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The dimension of tensor products of k-algebras arising from pullbacks

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Abstract

The purpose of this paper is to compute the Krull dimension of tensor products of k-algebras arising from pullbacks. We also state a formula for the valuative dimension. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

All rings and algebras considered in this paper are commutative with identity elements and, unless otherwise specified, are to be assumed to be non-trivial. All ring homomorphisms are unital. Let k be a field. We denote the class of commutative k-algebras with finite transcendence degree over k by C. Also, we shall use t.d.(A) to denote the transcendence degree of a k-algebra A over k, $A[n]$ to denote the polynomial ring $A[X_1,\dots,X_n]$, and $p[n]$ to denote the prime ideal $p[X_1,\dots,X_n]$ of $A[n]$, where p is a prime ideal of A. Recall that an integral domain R of finite (Krull) dimension n is a Jaffard domain if its valuative dimension, $\dim_v(R)$, is also n. Prüfer domains and noetherian domains are Jaffard domains. We assume familiarity with this concept, as in [1, 6, 10]. Suitable background on pullbacks is [4, 11, 12, 16]. Any unreferenced material is standard, as in [12, 17].

In [20] Sharp proved that if K and L are two extension fields of k, then $\dim(K \otimes_k L) = \min(\text{t.d.}(K), \text{t.d.}(L))$. This result provided a natural starting point to investigate

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dimensions of tensor products of somewhat general k-algebras. This was concretized by Wadsworth in [21], where the result of Sharp was extended to AF-domains, that is, integral domains A such that $\mathrm{ht}(p)+\mathrm{t.d.}(A/p)=\mathrm{t.d.}(A)$, for all prime ideals p of A. He showed that if A_1 and A_2 are AF-domains, then $\dim(A_1 \otimes_k A_2) = \min(\dim(A_1) + \mathrm{t.d.}(A_2), \dim(A_2) + \mathrm{t.d.}(A_1))$. He also stated a formula for $\dim(A \otimes_k R)$ which holds for an AF-domain A, with no restriction on R. We recall, at this point, that an AF-domain is a (locally) Jaffard domain [13].

In [5] we were concerned with AF-rings. A k-algebra A is said to be an AF-ring provided $\mathrm{ht}(p) + \mathrm{t.d.}(A/p) = \mathrm{t.d.}(A_p)$, for all prime ideals p of A (for nondomains, $\mathrm{t.d.}(A)=\sup\{\mathrm{t.d.}(A/p)/p$ prime ideal of A$\}$). A tensor product of AF-domains is perhaps the most natural example of an AF-ring. We then developed quite general results for AF-rings, showing that the results do not extend trivially from integral domains to rings with zero divisors.

Our aim in this paper is to extend Wadsworth's results in a different way, namely to tensor products of k-algebras arising from pullbacks. In order to do this, we use previous deep investigations of the prime ideal structure of various pullbacks, as in [1–4, 6, 8–10, 16]. Moreover, in [14] dimension formulas for the tensor product of two particular pullbacks are established and a conjecture on the dimension formulas for more general pullbacks is raised; in the present paper such conjecture is resolved.

Before presenting our main result of Section 1, Theorem 1.9, it is convenient to recall from [21] some notation. Let $A \in C$ and let d, s be integers with $0 \le d \le s$. Put $D(s,d,A)=\max\{\ \mathrm{ht}p[s]+\min(s,d+\mathrm{t.d.}(A/p))/p$ prime ideal of $A\}$. Our main result is the following: given $R_1=\varphi^{-1}(D_1)$ and $R_2=\varphi^{-1}(D_2)$ two pullbacks issued from T_1 and T_2, respectively. Assume that D_i, T_i are AF-domains and $\mathrm{ht}(M_i)=\dim(T_i)$, for $i=1,2$. Then

$$\dim(R_1 \otimes_k R_2) = \max\{\mathrm{ht}\, M_1[\mathrm{t.d.}(R_2)] + D(\mathrm{t.d.}(D_1), \dim(D_1), R_2),$$

$$\mathrm{ht}\, M_2[\mathrm{t.d.}(R_1)] + D(\mathrm{t.d.}(D_2), \dim(D_2), R_1)\}.$$

