BOUVIER'S CONJECTURE

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ABSTRACT. This paper deals with Bouvier's conjecture which sustains that finite-dimensional non-Noetherian Krull domains need not be Jaffard.

1. INTRODUCTION

All rings and algebras considered in this paper are commutative with identity element and, unless otherwise specified, are assumed to be non-zero. All ring homomorphisms are unital. If k is a field and A a domain which is a k-algebra, we use $\mathrm{qf}(A)$ to denote the quotient field of A and $\mathrm{t.d.}(A)$ to denote the transcendence degree of $\mathrm{qf}(A)$ over k. Finally, recall that an affine domain over a ring A is a finitely generated A-algebra that is a domain [28, p. 127]. Any unreferenced material is standard as in [17, 23, 25].

A finite-dimensional integral domain R is said to be Jaffard if

$$\dim(R[X_1,...,X_n]) = n + \dim(R)$$

for all $n \geq 1$; equivalently, if $\dim(R) = \dim_v(R)$, where $\dim(R)$ denotes the (Krull) dimension of R and $\dim_v(R)$ its valuative dimension (i.e., the supremum of dimensions of the valuation overrings of R). As this notion does not carry over to localizations, R is said to be locally Jaffard if R_p is a Jaffard domain for each prime ideal p of R (equiv., $S^{-1}R$ is a Jaffard domain for each multiplicative subset S of R). The class of Jaffard domains contains most of the well-known classes of rings involved in Krull dimension theory such as Noetherian domains, Prüfer domains, universally catenarian domains, and universally strong S-domains. We assume familiarity with these concepts, as in [3, 5, 7, 8, 13, 20, 21, 22, 24].

PID

UFD   Dedekind   UFD + Noetherian

Krull   Prüfer   Noetherian

U. Strong S-Domain

PVMD   Locally Jaffard

Strong S-domain   Jaffard
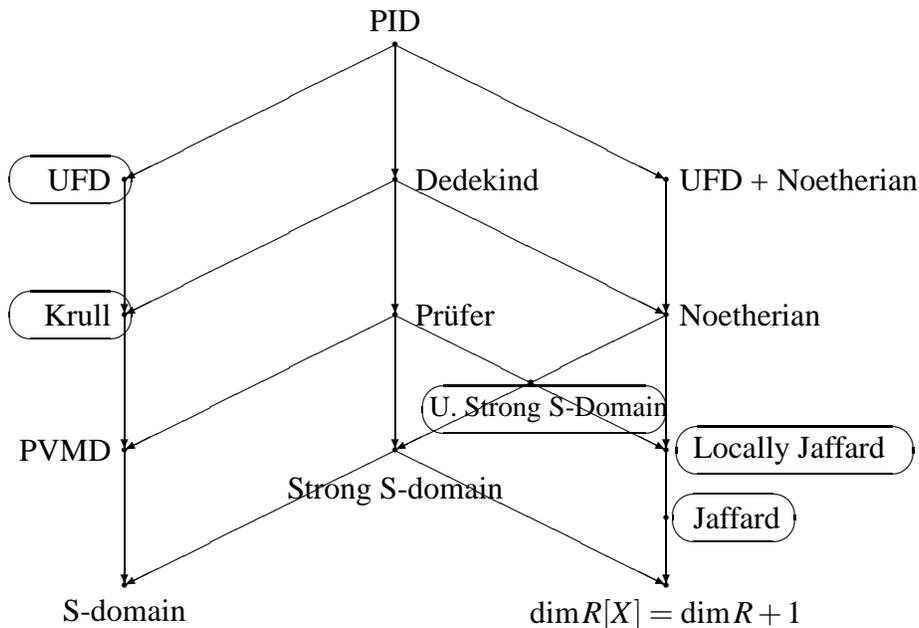
S-domain $\dim R[X] = \dim R + 1$

FIGURE 1. Diagram of Implications

It is an open problem to compute the dimension of polynomial rings over Krull domains in general. In this vein, Bouvier conjectured that "finite-dimensional Krull (or more particularly factorial) domains need not be Jaffard" [8, 15]. In Figure 1, a diagram of implications places this conjecture in its proper perspective and hence shows how it naturally arises. In particular, it indicates how the classes of (finite-dimensional) Noetherian domains, Prüfer domains, UFDs, Krull domains, and PVMDs [17] interact with the notion of Jaffard domain as well as with the (strong) S-domain properties of Kaplansky [22, 23, 24].

