $SD = D$, then the partial order on $\mathcal{E}(R)$ is induced by the inclusion between prime ideals and the reverse inclusion between fractional overrings. Moreover, we have:

Proposition 2.21 ([7, Proposition 4.1]). *Assume that R is a Prüfer domain of finite character. Let $L = P_1 \cdot P_2 \cdot \cdots \cdot P_n D$, $H = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_k S$ be nonzero idempotent fractional ideals of R satisfying the conditions of Proposition 2.17. Then $[L] \leq [H]$ if and only if*

(1) *$S \subseteq D$,*
(2) *For every $1 \leq j \leq k$ either $Q_j D = D$ or there exists $1 \leq i \leq n$ such that $Q_j = P_i$.*

To describe the bonding homomorphisms between the constituent groups of the class semigroup $\mathcal{S}(R)$ it is convenient to consider the properties of two special types of such homomorphisms, that is, those induced by multiplication by a fractional overring of R or by an idempotent prime ideal of R.

Lemma 2.22 ([7, Lemma 4.2]). *Let P, Q be nonzero idempotent prime ideals of the Prüfer domain of finite character R and let D and S be overrings of R such that $S \subseteq D$. Then:*

(1) *The maps*
$$\phi_D^S \colon G_S \to G_D \ \text{ and } \ \phi_{PD}^P \colon G_P \to G_{PD}$$
are surjective homomorphisms induced by multiplication by D.
(2) *Assume that $D \supseteq \operatorname{End}(QP)$ and that P, Q are non-comparable, then:*
$$\phi_{PD}^D \colon G_D \to G_{PD} \ \text{ and } \ \phi_{QPD}^{QD} \colon G_{QD} \to G_{QPD}$$
are injective homomorphisms induced by multiplication by P.

The bonding homomorphisms are then described by the following proposition.

Proposition 2.23 ([7, Proposition 4.3]). *Assume that R is a Prüfer domain of finite character. Let $L = P_1 \cdot P_2 \cdot \dots \cdot P_n D$, $H = Q_1 \cdot Q_2 \cdot \dots \cdot Q_k S$ be nonzero idempotent fractional ideals of R satisfying the conditions of Proposition 2.17 and such that $[L] \leq [H]$. Let $K = Q_1 \cdot Q_2 \cdot \dots \cdot Q_k D$, then the bonding homomorphism $\phi_L^H : G_H \to G_L$ is the composition of the bonding epimorphism ϕ_K^H and the bonding monomorphism ϕ_L^K, namely $\phi_L^H = \phi_L^K \circ \phi_K^H$.*

The results on the structure of the Clifford semigroup of a Prüfer domain of finite character have been generalized by Fuchs [28] by considering an arbitrary Prüfer domain R and restricting considerations to the subsemigroup $\mathcal{S}'(R)$ of $\mathcal{S}(R)$ consisting of the isomorphy classes of ideals containing at least one element of finite character.

2.6 Boole regular domains

Recall that a semigroup S (with 1) is said to be Boolean if for each $x \in S$, $x = x^2$. This subsection seeks ring-theoretic conditions of a domain R that reflects in the Boolean property of its class semigroup $\mathcal{S}(R)$. Precisely, it characterizes integrally closed domains with Boolean class semigroup; in this case, $\mathcal{S}(R)$ happens to identify with the Boolean semigroup formed of all fractional overrings of R. It also treats Noetherian-like settings where the Clifford and Boolean properties of $\mathcal{S}(R)$ coincide with stability conditions; a main feature is that the Clifford property forces t-locally Noetherian domains to be one-dimensional Noetherian domains. It closes with a study of the transfer of the Clifford and Boolean properties to various pullback constructions. These results lead to new families of domains with Clifford or Boolean class semigroup, moving therefore beyond the contexts of integrally closed domains or Noetherian domains.

By analogy with Clifford regularity, we define Boole regularity as follows:

Definition 2.24 ([38]). A domain R is Boole regular if $\mathcal{S}(R)$ is a Boolean semigroup.

Clearly, a PID is Boole regular and a Boole regular domain is Clifford regular. The integral closure of a Clifford regular domain is Prüfer [8, 59]. The next result is an analogue for Boole regularity.

Proposition 2.25 ([38, Proposition 2.3]). *The integral closure of a Boole regular domain is Bézout.*

A first application characterizes almost Krull domains subject to Clifford or Boole regularity as shown below:

Corollary 2.26 ([38, Corollary 2.4]). *A domain R is almost Krull and Boole (resp., Clifford) regular if and only if R is a PID (resp., Dedekind).*

A second application handles the transfer to polynomial rings:

Corollary 2.27 ([38, Corollary 2.5]). *Let R be a domain and X an indeterminate over R. Then:*

$$R \text{ is a field} \Longleftrightarrow R[X] \text{ is Boole regular} \Longleftrightarrow R[X] \text{ is Clifford regular}$$

One of the aims is to establish sufficient conditions for Boole regularity in integrally closed domains. One needs first to examine the valuation case. For this purpose, recall first a stability condition that best suits Boole regularity:

Definition 2.28. A domain R is strongly stable if each nonzero ideal I of R is principal in its endomorphism ring $(I:I)$.

Note that for a domain R, the set $\mathcal{F}_{OV}(R)$ of fractional *overrings* of R is a Boolean semigroup with identity equal to R. Recall that a domain R is said to be strongly discrete if $P^2 \subsetneqq P$ for every nonzero prime ideal P of R [26].

Theorem 2.29 ([38, 39, Theorem 3.2]). *Let R be an integrally closed domain. Then R is a strongly discrete Bézout domain of finite character if and only if R is strongly stable. Moreover, when any one condition holds, R is Boole regular with $\mathcal{S}(R) \cong \mathcal{F}_{OV}(R)$.*

The proof lies partially on the following lemmas.