It turns out ultimately from this theorem and via a result of Girolami [13] that one may compute (Krull) dimensions of tensor products of two k-algebras for a large class of (not necessarily AF-domains) k-algebras. The purpose of Section 2 is to prove the following theorem: with the above notation,

$$\dim_v(R_1 \otimes_k R_2)=\min\{\dim_v R_1 + \mathrm{t.d.}(R_2), \dim_v R_2 + \mathrm{t.d.}(R_1)\}.$$

In Section 3 Theorem 3.1 asserts that, with mild restrictions, tensor products of pullbacks preserve Jaffard rings. Theorem 3.2 states, under weak assumptions, a formula similar to that of Theorem 1.9. It establishes a satisfactory analogue of [4, Theorem 5.4] (also [1, Proposition 2.7, 9, Corollary 1]) for tensor products of pullbacks issued from AF-domains. We finally focus on the special case in which $R_1 = R_2$. Some examples illustrate the limits of our results and the failure of Wadsworth's results for non AF-domains.

1. The Krull dimension

The discussion which follows, concerning basic facts (and notations) connected with the prime ideal structure of pullbacks and tensor products of k-algebras, will provide some background to the main theorem of this section and will be of use in its proof. Notice first that we will be concerned with pullbacks (of commutative k-algebras) of the following type:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & K \end{array}$$

where T is an integral domain with maximal ideal M, $K=T/M$, φ is the canonical surjection from T onto K, D is a proper subring of K and $R=\varphi^{-1}(D)$. Clearly, $M=(R:T)$ and $D\cong R/M$. Let p be a prime ideal of R. If $M\not\subseteq p$, then there is a unique prime ideal q in T such that $q\cap R=p$ and $T_q=R_p$. However, if $M\subseteq p$, there is a unique prime ideal q in D such that $p=\varphi^{-1}(q)$ and the following diagram of canonical homomorphisms

$$\begin{array}{ccc} R_p & \longrightarrow & D_q \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & K \end{array}$$

is a pullback. Moreover, $\operatorname{ht} p=\operatorname{ht} M+\operatorname{ht} q$ (see [11] for additional evidence). We recall from [8, 1] two well-known results describing how dimension and valuative dimension behave under pullback: with the above notation, $\dim R=\max\{\dim T, \dim D+\dim T_M\}$, and $\dim_v R=\max\{\dim_v T, \dim_v D+\dim_v T_M+\text{t.d.}(K:D)\}$. However, while $\dim R[n]$ seems not to be effectively computable in general, questions of effective upper and lower bounds for $\dim R[n]$ were partially answered. The following lower bound will be useful in the sequel: $\dim R[n]\geq \dim D[n]+\dim T_M+\min(n, \text{t.d.}(K:D))$, where the equality holds if T is supposed to be a locally Jaffard domain with $\operatorname{ht} M=\dim T$ (see [9]). At last, it is a key result [13] that R is an AF-domain if and only if so are T and D and $\text{t.d.}(K:D)=0$. A combination of this result and Theorem 1.9 allows one to compute dimensions of tensor products of two k-algebras for a large class of (not necessarily AF-domains) k-algebras.

We turn now to tensor products. Let us recall from [21] the following functions: let A, A_1 and $A_2\in C$. Let $p\in\operatorname{Spec}(A)$, $p_1\in\operatorname{Spec}(A_1)$ and $p_2\in\operatorname{Spec}(A_2)$. Let d,s be integers with $0\leq d\leq s$. Set

- $S_{p_1,p_2}=\{P\in\operatorname{Spec}(A_1\otimes_k A_2)/p_1=P\cap A_1 \text{ and } p_2=P\cap A_2\}$.
- $\delta(p_1,p_2)=\max\{\operatorname{ht} P/P\in S_{p_1,p_2}\}$.
- $\Delta(s,d,p)=\operatorname{ht} p[s]+\min(s, d+\text{t.d.}(A/p))$.
- $\mathrm{D}(s,d,A)=\max\{\Delta(s,d,p)/p\in\operatorname{Spec}(A)\}$.

One can easily check that $\dim(A_1\otimes_k A_2)=\max\{\delta(p_1,p_2)/p_1\in\operatorname{Spec}(A_1) \text{ and } p_2\in\operatorname{Spec}(A_2)\}$ (see [21, p. 394]). Let $P\in\operatorname{Spec}(A_1\otimes_k A_2)$ with $p_1\subseteq P\cap A_1$ and $p_2\subseteq P\cap A_2$.