This paper scans all known families of examples of non-Noetherian finite dimensional Krull (or factorial) domains existing in the literature. In Section 2, we show that most of these examples are in fact locally Jaffard domains. One of these families which arises from David's second example [12] yields examples of Jaffard domains but it is still open whether these are locally Jaffard. Further, David's example turns out to be the first example of a 3-dimensional factorial domain which is not catenarian (i.e., prior to Fujita's example [16]). Section 3 is devoted to the last known family of examples which stem from the generalized fourteenth problem of Hilbert (also called Zariski-Hilbert problem): Let k be a field of characteristic zero, T a normal affine domain over k, and F a subfield of qf(T). The Hilbert-Zariski problem asks whether $R := F \cap T$ is an affine domain over k. Counterexamples on this problem were constructed by Rees [30], Nagata [27] and

Roberts [31, 32] where R wasn't even Noetherian. In this vein, Anderson, Dobbs, Eakin, and Heinzer [4] asked whether R and its localizations inherit from T the Noetherian-like main behavior of having Krull and valuative dimensions coincide (i.e., Jaffard). This problem will be addressed within the more general context of subalgebras of affine domains over Noetherian domains; namely, let $A \subseteq R$ be an extension of domains where A is Noetherian and R is a subalgebra of an affine domain over A. It turns out that R is Jaffard but it is still elusively open whether R is locally Jaffard.

2. EXAMPLES OF NON-NOETHERIAN KRULL DOMAINS

Obviously, Bouvier's conjecture (mentioned above) makes sense beyond the Noetherian context. As the notion of Krull domain is stable under formation of rings of fractions and adjunction of indeterminates, it merely claims "the existence of a Krull domain R and a multiplicative subset S (possibly equal to $\{1\}$) such that $1 + \dim(S^{-1}R) \lneqq \dim(S^{-1}R[X])$." However, finite-dimensional non-Noetherian Krull domains are scarce in the literature and one needs to test them and their localizations as well for the Jaffard property.

Next, we show that most of these families of examples are subject to the (locally) Jaffard property. This reflects the difficulty of proving or disproving Bouvier's conjecture.

Example 2.1. Nagarajan's example [26] arises as the ring R_0 of invariants of a finite group of automorphisms acting on $R := k[[X,Y]]$, where k is a field of characteristic $p \neq 0$. It turned out that R is integral over R_0. Therefore [24, Theorem 4.6] forces R_0 to be a universally strong S-domain, hence a locally Jaffard domain [3, 23].

Example 2.2. Nagata's example [28, p. 206] and David's example [11] arise as integral closures of Noetherian domains, which are necessarily universally strong S-domains by [24, Corollary 4.21] (hence locally Jaffard).

Example 2.3. Gilmer's example [18] and Brewer-Costa-Lady's example [9] arise as group rings (over a field and a group of finite rank), which are universally strong S-domains by [2] (hence locally Jaffard).

Example 2.4. Fujita's example [16] is a 3-dimensional factorial quasilocal domain (R, M) that arises as a directed union of 3-dimensional Noetherian domains, say $R = \bigcup R_n$. We claim R to be a locally Jaffard domain.

Indeed, the localization with respect to any height-one prime ideal is a DVR (i.e., discrete valuation ring) and hence a Jaffard domain. As, by [13, Theorem 2.3], R is a Jaffard domain, then R_M is locally Jaffard. Now, let P be a prime ideal of R with $\text{ht}(P) = 2$. Clearly, there exists $Q \in \text{Spec}(R)$ such that $(0) \subset Q \subset P \subset M$ is a saturated chain of prime ideals of R. As,

$\mathrm{ht}(M[n]) = \mathrm{ht}(M) = 3$ for each positive integer n, we obtain $\mathrm{ht}(P[n]) = \mathrm{ht}(P) = 2$ for each positive integer n. Then R_P is locally Jaffard, as claimed.

Example 2.5. David's second example [12] is a 3-dimensional factorial domain $J := \bigcup J_n$ which arises as an ascending union of 3-dimensional polynomial rings J_n in three indeterminates over a field k. We claim that J is a Jaffard domain. Moreover, J turns out to be non catenarian. Thus, David's example is the first example of a 3-dimensional factorial domain which is not catenarian (prior to Fujita's example).