Lemma 2.30. *Let R be a domain. Then:*

$$R \text{ is stable Boole regular} \Longleftrightarrow R \text{ is strongly stable.}$$

Lemma 2.31. *Let R be an integrally closed domain. Then:*

$$R \text{ is strongly discrete Clifford regular} \Longleftrightarrow R \text{ is stable.}$$

Lemma 2.32. *Let V be a valuation domain. The following are equivalent:*

(1) *V_P is a divisorial domain, for each nonzero prime ideal P of R;*
(2) *V is a stable domain;*
(3) *V is a strongly discrete valuation domain.*

Moreover, when any one condition holds, V is Boole regular.

This lemma gives rise to a large class of Boole regular domains that are not PIDs. For example, any strongly discrete valuation domain of dimension ≥ 2 (cf. [27]) is a Boole regular domain which is not Noetherian. The rest of this subsection studies the class semigroups for two large classes of Noetherian-like domains, that is, t-locally Noetherian domains and Mori domains. Precisely, it examines conditions under which stability and strong stability characterize Clifford regularity and Boole regularity, respectively.

Next, we review some terminology related to the w-operation. For a nonzero fractional ideal I of R, $I_w := \bigcup (I : J)$ where the union is taken over all finitely generated ideals J of R with $J^{-1} = R$. We say that I is a w-ideal if $I_w = I$. The domain R is said to be *Mori* if it satisfies the ascending chain condition on divisorial ideals [3] and *strong Mori* if it satisfies the ascending chain condition on w-ideals [23, 48]. Trivially, a Noetherian domain is strong Mori and a strong Mori domain is Mori. Finally, we say that R is *t-locally Noetherian* if R_M is Noetherian for each t-maximal ideal M of R [43]. Recall that strong Mori domains are t-locally Noetherian [23, Theorem 1.9].

The next result handles the t-locally Noetherian setting.

Theorem 2.33 ([38, Theorem 4.2]). *Let R be a t-locally Noetherian domain. Then R is Clifford (resp., Boole) regular if and only if R is stable (resp., strongly stable). Moreover, when any one condition holds, R is either a field or a one-dimensional Noetherian domain.*

The proof relies partially on the next lemma.

Lemma 2.34. *Let R be a Clifford regular domain. Then $I_t \subsetneqq R$ for each nonzero proper ideal I of R. In particular, every maximal ideal of R is a t-ideal.*

The above theorem asserts that a strong Mori Clifford regular domain is necessarily Noetherian. Here, Clifford regularity forces the w-operation to be trivial (see also [48, Proposition 1.3]). Also noteworthy is that while *a t-locally Noetherian stable domain is necessarily a one-dimensional L-stable domain*, the converse does not hold in general. For instance, consider an almost Dedekind domain which is not Dedekind and appeal to Corollary 2.26. However, the equivalence holds for Noetherian domains [8, Theorem 2.1] and [1, Proposition 2.4].

Corollary 2.35 ([38, Corollary 4.4]). *Let R be a local Noetherian domain such that the extension $R \subseteq \overline{R}$ is maximal, where $\overline{R}$ denotes the integral closure of R. The following are equivalent:*

(1) *R is Boole regular;*
(2) *R is strongly stable;*
(3) *R is stable and $\overline{R}$ is a PID.*

This result generates new families of Boole regular domains beyond the class of integrally closed domains.

Example 2.36. Let $R := k[X^2, X^3]_{(X^2,X^3)}$ where k is a field and X an indeterminate over k. Clearly, $\overline{R} = k[X]_{R\setminus(X^2,X^3)}$ is a PID and the extension $R \subseteq \overline{R}$ is maximal. Further, R is a Noetherian Warfield domain, hence stable (cf. [10]). Consequently, R is a one-dimensional non-integrally closed local Noetherian domain that is Boole regular.

The next results handle the Mori setting. In what follows, we shall use $\overline{R}$ and R^* to denote the integral closure and complete integral closure, respectively, of a domain R.

Theorem 2.37 ([38, Theorem 4.7]). *Let R be a Mori domain. Then the following are equivalent:*

(1) *R is one-dimensional Clifford (resp., Boole) regular and R^* is Mori;*
(2) *R is stable (resp., strongly stable).*

It is worth recalling that for a Noetherian domain R we have $\dim(R) = 1 \Leftrightarrow \dim(R^*) = 1 \Leftrightarrow R^*$ is Dedekind since here $R^* = \overline{R}$. The same result holds if R is a Mori domain such that $(R : R^*) \neq 0$ [4, Corollary 3.4(1) and Corollary 3.5(1)]. Also, it was stated that the "only if" assertion holds for seminormal Mori domains [4, Corollary 3.4(2)]. However, beyond these contexts, the problem remains open. This explains the cohabitation of "$\dim(R) = 1$" and "R^* is Mori" assumptions in the above theorem. In this vein, we set the following open question:
"Let R be a local Mori Clifford regular domain is it true that:

$$\dim(R) = 1 \Longleftrightarrow R^* \text{ is Dedekind?"}$$

The next result partly draws on the above theorem and treats two well-studied large classes of Mori domains [3]. Recall that a domain R is seminormal if $x \in R$ whenever $x \in K$ and $x^2, x^3 \in R$.

Theorem 2.38 ([38, Theorem 4.9]). *Let R be a Mori domain. Consider the following statements:*

(1) *The conductor $(R : R^*) \neq 0$,*
(2) *R is seminormal,*
(3) *The extension $R \subseteq R^*$ has at most one proper intermediate ring.*

Assume that either (1), (2), or (3) holds. Then R is Clifford (resp., Boole) regular if and only if R is stable (resp., strongly stable).