It is known [21] that P is minimal in S_{p_1,p_2} if and only if it is a minimal prime divisor of $p_1 \otimes A_2 + A_1 \otimes p_2$. This result will be used to prove a special chain lemma for tensor products of k-algebras, which establishes a somewhat analogue of the Jaffard's special chain theorem for polynomial rings (see [7, 15]).

These facts will be used frequently in the sequel without explicit mention.

The proof of our main theorem requires some preliminaries. The following two lemmas deal with properties of polynomial rings over pullbacks, which are probably well known, but we have not located references in the literature.

Lemma 1.1. *Let T be an integral domain with maximal ideal M, $K = T/M$, φ the canonical surjection from T onto K, D a proper subring of K and $R = \varphi^{-1}(D)$. Then* $\operatorname{ht} p[n] = \operatorname{ht}(p[n]/M[n]) + \operatorname{ht} M[n]$, *for each positive integer n and each prime ideal p of R such that $M \subseteq p$.*

Proof. Since $M \subseteq p$, there is a unique $q \in \operatorname{Spec}(D)$ such that $p = \varphi^{-1}(q)$ and the following diagram is a pullback

$$\begin{array}{ccc} R_p & \longrightarrow & D_q \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & K \end{array}$$

By [1, Lemma 2.1(c)] $MT_M = MR_p$ is a divided prime ideal of R_p. By [1, Lemma 2.2] $\operatorname{ht} p[n] = \operatorname{ht} pR_p[n] = \operatorname{ht}(pR_p[n]/MR_p[n]) + \operatorname{ht} MR_p[n] = \operatorname{ht}(p[n]/M[n]) + \operatorname{ht} M[n]$. □

Lemma 1.2. *Let T be an integral domain with maximal ideal M, $K = T/M$, φ the canonical surjection from T onto K, D a proper subring of K and $R = \varphi^{-1}(D)$. Assume T_M and D are locally Jaffard domains. Then* $\operatorname{ht} p[n] = \operatorname{ht} p + \min(n, \text{t.d.}(K:D))$, *for each positive integer n and each prime ideal p of R such that $M \subseteq p$.*

Proof. Since $M \subseteq p$, there is a unique $q \in \operatorname{Spec}(D)$ such that $p = \varphi^{-1}(q)$ and the following diagram is a pullback

$$\begin{array}{ccc} R_p & \longrightarrow & D_q \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & K \end{array}$$

By [3, Corollary 2.10] $\operatorname{ht} p[n] = \dim(R_p[n]) - n$. Furthermore,

$$\begin{aligned} \dim(R_p[n]) &= \operatorname{ht} M + \dim(D_q[n]) + \min(n, \text{t.d.}(K:D)) \\ &= \operatorname{ht} M + \dim D_q + n + \min(n, \text{t.d.}(K:D)) \\ &= \operatorname{ht} p + n + \min(n, \text{t.d.}(K:D)), \end{aligned}$$

completing the proof. □

The following corollary is an immediate consequence of (1.2) and will be useful in the proof of the theorem.

Corollary 1.3. *Let T be an integral domain with maximal ideal M, $K=T/M$, φ the canonical surjection from T onto K, D a proper subring of K and $R=\varphi^{-1}(D)$. Assume T_M is a locally Jaffard domain. Then* $\operatorname{ht} M[n]=\operatorname{ht} M+\min(n,\text{t.d.}(K:D))$, *for each positive integer n.*

We next analyse the heights of ideals of $A_1\otimes_k A_2$ of the form $p_1\otimes_k A_2$, where $p_1\in\operatorname{Spec}(A_1)$ and A_2 is an integral domain.

Lemma 1.4. *Let $A_1,A_2\in C$ and p_1 be a prime ideal of A_1. Assume A_2 is an integral domain. Then* $\operatorname{ht}(p_1\otimes_k A_2)=\operatorname{ht} p_1[\text{t.d.}(A_2)]$.