Indeed, we have $J_n := k[X, \beta_{n-1}, \beta_n]$ for each positive integer n, where the indeterminates β_n satisfy the following condition: For $n \geq 2$,

(1) $$\beta_n = \frac{-\beta_{n-1}^{s(n)} + \beta_{n-2}}{X}$$

where the $s(n)$ are positive integers. Also, $J_n \subseteq J \subseteq J_n[X^{-1}]$ for each positive integer n. By [13, Theorem 2.3], J is a Jaffard domain, as the J_n are affine domains. Notice, at this point, we weren't able to prove or disprove that J is locally Jaffard.

Next, fix a positive integer n. We have $\dfrac{J_n}{XJ \cap J_n} = k[\overline{\beta_{n-1}}, \overline{\beta_n}]$. On account of (1), we get

(2) $$\overline{\beta_{n-1}} = \overline{\beta_n}^{s(n+1)}.$$

Therefore

$$\frac{J_n}{XJ \cap J_n} = k[\overline{\beta_n}].$$

Iterating the formula in (2), it is clear that for each positive integers $n \leq m$, there exists a positive integer r such that $\overline{\beta_n} = \overline{\beta_m}^r$ with respect to the integral domain $\dfrac{J}{XJ}$. It follows that $\dfrac{J}{XJ}$ is integral over $k[\overline{\beta_n}]$ for each positive integer n. Surely, $\overline{\beta_n}$ is transcendental over k, for each positive integer n, since $(0) \subset XJ \subset M := (X, \beta_0, \beta_1, ..., \beta_n, ...)$ is a chain of distinct prime ideals of J. Then $\dim(\dfrac{J}{XJ}) = 1$ and thus $(0) \subset XJ \subset M := (X, \beta_0, \beta_1, ..., \beta_n, ...)$ is a saturated chain of prime ideals of J. As $\mathrm{ht}(M) = 3$, it follows that J is not catenarian, as desired.

Example 2.6. Anderson-Mulay's example [6] draws from a combination of techniques of Abhyankar [1] and Nagata [28] and arises as a directed union of polynomial rings over a field. Let k be a field, d an integer ≥ 1, and $X, Z, Y_1, ..., Y_d$ $d + 2$ indeterminates over k. Let $\{\beta_i := \sum_{n \geq 0} b_{in} X^n \mid 1 \leq i \leq d\} \subset k[[X]]$ be a set of algebraically independent elements over $k(X)$ (with

$b_{in} \neq 0$ for all i and n). Define $\{U_{in} \mid 1 \leq i \leq d, \ 0 \leq n\}$ by

$$U_{i0} := Y_i$$

$$U_{in} := \frac{Y_i + Z(\sum_{0 \leq k \leq n-1} b_{ik}X^k)}{X^n}, \text{ for } n \geq 1.$$

For any i, n we have

(3) $$U_{in} = XU_{i(n+1)} - b_{in}Z.$$

Let $R_n := k[X, Z, U_{1n}, ..., U_{dn}]$, a polynomial ring in $d+2$ indeterminates (by (3)); and let $R := \bigcup R_n = k[X, Z, \{U_{1n}, ..., U_{dn} \mid n \geq 0\}]$. They proved that R is a $(d+2)$-dimensional non-Noetherian Jaffard and factorial domain. We claim that R is locally Jaffard. For this purpose, we envisage two cases.

Case 1: k is algebraically closed. Let P be a prime ideal of R. We may suppose $\mathrm{ht}(P) \geq 2$ (since R is factorial). Assume $X \notin P$. Clearly, $R_0 \subset R \subset R_0[X^{-1}]$, then $R_P \cong (R[X^{-1}])_{PR[X^{-1}]} = (R_0[X^{-1}])_{PR_0[X^{-1}]}$ is Noetherian (hence Jaffard). Assume $X \in P$. By (3), $\frac{R}{XR} \cong k[Z]$. Then $P = (X, f)$ for some irreducible polynomial f in $k[Z]$. As k is algebraically closed, we get $f = Z - \alpha$ for some $\alpha \in k$. For any positive integer n and $i = 1, ..., d$, define

$$V_{in} := U_{in} + b_{in}\alpha.$$

Observe that, for each n and i, we have

$$R_n = k[X, Z - \alpha, V_{1n}, ..., V_{dn}]$$
$$V_{in} = XU_{i(n+1)} - b_{in}(Z - \alpha).$$