2.7 Pullbacks

The purpose here is to examine Clifford regularity and Boole regularity in pullback constructions. This allows for the construction of new families of domains with Clifford or Boolean class semigroup, beyond the contexts of integrally closed or Noetherian domains.

Let us fix the notation for the rest of this subsection. Let T be a domain, M a maximal ideal of T, K its residue field, $\phi : T \longrightarrow K$ the canonical surjection, D a proper subring of K with quotient field k. Let $R := \phi^{-1}(D)$ be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

Next, we announce the first theorem which provides a necessary and sufficient condition for a pseudo-valuation domain (i.e., PVD) to inherit Clifford or Boole regularity.

Theorem 2.39 ([38, Theorem 5.1]).

(1) *If R is Clifford (resp., Boole) regular, then so are T and D, and $[K:k] \leq 2$.*
(2) *Assume $D = k$ and T is a valuation (resp., strongly discrete valuation) domain. Then R is Clifford (resp., Boole) regular if and only if $[K:k] = 2$.*

The following example shows that this theorem does not hold in general, and hence nor does the converse of (1).

Example 2.40. Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and field of rational numbers, respectively, and let X and Y be indeterminates over $\mathbb{Q}$. Set $V := \mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$, $M := X\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$, $T := \mathbb{Q}(\sqrt{2}) + M$, and $R := \mathbb{Q} + M$. Both T and R are one-dimensional local Noetherian domains arising from the DVR V, with $\overline{T} = V$ and $\overline{R} = T$. By the above theorem, T is Clifford (actually, Boole) regular, whereas R is not. More specifically, the isomorphy class of the ideal $I := X(\mathbb{Q} + \sqrt{2}\mathbb{Q} + \sqrt{3}\mathbb{Q} + M)$ is not regular in $\mathcal{S}(R)$.

Now, one can build original example using the above theorem as follows:

Example 2.41. Let n be an integer ≥ 1. Let R be a PVD associated with a non-Noetherian n-dimensional valuation (resp., strongly discrete valuation) domain (V,M) with $[V/M : R/M] = 2$. Then R is an n-dimensional local Clifford (resp., Boole) regular domain that is neither integrally closed nor Noetherian.

Recall that a domain A is said to be *conducive* if the conductor $(A:B)$ is nonzero for each overring B of A other than its quotient field. Examples of conducive domains include arbitrary pullbacks of the form $R := D + M$ arising from a valuation domain $V := K + M$ [19, Propositions 2.1 and 2.2]. We are now able to announce the last theorem of this subsection. It treats Clifford regularity, for the remaining case "$k = K$", for pullbacks $R := \phi^{-1}(D)$ where D is a conducive domain.

Theorem 2.42 ([38, Theorem 5.6]). *Under the same notation as above, consider the following statements:*

(1) *T is a valuation domain and $R := \phi^{-1}(D)$,*
(2) *$T := K[X]$ and $R := D + XK[X]$, where X is an indeterminate over K.*

Assume that D is a semilocal conducive domain with quotient field $k = K$ and either (1) or (2) holds. Then R is Clifford regular if and only if so is D.

Now a combination of Theorems 2.39 and 2.42 generates new families of examples of Clifford regular domains, as shown by the following construction [38, Example 5.8]:

Example 2.43. For every positive integer $n \geq 2$, there exists an example of a domain R satisfying the following conditions:

(1) $\dim(R) = n$,
(2) R is neither integrally closed nor Noetherian,
(3) R is Clifford regular,
(4) Each overring of R is Clifford regular,
(5) R has infinitely many maximal ideals.

2.8 Open problems

By Proposition 2.5 the class of Clifford regular domains contains the class of stable domains and is contained in the class of finitely stable domains. Both inclusions are proper. In fact, every Prüfer domain is finitely stable, but only the Prüfer domains of finite character are Clifford regular. Moreover, a Prüfer domain is stable if and only if it is of finite character and strongly discrete, that is, every nonzero prime ideal is not idempotent (cf. [49, Theorem 4.6]), hence there exists a large class of nonstable integrally closed Clifford regular domains. The classification of stable domains obtained by B. Olberding in [50], shows that there are stable domains which are neither Noetherian nor integrally closed. Furthermore, there is an example of a non-coherent stable domain ([50, Section 5]), hence there exist non-coherent Clifford regular domains.

There are also examples of Clifford regular domains which are neither stable nor integrally closed, as illustrated by [8, Example 6.1].

Example 2.44. Let k_0 be a field and let K be an extension field of k_0 such that $[K : k_0] = 2$. Consider a valuation domain V of the form $K+M$ where M is the maximal ideal of V and assume $M^2 = M$. Let R be the domain k_0+M. The ideals of R can be easily described: they are either ideals of V or principal ideals of R. Thus, R is Clifford regular, but it is not stable, since M is an idempotent ideal of R; moreover the integral closure of R is V.

There are still many questions related to the problem of characterizing the class of Clifford regular domains in general. Note that if a domain R is stable, then R is of finite character and every overring of R is again stable ([51, Theorems 3.3 and 5.1]). If R is an integrally closed Clifford regular domain, then R is a Prüfer domain of finite character (Theorem 2.16) and thus the same holds for every overring of R. Hence, the two subclasses of Clifford regular domains consisting of the stable domains and of the integrally closed domains are closed for overrings and their members are domains of finite character. We may ask the following major questions concerning Clifford regular domains:

Question 2.45 *Is every Clifford regular domain of finite character?*

Question 2.46 (a) *Is every overring of a Clifford regular domain again Clifford regular?*

(b) *In particular, is the integral closure of a Clifford regular domain a Clifford regular domain?*

In [56], Sega gives partial answers to part (a) of this question. In particular, he proves that if R is a Clifford regular domain such that the integral closure of R is a fractional overring, then every overring of R is Clifford regular. An affirmative answer to part (b) would imply that a Clifford regular domain is necessarily of finite character, since the integral closure of a Clifford regular domain is a Prüfer domain.