Proof. Put $t_2=\text{t.d.}(A_2)$. Let Q be a minimal prime divisor of $p_1\otimes A_2$ in $A_1\otimes A_2$. Then Q is minimal in $S_{p_1,(0)}$, and hence $\text{t.d.}((A_1\otimes A_2)/Q)=\text{t.d.}(A_1/p_1)+t_2$ by [21, Proposition 2.3]. Furthermore, Q survives in $A_1\otimes F_2$, where F_2 is the quotient field of A_2, whence $\operatorname{ht} Q+\text{t.d.}((A_1\otimes A_2)/Q)=t_2+\operatorname{ht} p_1[t_2]+\text{t.d.}(A_1/p_1)$ by [21, Remark 1.b], completing the proof. □

With the further assumption that A_2 is an AF-domain, we obtain the following.

Lemma 1.5 (Special chain lemma). *Let $A_1,A_2\in C$ and p_1 be a prime ideal of A_1. Assume A_2 is an AF-domain. Let $P\in\operatorname{Spec}(A_1\otimes_k A_2)$ such that $p_1=P\cap A_1$. Then* $\operatorname{ht} P=\operatorname{ht}(p_1\otimes_k A_2)+\operatorname{ht}(P/(p_1\otimes_k A_2))$.

Proof. Since A_2 is an AF-domain, by [21, Remark 1.b] $\operatorname{ht} P+\text{t.d.}((A_1\otimes A_2)/P)=t_2+\operatorname{ht} p_1[t_2]+\text{t.d.}(A_1/p_1)$, where $t_2=\text{t.d.}(A_2)$. A similar argument with $(A_1/p_1)\otimes_k A_2$ in place of $A_1\otimes_k A_2$ shows that $\operatorname{ht}(P/(p_1\otimes_k A_2))+\text{t.d.}((A_1\otimes A_2)/P)=t_2+\text{t.d.}(A_1/p_1)$, whence $\operatorname{ht} P=\operatorname{ht} p_1[t_2]+\operatorname{ht}(P/(p_1\otimes_k A_2))$. The proof is complete via Lemma 1.4. □

An important case of Lemma 1.5 occurs when $A_2=k[X_1,\dots,X_n]$ and hence if P is a prime ideal of $A_1\otimes A_2\cong A_1[X_1,\dots,X_n]$ with $p=P\cap A_1$, then $\operatorname{ht} P=\operatorname{ht} p[n]+\operatorname{ht} P/p[n]$. Our special chain lemma may be then viewed as an analogue of the Jaffard's special chain theorem (see [7, 15]). Notice for convenience that Jaffard's theorem holds for any (commutative) ring, while here we are concerned with k-algebras.

To avoid unnecessary repetition, let us fix notation for the rest of this section and also for much of Sections 2 and 3. Data will consist of two pullbacks of k-algebras

$$\begin{array}{ccc} R_1 & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ T_1 & \longrightarrow & K_1 \end{array} \qquad \begin{array}{ccc} R_2 & \longrightarrow & D_2 \\ \downarrow & & \downarrow \\ T_2 & \longrightarrow & K_2 \end{array}$$

where, for $i=1,2$, T_i is an integral domain with maximal ideal M_i, $K_i=T_i/M_i$, φ_i is the canonical surjection from T_i onto K_i, D_i is a proper subring of K_i and $R_i=\varphi_i^{-1}(D_i)$. Let $d_i=\dim T_i$, $d_i'=\dim D_i$, $t_i=\text{t.d.}(T_i)$, $r_i=\text{t.d.}(K_i)$ and $s_i=\text{t.d.}(D_i)$.

The next result deals with the function $\delta(p_1, p_2)$ according to inclusion relations between p_i and M_i $(i=1,2)$.

Lemma 1.6. *Assume T_1 and T_2 are AF-domains. If $p_1 \in \mathrm{Spec}(R_1)$ and $p_2 \in \mathrm{Spec}(R_2)$ are such that $M_1 \not\subseteq p_1$ and $M_2 \not\subseteq p_2$, then*

$$\delta(p_1, p_2) = \min(\mathrm{ht}\, p_1 + t_2, t_1 + \mathrm{ht}\, p_2) \leq \min(d_1 + t_2, t_1 + d_2).$$