Then $P \cap R_n = (X, Z - \alpha, \{V_{1n}, ..., V_{dn}\})$ is a maximal ideal of R_n for each positive integer n. For each $0 \leq i \leq d$, set

$$P_i := (Z - \alpha, \{V_{rn}\}_{1 \leq r \leq i, \ 0 \leq n})R.$$

Each P_i is a prime ideal of R since $P_i \cap R_n = (Z - \alpha, V_{1n}, \ldots, V_{in})$ is a prime ideal of R_n. This gives rise to the following chain of prime ideals of R

$$0 \subset (Z - \alpha)R = P_0 \subset P_1 \subset ... \subset P_d \subset P.$$

Each inclusion is proper since the P_i's contract to distinct ideals in each R_n. Hence $\mathrm{ht}(P) \geq d+2$, whence $\mathrm{ht}(P) = d+2$ as $\dim(R) = d+2$. Since R is a Jaffard domain, we get $\mathrm{ht}(P[n]) = \mathrm{ht}(P)$ for each positive integer n. Therefore, R is locally Jaffard, as desired.

Case 2: k is an arbitrary field. Let K be an algebraic closure of k. Let $T_n = K[X, Z, U_{1n}, ..., U_{dn}]$ for each positive integer n and let

$$T := \bigcup_{n \geq 0} T_n = K[X, Z, \{U_{1n}, ..., U_{dn} : n \geq 0\}].$$

Let Q be a minimal prime ideal of PT. Then $Q = (X, Z - \beta)$ with $\beta \in K$, as $\dfrac{T}{XT} \cong K[Z]$. By the above case, we have $\text{ht}(Q) = d + 2$. Hence $\text{ht}(PT) = d + 2$. As $T_n \cong K \otimes_k R_n$, we get,

$$T = \bigcup_{n \geq 0} T_n = \bigcup_{n \geq 0} K \otimes_k R_n = K \otimes_k \bigcup_{n \geq 0} R_n = K \otimes_k R.$$

Then T is a free and hence faithfully flat R-module. A well-known property of faithful flatness shows that $PT \cap R = P$. Further, T is an integral and flat extension of R. It follows that $\text{ht}(PT) = \text{ht}(P) = d + 2$, and thus R_P is a Jaffard domain.

Example 2.7. Eakin-Heinzer's 3-dimensional non-Noetherian Krull domain, say R, arises -via [30] and [14, Theorem 2.2]- as the symbolic Rees algebra with respect to a minimal prime ideal P of the 2-dimensional homogeneous coordinate ring A of a nonsingular elliptic cubic defined over the complex numbers. We claim that this construction, too, yields locally Jaffard domains. Indeed, let $K := \text{qf}(A)$, t be an indeterminate over A, and $P^{(n)} := P^n A_P \cap A$, the nth symbolic power of P, for $n \geq 2$. Set $R := A[t^{-1}, Pt, P^{(2)}t^2, ..., P^{(n)}t^n, ...]$, the 3-dimensional symbolic Rees algebra with respect to P. We have

$$A \subset A[t^{-1}] \subset R \subset A[t, t^{-1}] \subset K(t^{-1}).$$

Let Q be a prime ideal of R, $Q' := Q \cap A[t^{-1}]$, and $q := Q \cap A = Q' \cap A$. We envisage three cases.

   Case 1: $\text{ht}(Q) = 1$. Then R_Q is a DVR hence a Jaffard domain.

   Case 2: $\text{ht}(Q) = 3$. Then $3 = \dim(R_Q) \leq \dim_v(R_Q) \leq \dim_v(A[t^{-1}]_{Q'}) = \dim(A[t^{-1}]_{Q'}) \leq \dim(A[t^{-1}] = 1 + \dim(A) = 3$. Hence R_Q is a Jaffard domain.