In view of the validity of the conjecture about the finite character of Prüfer domains with the local invertibility property proved in [36], Question 2.46 (b) may be weakened by asking if the integral closure of a Clifford regular domain satisfies the local invertibility property. More generally we may ask:

Question 2.47 *If a finitely stable domain satisfies the local invertibility property, is it true that its integral closure satisfies the same property?*

A positive answer to the above question would imply that a finitely stable domain satisfying the local invertibility property has finite character.

Another interesting problem is to characterize the local Clifford regular domains. The next example shows that not every finitely stable local domain is Clifford regular.

Example 2.48. Let A be a DVR with quotient field Q and let B be the ring $Q[[X^2,X^3]]$. Denote by P the maximal ideal of B and let $R = A+P$. By [50, Proposition 3.6], R is finitely stable but it is not L-stable. In fact, $J = Q+AX+P$ is a fractional ideal of R, since $JP \subseteq P \subseteq R$ and $(J:J) = R$, but $J^2 = Q[[X]]$. Thus, by Proposition 2.5, R is not Clifford regular.

However, the following result holds.

Proposition 2.49 ([8, Corollary 5.6]). *Let R be a local Clifford regular domain with principal maximal ideal. Then R is a valuation domain.*

In the case of a Clifford regular domain R of finite character a description of the idempotent elements of $\mathcal{S}(R)$ is available. It generalizes the situation illustrated in Proposition 2.17 for Clifford regular Prüfer domains.

Lemma 2.50. *Let R be a Clifford regular domain of finite character and let T be a nonzero idempotent fractional ideal of R. If* $E = \mathrm{End}(T)$*, then either* $T = E$ *or T is a product of idempotent maximal ideals of E.*

We end this subsection by recalling a partial result regarding the finite character of Clifford regular domains. We denote by $\mathcal{T}(R)$ the set of maximal ideals m of R for which there exists a finitely generated ideal with the property that m is the only maximal ideal containing it.

Proposition 2.51. *Let R be a finitely stable domain satisfying the local stability property. Then every nonzero element of R is contained in at most a finite number of maximal ideals of* $\mathcal{T}(R)$*. In particular the result holds for every Clifford regular domain.*

3 *t*-Class semigroups of integral domains

A domain R is called a PVMD (for Prüfer v-multiplication domain) if the v-finite v-ideals form a group under the t-multiplication; equivalently, if R_M is a valuation domain for each t-maximal ideal M of R. Ideal t-multiplication converts ring notions such as PID, Dedekind, Bézout, Prüfer, and integrality to UFD, Krull, GCD, PVMD, and pseudo-integrality, respectively. The pseudo-integrality (i.e., t-integrality) was introduced and studied in 1991 by D. F. Anderson, Houston, and Zafrullah [2].

The *t-class semigroup* of R is defined by

$$\mathcal{S}_t(R) := \mathcal{F}_t(R)/\mathcal{P}(R)$$

where $\mathcal{P}(R)$ is the subsemigroup of $\mathcal{F}_t(R)$ consisting of nonzero principal fractional ideals of R. Thus, $\mathcal{S}_t(R)$ stands as the t-analogue of $\mathcal{S}(R)$, the class semigroup of R. For the reader's convenience we recall from the introduction the set-theoretic inclusions:

$$\mathcal{C}(R) \subseteq \mathrm{Cl}(R) \subseteq \mathcal{S}_t(R) \subseteq \mathcal{S}(R).$$

By analogy with Clifford regularity and Boole regularity (Section 2), we define t-regularity as follows:

Definition 3.1 ([40]). A domain R is Clifford (resp., Boole) t-regular if $\mathcal{S}_t(R)$ is a Clifford (resp., Boolean) semigroup.

This section reviews recent works that examine ring-theoretic conditions of a domain R that reflect reciprocally in semigroup-theoretic properties of its t-class semigroup $\mathcal{S}_t(R)$. Contexts that suit best t-regularity are studied in [40–42] in an attempt to parallel analogous developments and generalize the results on class semigroups (reviewed in Section 2).

Namely, [40] treats the case of PVMDs extending Bazzoni's results on Prüfer domains [5, 8]; [41] describes the idempotents of $S_t(R)$ and the structure of their associated groups recovering well-known results on class semigroups of valuation domains [9] and Prüfer domains [6, 7]; and [42] studies the t-class semigroup of a Noetherian domain. All results are illustrated by original examples distinguishing between the two concepts of class semigroup and t-class semigroup. Notice that in Prüfer domains, the t- and trivial operations (and hence the t-class and class semigroups) coincide.

3.1 Basic results on t-regularity

Here, we discuss t-analogues of basic results on t-regularity. First we notice that Krull domains and UFDs are Clifford and Boole t-regular, respectively. These two classes of domains serve as a starting ground for t-regularity as Dedekind domains and PIDs do for regularity. Also, we will see that t-regularity stands as a default measure for some classes of Krull-like domains, e.g., "UFD $=$ Krull $+$ Boole t-regular." Moreover, while an integrally closed Clifford regular domain is Prüfer (Proposition 2.14), an integrally closed Clifford t-regular domain need not be a PVMD. An example is built to this end, as an application of the main theorem of this subsection, which examines the transfer of t-regularity to pseudo-valuation domains.

The first result displays necessary and/or sufficient ideal-theoretic conditions for the isomorphy class of an ideal to be regular in the t-class semigroup.