Proof. By [1, Lemma 2.1(e)], for $i=1,2$, there exists $q_i \in \mathrm{Spec}(T_i)$ such that $p_i = q_i \cap R_i$ and $T_{iq_i} = R_{ip_i}$. So that R_{1p_1} and R_{2p_2} are AF-domains, whence $\delta(p_1, p_2) = \min(\mathrm{ht}\, p_1 + t_2,\ t_1 + \mathrm{ht}\, p_2)$ by [21, Theorem 3.7]. Further, $\mathrm{ht}\, p_1 \leq d_1$ and $\mathrm{ht}\, p_2 \leq d_2$, completing the proof. □

Lemma 1.7. *Assume T_1 and T_2 are AF-domains. Let $P \in \mathrm{Spec}(R_1 \otimes_k R_2)$, $p_1 = P \cap R_1$ and $p_2 = P \cap R_2$. If $M_1 \subseteq p_1$ and $M_2 \not\subseteq p_2$, then $\mathrm{ht}\, P = \mathrm{ht}\, M_1[t_2] + \mathrm{ht}(P/(M_1 \otimes R_2))$.*

Proof. Since $M_2 \not\subseteq p_2$, R_{2p_2} is an AF-domain. By Lemma 1.5 $\mathrm{ht}\, P = \mathrm{ht}\, p_1[t_2] + \mathrm{ht}(P/(p_1 \otimes R_2))$. Since $M_1 \subseteq p_1$, $\mathrm{ht}\, p_1[t_2] = \mathrm{ht}(p_1[t_2]/M_1[t_2]) + \mathrm{ht}\, M_1[t_2]$ by Lemma 1.1. Hence,

$$\begin{aligned}
\mathrm{ht}\, P &= \mathrm{ht}(p_1[t_2]/M_1[t_2]) + \mathrm{ht} M_1[t_2] + \mathrm{ht}(P/(p_1 \otimes R_2)) \\
&= \mathrm{ht}((p_1 \otimes R_2)/(M_1 \otimes R_2)) + \mathrm{ht} M_1[t_2] + \mathrm{ht}(P/(p_1 \otimes R_2)) \\
&\leq \mathrm{ht}\, M_1[t_2] + \mathrm{ht}(P/(M_1 \otimes R_2)) \\
&= \mathrm{ht}\,(M_1 \otimes R_2) + \mathrm{ht}(P/(M_1 \otimes R_2)) \\
&\leq \mathrm{ht}\, P. \quad \square
\end{aligned}$$

A similar argument with the roles of p_1 and p_2 reversed shows that if $M_1 \not\subseteq p_1$ and $M_2 \subseteq p_2$, then $\mathrm{ht}\, P = \mathrm{ht}\, M_2[t_1] + \mathrm{ht}(P/(R_1 \otimes M_2))$.

Now, we state our last preparatory result, by giving a formula for $\dim((R_1/M_1) \otimes (R_2/M_2))$ and useful lower bounds for $\dim((R_1/M_1) \otimes R_2)$ and $\dim(R_1 \otimes (R_2/M_2))$.

Lemma 1.8. *Assume T_1, T_2, D_1 and D_2 are AF-domains with $\dim T_1 = \mathrm{ht}\, M_1$ and $\dim T_2 = \mathrm{ht}\, M_2$. Then*

(a) $\dim((R_1/M_1) \otimes R_2) \geq d_2 + \min(s_1, r_2 - s_2) + \min(s_1 + d'_2, d'_1 + s_2)$.

(b) $\dim(R_1 \otimes (R_2/M_2)) \geq d_1 + \min(s_2, r_1 - s_1) + \min(s_1 + d'_2, d'_1 + s_2)$.

(c) $\dim((R_1/M_1) \otimes (R_2/M_2)) = \min(s_1 + d'_2, d'_1 + s_2)$.