   Case 3: $\text{ht}(Q) = 2$. If $t^{-1} \notin Q$, then R_Q is a localization of $A[t, t^{-1}]$, hence a Jaffard domain. Next, assume that $t^{-1} \in Q$. If Q is a homogeneous prime ideal, then $Q \subset M := (m[t^{-1}] + t^{-1}A[t^{-1}]) \oplus pt \oplus ... \oplus p^{(n)}t^n \oplus ...$ and $\text{ht}(M) = 3$, where m is the unique maximal ideal of A. As R is a Jaffard domain, we get $\text{ht}(M[X_1, ..., X_n]) = \text{ht}(M) = 3$ for each positive integer n. Hence $\text{ht}(Q[X_1, ..., X_n]) = \text{ht}(Q) = 2$ for each positive integer n, so that R_Q is Jaffard. Now, assume that Q is not homogeneous. As $t^{-1} \in Q$ and $\text{ht}(Q) = 1 + \text{ht}(Q^*)$, where Q^* is the ideal generated by all homogeneous elements of Q, we get $Q^* = t^{-1}R$ which is a height one prime ideal of the Krull domain R. Also, for each positive integer n, note that $Q[X_1, X_2, ..., X_n]^* =$

$Q^*[X_1, ..., X_n]$. Therefore, for each positive integer n, we have

$$
\begin{aligned}
\operatorname{ht}(Q[X_1, ..., X_n]) &= 1 + \operatorname{ht}(Q[X_1, ..., X_n]^*) \\
&= 1 + \operatorname{ht}(Q^*[X_1, ..., X_n]) \\
&= 1 + \operatorname{ht}(t^{-1}R[X_1, ... X_n]) \\
&= 1 + \operatorname{ht}(t^{-1}R) = 2 \\
&= \operatorname{ht}(Q).
\end{aligned}
$$

It follows that R_Q is Jaffard, completing the proof. Notice that Anderson-Dobbs-Eakin-Heinzer's example [4, Example 5.1] is a localization of R (by a height 3 maximal ideal), then locally Jaffard.

Also, Eakin-Heinzer's second example [14] is a universally strong S-domain; in fact, it belongs to the same family as Example 2.1. Another family of non-Noetherian finite-dimensional Krull domains stems from the generalized fourteenth problem of Hilbert (also called Zariski-Hilbert problem). This is the object of our investigation in the following section.

3. KRULL DOMAINS ISSUED FROM THE HILBERT-ZARISKI PROBLEM

Let k be a field of characteristic zero and let T be a normal affine domain over k. Let F be a subfield of the field of fractions of T. Set $R := F \cap T$. The Hilbert-Zariski problem asks whether R is an affine domain over k. Counterexamples on this problem were constructed by Rees [30], Nagata [27] and Roberts [31, 32], where it is shown that R does not inherit the Noetherian property from T in general. In this vein, Anderson, Dobbs, Eakin, and Heinzer [4] asked whether R inherits from T the Noetherian-like main behavior of being locally Jaffard. We investigate this problem within a more general context; namely, extensions of domains $A \subseteq R$, where A is Noetherian and R is a subalgebra of an affine domain over A.

The next result characterizes the subalgebras of affine domains over a Noetherian domain. It allows one to reduce the study of the prime ideal structure of these constructions to those domains R between a Noetherian domain B and its localization $B[b^{-1}]$ ($0 \neq b \in B$).

Proposition 3.1. *Let $A \subseteq R$ be an extension of domains where A is Noetherian. Then the following statements are equivalent:*

(1) *R is a subalgebra of an affine domain over A;*
(2) *There is $r \neq 0 \in R$ such that $R[r^{-1}]$ is an affine domain over A;*
(3) *There is an affine domain B over A and $b \neq 0 \in B$ such that $B \subseteq R \subseteq B[b^{-1}]$.*

Proof. (1) $\Rightarrow$ (2) This is [19, Proposition 2.1(b)].

$(2) \Rightarrow (3)$ Let $r \neq 0 \in R$ and $x_1, ..., x_n \in R[r^{-1}]$ such that $R[r^{-1}] = A[x_1, ..., x_n]$. For each $i = 1, ..., n$, write $x_i = \sum_{j=0}^{n_i} r_{ij} r^{-j}$ with $r_{ij} \in R$ and $n_i \in \mathbb{N}$. Let $B := A[\{r_{ij} : i = 1, ..., n \text{ and } j = 0, ..., n_i\}]$ and let $b := r$. Clearly, B is an affine domain over A such that $B \subseteq R \subseteq B[b^{-1}]$.