Lemma 3.2 ([40, Lemma 2.1]). *Let I be a t-ideal of a domain R. Then*

(1) *$[I]$ is regular in $\mathcal{S}_t(R)$ if and only if $I = (I^2(I : I^2))_t$.*
(2) *If I is t-invertible, then $[I]$ is regular in $\mathcal{S}_t(R)$.*

A domain R is Krull if every t-ideal of R is t-invertible. From the lemma one can obviously see that a Krull domain is Clifford t-regular. Recall that a domain R is t-almost Dedekind if R_M is a rank-one DVR for each t-maximal ideal M of R; t-almost Dedekind domains lie strictly between Krull domains and general PVMDs [43]. A domain R is said to be strongly t-discrete if it has no t-idempotent t-prime ideals (i.e., for every t-prime ideal P, $(P^2)_t \subsetneqq P$) [22]. The next results (cf. [40, Proposition 2.3]) show that t-regularity measures how far some Krull-like domains are from being Krull or UFDs.

Proposition 3.3. *Let R be a domain. The following are equivalent:*

(1) *R is Krull;*
(2) *R is t-almost Dedekind and Clifford t-regular;*
(3) *R is strongly t-discrete, completely integrally closed, and Clifford t-regular.*

Proposition 3.4. *Let R be a domain. The following are equivalent:*

(1) *R is a UFD;*
(2) *R is Krull and Boole t-regular;*
(3) *R is t-almost Dedekind and Boole t-regular;*
(4) *R is strongly t-discrete, completely integrally closed, and Boole t-regular.*

Note that the assumptions in the previous results are not superfluous. For, the (Bézout) ring of all entire functions in the complex plane is strongly (t-)discrete [26, Corollary 8.1.6] and completely integrally closed, but it is not (t-)almost Dedekind (since it has an infinite Krull dimension). Also, a non-discrete rank-one valuation domain is completely integrally closed and Clifford (t-)regular [9], but it is not Krull.

The t-regularity transfers to polynomial rings and factor rings providing more examples of Clifford or Boole t-regular domains, as shown in the next result. Recall that Clifford regularity of $R[X]$ forces R to be a field (Corollary 2.27).

Proposition 3.5 ([40, Propositions 2.4 and 2.5]). *Let R be a domain, X an indeterminate over R, and S a multiplicative subset of R.*

(1) *Assume R is integrally closed. Then R is Clifford (resp., Boole) t-regular if and only if so is $R[X]$.*
(2) *If R is Clifford (resp., Boole) t-regular, then so is R_S.*

Now, one needs to examine the integrally closed setting. At this point, recall that an integrally closed Clifford (resp., Boole) regular domain is necessarily Prüfer (resp., Bézout) [38,59]. This fact does not hold for t-regularity; namely, an integrally closed Clifford (or Boole) t-regular domain need not be a PVMD (i.e., t-Prüfer). Examples stem from the following theorem on the inheritance of t-regularity by PVDs (for pseudo-valuation domains).

Theorem 3.6 ([40, Theorem 2.7]). *Let R be a PVD issued from a valuation domain V. Then:*

(1) *R is Clifford t-regular.*
(2) *R is Boole t-regular if and only if V is Boole regular.*

Contrast this result with Theorem 2.39 about regularity; which asserts that if R is a PVD issued from a valuation (resp., strongly discrete valuation) domain (V,M), then R is a Clifford (resp., Boole) regular domain if and only if $[V/M : R/M] = 2$.

Now, using Theorem 3.6, one can build integrally closed Boole (hence Clifford) t-regular domains which are not PVMDs. For instance, let k be a field and X,Y two indeterminates over k. Let $R := k + M$ be the PVD associated to the rank-one DVR $V := k(X)[[Y]] = k(X) + M$, where $M = YV$. Clearly, R is an integrally closed Boole t-regular domain but not a PVMD [25, Theorem 4.1].

3.2 The PVMD case

A domain R is of finite t-character if each proper t-ideal is contained in only finitely many t-maximal ideals. It is worthwhile recalling that the PVMDs of finite t-character are exactly the Krull-type domains introduced and studied by Griffin in 1967–1968 [31, 32]. This subsection discusses the t-analogue for Bazzoni's result that "*an integrally closed domain is Clifford regular if and only if it is a Prüfer domain of finite character*" (Theorem 2.16).

Recall from [2] that the pseudo-integral closure of a domain R is defined as $\widetilde{R} = \bigcup(I_t : I_t)$, where I ranges over the set of finitely generated ideals of R; and R is said to be pseudo-integrally closed if $R = \widetilde{R}$. This is equivalent to saying that R is a v-domain, i.e. a domain such that $(I_v : I_v) = R$ for each nonzero finitely generated ideal I of R. A domain with this property is called in Bourbaki's language regularly integrally closed [11, Chap. VII, Exercise 30]. Clearly $\overline{R} \subseteq \widetilde{R} \subseteq R^{\star}$, where $\overline{R}$ and $R^{\star}$ are respectively the integral closure and the complete integral closure of R. In view of the example provided in the previous subsection, one has to elevate the "integrally closed" assumption in Bazzoni's result to "pseudo-integrally closed." Accordingly, in [40, Conjecture 3.1], the authors sustained the following:

Conjecture 3.7. A pseudo-integrally closed domain (i.e., v-domain) is Clifford t-regular if and only if it is a PVMD of finite t-character.

The next result presented a crucial step towards a satisfactory t-analogue.

Theorem 3.8 ([40, Theorem 3.2]). *A PVMD is Clifford t-regular if and only if it is a Krull-type domain.*

Since in Prüfer domains the t- and trivial operations coincide, this theorem recovers Bazzoni's result (mentioned above) and also uncovers the fact that in the class of PVMDs, Clifford t-regularity coincides with the finite t-character condition.

The proof involves several preliminary lemmas, some of which are of independent interest and their proofs differ in form from their respective analogues – if any – for the trivial operation. These lemmas are listed below.