Proof. (a) Since $R_1/M_1 \cong D_1$ is an AF-domain, by [21, Theorem 3.7]

$$\dim((R_1/M_1) \otimes R_2) = D(s_1, d'_1, R_2) = \max\{\Delta(s_1, d'_1, p_2)/p_2 \in \mathrm{Spec}(R_2)\}.$$

Let $p_2 \in \mathrm{Spec}(R_2)$ such that $M_2 \subseteq p_2$. Then there is a unique $q_2 \in \mathrm{Spec}(D_2)$ such that $p_2 = \varphi_2^{-1}(q_2)$ and the following diagram is a pullback

$$\begin{array}{ccc} R_{2p_2} & \longrightarrow & D_{2q_2} \\ \downarrow & & \downarrow \\ T_{2M_2} & \longrightarrow & K_2 \end{array}$$

By Lemma 1.2 $\mathrm{ht}\, p_2[s_1] = \mathrm{ht}\, p_2 + \min(s_1, r_2 - s_2)$. Since R_2/p_2 and D_2/q_2 are isomorphic k-algebras, $\mathrm{t.d.}(R_2/p_2) = \mathrm{t.d.}(D_2/q_2) = s_2 - \mathrm{ht}\, p_2 + \mathrm{ht}\, M_2$, so that

$$\begin{aligned} \Delta(s_1, d_1', p_2) &= \mathrm{ht}\, p_2[s_1] + \min(s_1, d_1', \mathrm{t.d.}(R_2/p_2)) \\ &= \mathrm{ht}\, p_2 + \min(s_1, r_2 - s_2) + \min(s_1, d_1' + s_2 - \mathrm{ht}\, p_2 + \mathrm{ht}\, M_2) \\ &= \min(s_1, r_2 - s_2) + \min(s_1 + \mathrm{ht}\, p_2, d_1' + s_2 + \mathrm{ht}\, M_2) \\ &= \mathrm{ht}\, M_2 + \min(s_1, r_2 - s_2) + \min(s_1 + \mathrm{ht}\, q_2, d_1' + s_2) \\ &= d_2 + \min(s_1, r_2 - s_2) + \mathit{min}(s_1 + \mathrm{ht}\, q_2, d_1' + s_2). \end{aligned}$$

(b) As in (a) with the roles of R_1 and R_2 reversed.

(c) It is immediate from [21, Theorem 3.7]. □

The facts stated above provide motivation for setting:

$$\begin{aligned} \alpha_1 &= d_1 + \min(t_2, r_1 - s_1) + d_2 + \min(s_1, r_2 - s_2) + \min(s_1 + d_2', d_1' + s_2), \\ \alpha_2 &= d_2 + \min(t_1, r_2 - s_2) + d_1 + \min(s_2, r_1 - s_1) + \mathit{min}(s_1 + d_2', d_1' + s_2), \\ \alpha_3 &= d_1 + d_2 + \min(r_1, r_2) + \min(s_1 + d_2', d_1' + s_2). \end{aligned}$$

We shall use these numbers in the proof of the next theorem and in Section 3.

We now are able to state our main result of this section.

Theorem 1.9. *Assume T_1, T_2, D_1 and D_2 are AF-domains with* $\dim T_1 = \mathrm{ht}\, M_1$ *and* $\dim T_2 = \mathrm{ht}\, M_2$. *Then*

$$\begin{aligned} \dim(R_1 \otimes_k R_2) = \max\{&\mathrm{ht}\, M_1[\mathrm{t.d.}(R_2)] + D(\mathrm{t.d.}(D_1), \dim(D_1), R_2), \\ &\mathrm{ht}\, M_2[\mathrm{t.d.}(R_1)] + D(\mathrm{t.d.}(D_2), \dim(D_2), R_1)\}. \end{aligned}$$

Proof. Since $\dim(R_1 \otimes R_2) \geq \mathrm{ht}(M_1 \otimes R_2) + \dim((R_1/M_1) \otimes R_2)$, we have $\dim(R_1 \otimes R_2) \geq \mathrm{ht}\, M_1[t_2] + \dim((R_1/M_1) \otimes R_2)$ by Lemma 1.4. Similarly, $\dim(R_1 \otimes R_2) \geq \mathrm{ht}\, M_2[t_1] + \dim(R_1 \otimes (R_2/M_1))$. Therefore, it suffices to show that $\dim(R_1 \otimes R_2) \leq \max\{\mathrm{ht}\, M_1[t_2] + \dim((R_1/M_1) \otimes R_2), \mathrm{ht}\, M_2[t_1] + \dim(R_1 \otimes (R_2/M_2))\}$.