The implication $(3) \Rightarrow (1)$ is trivial, completing the proof of the proposition. $\square$

Corollary 3.2. *Let $A \subseteq R$ be an extension of domains where A is Noetherian and R is a subalgebra of an affine domain over A. Then there exists an affine domain T over A such that $R \subseteq T$ and R_p is Noetherian (hence Jaffard) for each prime ideal p of R that survives in T.*

Proof. By Proposition 3.1, there exists an affine domain B over A and a nonzero element b of B such that $B \subseteq R \subseteq B[b^{-1}]$. Put $T = B[b^{-1}]$. Let p be a prime ideal of R that survives in T (i.e., $b \notin p$). Then it is easy to see that

$$R_p \cong R[b^{-1}]_{pR[b^{-1}]} = B[b^{-1}]_{pB[b^{-1}]} = T_{pT}$$

is a Noetherian domain, as desired. $\square$

Corollary 3.3. *Let R be a subalgebra of an affine domain T over a field k. Then:*

(1) $\dim(R) = \mathrm{t.\,d.}(R)$ *and R is a Jaffard domain.*
(2) $\dim(R) = \mathrm{ht}(P \cap R) + \mathrm{t.\,d.}(\frac{R}{P \cap R})$ *for each prime ideal P of T. In particular, $\dim(R) = \mathrm{ht}(M)$ for each maximal ideal M of R that survives in T.*

Proof. (1) This is [10, Proposition 5.1] which is a consequence of a more general result on valuative radicals [10, Théorème 4.4]. Also the statement "$\dim(R) = \mathrm{t.\,d.}(R)$" is [29, Corollary 1.2]. We offer here an alternate proof: By Proposition 3.1, there exists an affine domain B over k and a nonzero element b of B such that $B \subseteq R \subseteq B[b^{-1}]$. By [28, Corollary 14.6], $\dim(B[b^{-1}]) = \dim_v(B[b^{-1}]) = \dim_v(B) = \dim(B) = \mathrm{t.\,d.}(B) = \mathrm{t.\,d.}(R)$. Further, observe that $B[b^{-1}] = R[b^{-1}]$ is a localization of R. Hence $\dim(B[b^{-1}]) = \dim(R[b^{-1}]) \leq \dim(R) \leq \dim_v(R) \leq \dim_v(B)$. Consequently, $\dim(R) = \dim_v(R) = \mathrm{t.\,d.}(R)$, as desired.

(2) Let P be a prime ideal of T with $p := P \cap R$. By [10, Théorème 1.2], the extension $R \subseteq T$ satisfies the altitude inequality formula. Hence

$$ht(P) + \mathrm{t.d.}(\frac{T}{P} : \frac{R}{p}) \leq ht(p) + \mathrm{t.d.}(T : R).$$

By [28, Corollary 14.6], we obtain

$$\mathrm{t.d.}(T : k) - \mathrm{t.d.}(\frac{R}{p} : k) \leq ht(p) + \mathrm{t.d.}(T : k) - \mathrm{t.d.}(R : k).$$

Then t.d.$(R) \leq ht(p)+$t.d.$(\frac{R}{p}:k)$. Moreover, it is well known that

$$ht(p)+\text{t.d.}(\frac{R}{p}:k) \leq \text{t.d.}(R) \text{ [33, p. 10]}.$$

Applying (1), we get

$$\dim(R) = \text{t.d.}(R:k) = ht(p)+\text{t.d.}(\frac{R}{p}:k).$$

Finally, notice that if $M \in \text{Spec}(R)$ with $MT \neq T$, then there exists $M' \in \text{Spec}(T)$ contracting to M, so that

$$\text{t.d.}(\frac{R}{M}) \leq \text{t.d.}(\frac{T}{M'}) = 0 \text{ [28, Corollary 14.6],}$$

completing the proof. $\square$

The above corollaries shed some light on the dimension and prime ideal structure of the non-Noetherian Krull domains emanating from the Hilbert-Zariski problem. In particular, these are necessarily Jaffard. But we are unable to prove or disprove if they are locally Jaffard. An in-depth study is to be carried out on (some contexts of) subalgebras of affine domains over Noetherian domains in line with Rees, Nagata, and Roberts constructions.

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