Lemma 3.9. *Let R be a PVMD and I a nonzero fractional ideal of R. Then for every t-prime ideal P of R, $I_tR_P = IR_P$.*

Lemma 3.10. *Let R be a PVMD which is Clifford t-regular and I a nonzero fractional ideal of R. Then I is t-invertible if and only if I is t-locally principal.*

Lemma 3.11. *Let R be a PVMD which is Clifford t-regular and let $P \subsetneq Q$ be two t-prime ideals of R. Then there exists a finitely generated ideal I of R such that $P \subsetneq I_t \subseteq Q$.*

Lemma 3.12. *Let R be a PVMD which is Clifford t-regular and P a t-prime ideal of R. Then $(P:P)$ is a PVMD which is Clifford t-regular and P is a t-maximal ideal of $(P:P)$.*

Lemma 3.13. *Let R be a PVMD which is Clifford t-regular and Q a t-prime ideal of R. Suppose there is a nonzero prime ideal P of R such that $P \subsetneq Q$ and* $\mathrm{ht}(Q/P) = 1$. *Then there exists a finitely generated subideal I of Q such that* $\mathrm{Max}_t(R,I) = \mathrm{Max}_t(R,Q)$, *where* $\mathrm{Max}_t(R,I)$ *consists of t-maximal ideals containing I.*

As a consequence of Theorem 3.8, the next result handles the context of strongly t-discrete domains.

Corollary 3.14 ([40, Corollary 3.12]). *Assume R is a strongly t-discrete domain. Then R is a pseudo-integrally closed Clifford t-regular domain if and only if R is a PVMD of finite t-character.*

Recently, Halter-Koch solved Conjecture 3.7 by using the language of ideal systems on cancellative commutative monoids. Precisely, he proved that "*every t-Clifford regular v-domain is a Krull-type domain*" [35, Propositions 6.11 and 6.12]. This result combined with the "if" statement of Theorem 3.8 provides a t-analogue for Bazzoni's result (mentioned above):

Theorem 3.15. *A v-domain is Clifford t-regular if and only if it is a Krull-type domain.*

The rest of this subsection is devoted to generating examples. For this purpose, two results will handle the possible transfer of the PVMD notion endowed with the finite t-character condition to pullbacks and polynomial rings, respectively. This will allow for the construction of original families of Clifford t-regular domains via PVMDs.

Proposition 3.16 ([40, Proposition 4.1]). *Let T be a domain, M a maximal ideal of T, K its residue field, $\phi : T \longrightarrow K$ the canonical surjection, and D a proper* *subring of K. Let $R = \phi^{-1}(D)$ be the pullback issued from the following diagram of canonical homomorphisms:*

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

Then R is a PVMD of finite t-character if and only if D is a semilocal Bézout domain with quotient field K and T is a Krull-type domain such that T_M is a valuation domain.

Proposition 3.17 ([40, Proposition 4.2]). *Let R be an integrally closed domain and X an indeterminate over R. Then R has finite t-character if and only if so does $R[X]$.*

Note that the "integrally closed" condition is unnecessary in the above result, as pointed out recently in [30]. Now one can build new families of Clifford t-regular domains originating from the class of PVMDs via a combination of the two previous results and Theorem 3.8 (cf. [40, Example 4.3]).

Example 3.18. For each integer $n \geq 2$, there exists a PVMD R_n subject to the following conditions:

(1) $\dim(R_n) = n$.
(2) R_n is Clifford t-regular.
(3) R_n is not Clifford regular.
(4) R_n is not Krull.

Here are two ways to realize this. Let V_0 be a rank-one valuation domain with quotient field K. Let $V = K + N$ be a rank-one non-strongly discrete valuation domain (cf. [21, Remark 6(b)]). Take $R_n = V[X_1, \ldots, X_{n-1}]$.

For $n \geq 4$, the classical $D + M$ construction provides more examples. Indeed, consider an increasing sequence of valuation domains $V = V_1 \subset V_2 \subset, \ldots, \subset V_{n-2}$ such that, for each $i \in \{2, \ldots, n-2\}$, $\dim(V_i) = i$ and $V_i/M_i = V/N = K$, where M_i denotes the maximal ideal of V_i. Set $T = V_{n-2}[X]$ and $M = (M_{n-2}, X)$. Therefore $R_n = V_0 + M$ is the desired example.

3.3 The structure of the t-class semigroup of a Krull-type domain

This subsection extends Bazzoni and Salce's study of groups in the class semigroup of a valuation domain [9] and recovers Bazzoni's results on the constituents groups of the class semigroup of a Prüfer domain of finite character [6, 7] to a larger class of domains. Precisely, it describes the idempotents of $\mathcal{S}_t(R)$ and the structure of their associated groups when R is a Krull-type domain (i.e., PVMD of finite t-character). Indeed, it states that there are two types of idempotents in $\mathcal{S}_t(R)$: those represented by fractional overrings of R and those represented by finite intersections of t-maximal ideals of fractional overrings of R. Further, it shows that the group associated with an idempotent of the first type equals the class group of the fractional overring, and characterizes the elements of the group associated with an idempotent of the second type in terms of their localizations at t-prime ideals.

In any attempt to extend classical results on Prüfer domains to PVMDs (via t-closure), the t-linked notion plays a crucial role in order to make the t-move possible. An overring T of a domain R is *t-linked* over R if, for each finitely generated ideal I of R, $I^{-1} = R \Rightarrow (T : IT) = T$ [2,45]. In Prüfer domains, the t-linked property coincides with the notion of overring (since every finitely generated proper ideal is invertible and then its inverse is a fortiori different from R). Recall also that an overring T of R is fractional if T is a fractional ideal of R. Of significant importance too for the study of t-class semigroups is the notion of t-idempotence; namely, a t-ideal I is t-idempotent if $(I^2)_t = I$.