It is well known that $\dim(R_1 \otimes R_2) = \max\{\delta(p_1, p_2) \mid p_1 \in \mathrm{Spec}(R_1), p_2 \in \mathrm{Spec}(R_2)\}$. Let $p_1 \in \mathrm{Spec}(R_1)$ and $p_2 \in \mathrm{Spec}(R_2)$. There are four cases:

1. If $M_1 \not\subseteq p_1$ and $M_2 \not\subseteq p_2$, by Lemma 1.6 $\delta(p_1, p_2) = \min(\mathrm{ht} p_1 + t_2, t_1 + \mathrm{ht} p_2) \leq \alpha_3$.
2. If $M_1 \subseteq p_1$ and $M_2 \not\subseteq p_2$, by Lemma 1.7 $\delta(p_1, p_2) \leq \mathrm{ht}\, M_1[t_2] + \dim((R_1/M_1) \otimes R_2)$.
3. If $M_1 \not\subseteq p_1$ and $M_2 \subseteq p_2$, by Lemma 1.7 $\delta(p_1, p_2) \leq \mathrm{ht}\, M_2[t_1] + \dim(R_1 \otimes (R_2/M_2))$.
4. If $M_1 \subseteq p_1$ and $M_2 \subseteq p_2$, then $\delta(p_1, p_2) \leq \max\{\mathrm{ht}\, M_1[t_2] + \dim((R_1/M_1) \otimes R_2), \mathrm{ht}\, M_2[t_1] + \dim(R_1 \otimes (R_2/M_2)), \alpha_3\}$. Indeed, put $h = \delta(p_1, p_2)$. Pick a chain $P_0 \subset$

$P_1 \subset \cdots \subset P_h$ of $h+1$ distinct prime ideals in $R_1 \otimes R_2$ with $P_h \in S_{p_1,p_2}$. If $M_1 \subset P_0 \cap R_1$ and $M_2 \subset P_0 \cap R_2$, then $h = \operatorname{ht} P_h/P_0 \le \dim((R_1/M_1) \otimes (R_2/M_2)) \le \alpha_3$. Otherwise, let i be the largest integer such that $M_1 \not\subset P_i \cap R_1$ and let j be the largest integer such that $M_2 \ne P_j \cap R_2$. If $i \ne j$, say $i < j$, by Lemma 1.7 $\operatorname{ht} P_j = \operatorname{ht} M_1[t_2] + \operatorname{ht}(P_j/(M_1 \otimes R_2))$, whence $h \le \operatorname{ht} M_1[t_2] + \operatorname{ht}(P_h/(M_1 \otimes R_2)) \le \operatorname{ht} M_1[t_2] + \dim((R_1/M_1) \otimes R_2)$. If $i = j$, since $M_1 \subseteq p_1$, there is a unique $q_1 \in \operatorname{Spec}(D_1)$ such that $p_1 = \varphi_1^{-1}(q_1)$ and the following diagramm is a pullback:

$$\begin{array}{ccc} R_{1p_1} & \longrightarrow & D_{1q_1} \\ \downarrow & & \downarrow \\ T_{1M_1} & \longrightarrow & K_1 \end{array}$$

Since $M_1 \not\subset P_i \cap R_1$, it follows that $(P_i \cap R_1)R_{1p_1} \subset M_1 T_{1M_1} = (R_{1p_1} : T_{1M_1})$ by [1, Lemma 2.1(c)], whence $\operatorname{ht}(P_i \cap R_1) \le \operatorname{ht} M_1 - 1 = d_1 - 1$. Similarly, $\operatorname{ht}(P_i \cap R_2) \le \operatorname{ht} M_2 - 1 = d_2 - 1$. Finally, we get via Lemma 1.6

$$\begin{aligned} h &= \operatorname{ht} P_i + 1 + \operatorname{ht}(P_h/P_{i+1}) \\ &\le \delta(P_i \cap R_1, P_i \cap R_2) + 1 + \dim((R_1/M_1) \otimes (R_2/M_2) \\ &= \min(\operatorname{ht}(P_i \cap R_1) + t_2, t_1 + \operatorname{ht}(P_i \cap R_2)) + 1 + \dim((R_1/M_1) \otimes (R_2/M_2) \\ &\le \min(d_1 - 1 + t_2, t_1 + d_2 - 1) + 1 + \dim((R_1/M_1) \otimes (R_2/M_2)) \\ &= \alpha_3. \text{ The fourth case is done.} \end{aligned}$$