Let R be a PVMD. Note that T is a t-linked overring of R if and only if T is a subintersection of R, that is, $T = \bigcap R_P$, where P ranges over some set of t-prime ideals of R [44, Theorem 3.8] or [15, p. 206]. Further, every t-linked overring of R is a PVMD [44, Corollary 3.9]; in fact, this condition characterizes the notion of PVMD [20, Theorem 2.10]. Finally, let I be a t-ideal of R. Then $(I : I)$ is a fractional t-linked overring of R and hence a PVMD.

Theorem 3.8 asserts that if R is a Krull-type domain, then $\mathcal{S}_t(R)$ is Clifford and hence a disjoint union of subgroups $G_{[J]}$, where $[J]$ ranges over the set of idempotents of $\mathcal{S}_t(R)$ and $G_{[J]}$ is the largest subgroup of $\mathcal{S}_t(R)$ with unit $[J]$. At this point, it is worthwhile recalling Bazzoni-Salce's result that valuation domains have Clifford class semigroup [9]. To the main result of this subsection:

Theorem 3.19 ([41, Theorem 2.1]). *Let R be a Krull-type domain and I a t-ideal of R. Set $T := (I : I)$ and $\Gamma(I) := \{$finite intersections of t-idempotent t-maximal ideals of $T\}$. Then $[I]$ is an idempotent of $\mathcal{S}_t(R)$ if and only if there exists a unique $J \in \{T\} \cup \Gamma(I)$ such that $[I] = [J]$. Moreover,*
(1) If $J = T$, then $G_{[J]} \cong \mathrm{Cl}(T)$;
(2) If $J = \bigcap_{1 \leq i \leq r} Q_i \in \Gamma(I)$, then the sequence

$$0 \longrightarrow \mathrm{Cl}(T) \xrightarrow{\phi} G_{[J]} \xrightarrow{\psi} \prod_{1 \leq i \leq r} G_{[Q_i T_{Q_i}]} \longrightarrow 0$$

of natural group homomorphisms is exact, where $G_{[Q_i T_{Q_i}]}$ denotes the constituent group of the Clifford semigroup $\mathcal{S}(T_{Q_i})$ associated with $[Q_i T_{Q_i}]$.

The proof of the theorem draws partially on the following lemmas, which are of independent interest.

Lemma 3.20. *Let R be a PVMD. Let T be a t-linked overring of R and Q a t-prime ideal of T. Then $P := Q \cap R$ is a t-prime ideal of R with $R_P = T_Q$. If, in addition, Q is t-idempotent in T, then so is P in R.*

Lemma 3.21. *Let R be a PVMD and T a t-linked overring of R. Let J be a common (fractional) ideal of R and T. Then:*

(1) *$J_{t_1} = J_t$, where t_1 denotes the t-operation with respect to T.*
(2) *J is a t-idempotent t-ideal of $R \Longleftrightarrow J$ is a t-idempotent t-ideal of T.*

Lemma 3.22. *Let R be a PVMD, I a t-ideal of R, and $T := (I : I)$. Let $J := \bigcap_{1\leq i\leq r} Q_i$, where each Q_i is a t-idempotent t-maximal ideal of T. Then J is a fractional t-idempotent t-ideal of R.*

Lemma 3.23. *Let R be a PVMD, I a t-idempotent t-ideal of R, and $M \supseteq I$ a t-maximal ideal of R. Then IR_M is an idempotent (prime) ideal of R_M.*

Lemma 3.24. *Let R be a Krull-type domain, L a t-ideal of R, and J a t-idempotent t-ideal of R. Then:*

$$[L] \in G_{[J]} \Longleftrightarrow (L : L) = (J : J) \text{ and } (JL(L : L^2))_t = (L(L : L^2))_t = J.$$

Lemma 3.25. *Let R be a PVMD and I a t-ideal of R. Then:*

(1) *I is a t-ideal of $(I : I)$.*
(2) *If R is Clifford t-regular, then so is $(I : I)$.*

Since in a Prüfer domain the t-operation collapses to the trivial operation, Theorem 3.19 recovers Bazzoni's results on Prüfer domains of finite character (Proposition 2.17 and Theorem 2.19). Moreover, there is the following consequence:

Corollary 3.26 ([41, Corollary 2.9]). *Let R be a Krull-type domain which is strongly t-discrete. Then $\mathcal{S}_t(R)$ is a disjoint union of subgroups* $\mathrm{Cl}(T)$*, where T ranges over the set of fractional t-linked overrings of R.*

Now one can develop numerous illustrative examples via Theorem 3.19 and Corollary 3.26. Two families of such examples can be provided by means of polynomial rings over valuation domains. First, the following lemma investigates this setting:

Lemma 3.27 ([41, Lemma 3.1]). *Let V be a nontrivial valuation domain and X an indeterminate over V. Then:*

(1) *$R := V[X]$ is a Krull-type domain which is not Prüfer.*
(2) *Every fractional t-linked overring of R has the form $V_p[X]$ for some nonzero prime ideal p of V.*
(3) *Every t-idempotent t-prime ideal of R has the form $p[X]$ for some idempotent prime ideal p of V.*

Example 3.28. Let n be an integer ≥ 1. Let V be an n-dimensional strongly discrete valuation domain and let $(0) \subset p_1 \subset p_2 \subset ... \subset p_n$ denote the chain of its prime ideals. Let $R := V[X]$, a Krull-type domain. A combination of Lemma 3.27 and Corollary 3.26 yields

$$\mathcal{S}_t(R) = \{V_{p_1}[X], V_{p_2}[X], ..., V_{p_n}[X]\}$$

where, for each i, the class $[V_{p_i}[X]]$ in $\mathcal{S}_t(R)$ is identified with $V_{p_i}[X]$ (due to the uniqueness stated by the main theorem).

Example 3.29. Let V be a one-dimensional valuation domain with idempotent maximal ideal M and $R := V[X]$, a Krull-type domain. By Theorem 3.19 and Lemma 3.27, we have:

$$\mathcal{S}_t(R) = \{[R]\} \cup \{[I] \mid I \ t\text{-ideal of } R \text{ with } (II^{-1})_t = M[X]\}.$$

3.4 The Noetherian case

A domain R is called *strong Mori* if R satisfies the ascending chain condition on w-ideals (cf. Section 2.6). Recall that the *t-dimension* of R, abbreviated t-dim(R), is by definition equal to the length of the longest chain of t-prime ideals of R.

This subsection discusses t-regularity in Noetherian and Noetherian-like domains. Precisely, it studies conditions under which t-stability (see definition below) characterizes t-regularity. Unlike regularity, t-regularity over Noetherian domains does not force the t-dimension to be one. However, Noetherian strong t-stable domains happen to have t-dimension 1.

Recall that an ideal I of a domain R is said to be L-stable if $R^I := \bigcup_{n\geq 1}(I^n : I^n)$ $= (I : I)$.

The next result compares Clifford t-regularity to two forms of stability.

Theorem 3.30 ([42, Theorem 2.2]). *Let R be a Noetherian domain and consider the following:*

(1) *R is Clifford t-regular,*
(2) *Each t-ideal I of R is t-invertible in $(I : I)$,*
(3) *Each t-ideal is L-stable.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$. *If* t-dim$(R) = 1$, *then the 3 conditions are equivalent.*

Recall that an ideal I of a domain R is said to be *stable* (resp., *strongly stable*) if I is invertible (resp., principal) in $(I : I)$, and R is called a stable (resp., strongly stable) domain provided each nonzero ideal of R is stable (resp., strongly stable). A stable domain is L-stable [1, Lemma 2.1]. By analogy, t-stability is defined in [42] as follows:

Definition 3.31. A domain R is *t-stable* if each t-ideal of R is stable, and R is *strongly t-stable* if each t-ideal of R is strongly stable.

Recall that a Noetherian domain R is Clifford regular if and only if R is stable if and only if R is L-stable and $\dim(R) = 1$ [8, Theorem 2.1] and [38, Corollary 4.3]. Unlike Clifford regularity, Clifford (or even Boole) t-regularity does not force a Noetherian domain R to be of t-dimension one. In order to illustrate this fact with an example, a result first establishes the transfer of Boole t-regularity to pullbacks issued from local Noetherian domains.

Proposition 3.32 ([42, Proposition 2.3]). *Let (T,M) be a local Noetherian domain with residue field K and $\phi : T \longrightarrow K$ the canonical surjection. Let k be a proper subfield of K and $R := \phi^{-1}(k)$ the pullback issued from the following diagram of canonical homomorphisms:*

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

Then R is Boole t-regular if and only if T is Boole t-regular.

Now the next example provides a Boole t-regular Noetherian domain with t-dimension $\gneqq 1$.

Example 3.33. Let K be a field, X and Y two indeterminates over K, and k a proper subfield of K. Let $T := K[[X,Y]] = K + M$ and $R := k + M$ where $M := (X,Y)$. Since T is a UFD, then T is Boole t-regular (Proposition 3.4). Further, R is a Boole t-regular Noetherian domain by the above proposition. Further M is a v-ideal of R, so that $t\text{-dim}(R) = \dim(R) = 2$, as desired.

Next, the main result of this subsection presents a t-analogue for Boole regularity as stated in Theorem 2.33.

Theorem 3.34 ([42, Theorem 2.6]). *Let R be a Noetherian domain. Then:*

$$R \textit{ is strongly } t\textit{-stable} \Longleftrightarrow R \textit{ is Boole } t\textit{-regular and } t\text{-dim}(R) = 1.$$

An analogue of this result does not hold for Clifford t-regularity. For, there exists a Noetherian Clifford t-regular domain with t-dim$(R) = 1$ such that R is not t-stable. Indeed, recall first that a domain R is said to be pseudo-Dedekind [43] (or generalized Dedekind [57]) if every v-ideal is invertible. In [55], P. Samuel gave an example of a Noetherian UFD R for which $R[[X]]$ is not a UFD. In [43], Kang noted that $R[[X]]$ is a Noetherian Krull domain which is not pseudo-Dedekind (otherwise, $\mathrm{Cl}(R[[X]]) = \mathrm{Cl}(R) = 0$ forces $R[[X]]$ to be a UFD, absurd). Moreover, $R[[X]]$ is a Clifford t-regular domain with t-dimension 1 (since Krull). But $R[[X]]$ not being a UFD translates into the existence of a v-ideal of $R[[X]]$ that is not invertible, as desired.

The next result extends the above theorem to the larger class of strong Mori domains.

Theorem 3.35 ([42, Theorem 2.10]). *Let R be a strong Mori domain. Then:*

$$R \textit{ is strongly } t\textit{-stable} \Longleftrightarrow R \textit{ is Boole } t\textit{-regular and } t\text{-dim}(R) = 1.$$

Unlike Clifford regularity, Clifford (or even Boole) t-regularity does not force a strong Mori domain to be Noetherian. Indeed, it suffices to consider a UFD which is not Noetherian. We close with the following discussion about the limits and possible extensions of the above results.

Remark 3.36. (1) It is not known whether the assumption "t-dim$(R) = 1$" in Theorem 3.30 can be omitted.

(2) Following Proposition 2.25, the integral closure $\overline{R}$ of a Noetherian Boole regular domain R is a PID. By analogy, it is not known if $\overline{R}$ is a UFD in the case of Boole t-regularity. (It is the case if the conductor $(R:\overline{R}) \neq 0$.)

(3) It is not known if the assumption "R strongly t-discrete, i.e., R has no t-idempotent t-prime ideals" forces a Clifford t-regular Noetherian domain to be of t-dimension one.

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