Now, let us assume $s_1 \le r_2 - s_2$. Then

$$\begin{aligned} \alpha_1 &= d_1 + \min(t_2, r_1 - s_1) + d_2 + s_1 + \min(s_1 + d_2', d_1' + s_2) \\ &= d_1 + \min(t_2 + s_1, r_1) + d_2 + \min(s_1 + d_2', d_1' + s_2) \\ &\ge d_1 + d_2 + \min(r_1, r_2) + \min(s_1 + d_2', d_1' + s_2) = \alpha_3. \end{aligned}$$

If $s_2 \le r_1 - s_1$, in a similar manner we obtain $\alpha_2 \ge \alpha_3$. Finally, assume $r_1 - s_1 < s_2$ and $r_2 - s_2 < s_1$, so that

$$\begin{aligned} \alpha_1 &= \alpha_2 \\ &= t_1 - s_1 + t_2 - s_2 + \min(s_1 + d_2', d_1' + s_2) \\ &= \min(t_1 + t_2 - s_2 + d_2', t_1 + t_2 - s_1 + d_1') \\ &= \min(\dim_v R_1 + t_2, t_1 + \dim_v R_2). \end{aligned}$$

Hence, by [13, Proposition 2.1]

$$\begin{aligned} \dim(R_1 \otimes R_2) &\le \dim_v(R_1 \otimes R_2) \\ &\le \min(\dim_v R_1 + t_2,\ \dim_v R_2 + t_1) \\ &= \alpha_1 = \alpha_2 \\ &\le \dim(R_1 \otimes R_2). \end{aligned}$$

Finally, one may easily check, via Corollary 1.3 and Lemma 1.8, that $\alpha_1 \le \operatorname{ht} M_1[t_2] + \dim((R_1/M_1) \otimes R_2)$ and $\alpha_2 \le \operatorname{ht} M_2[t_1] + \dim(R_1 \otimes (R_2/M_2))$. □

It is still an open problem to compute $\dim(R_1 \otimes R_2)$ when only T_1 (or T_2) is assumed to be an AF-domain. However, if none of the T_i is an AF-domain ($i=1,2$), then the formula of Theorem 1.9 may not hold (see [21, Examples 4.3]).

Now assume R_i is an AF-domain and $\dim T_i = \operatorname{ht} M_i = d_i$, for each $i=1,2$. By [13], T_i and D_i are AF-domains and $\text{t.d.}(K_i : D_i) = 0$ (that is, $r_i = s_i$). Further, by [1] $\dim R_i = \dim T_i + \dim D_i = d_i + d_i'$. Therefore, Theorem 1.9 yields:

$$\begin{aligned}\dim(R_1 \otimes R_2) &= \max\{\operatorname{ht} M_1[t_2] + \dim(D_1 \otimes R_2),\ \operatorname{ht} M_2[t_1] + \dim(R_1 \otimes D_2)\} \\ &= \max\{d_1 + \min(\dim R_2 + s_1, t_2 + d_1'), \\ &\qquad d_2 + \min(\dim R_1 + s_2, t_1 + d_2')\} \\ &= \max\{\min(\dim R_2 + r_1 + d_1, t_2 + d_1' + d_1), \\ &\qquad \min(\dim R_1 + r_2 + d_2, t_1 + d_2' + d_2)\} \\ &= \min(t_1 + \dim R_2, t_2 + \dim R_1).\end{aligned}$$

The upshot is that the formula stated in Theorem 1.9 and Wadsworth's formula match in the particular case where R_1 and R_2 are AF-domains.

2. The valuative dimension

It is worth reminding the reader that the valuative dimension behaves well with respect to polynomial rings, that is, $\dim_v R[n] = \dim_v R + n$, for each positive integer n and for any ring R [15, Theorem 2]. Whereas $\dim_v(R_1 \otimes R_2)$ seems not to be effectively computable in general. In [13] the following useful result is proved: given A_1 and A_2 two k-algebras, then $\dim_v(A_1 \otimes A_2) \le \min(\dim_v A_1 + \text{t.d.}(A_2),\ \dim_v A_2 + \text{t.d.}(A_1))$. This section's goal is to compute the valuative dimension for a large class of tensor products of (not necessarily AF-domains) k-algebras. We are still concerned with those arising from pullbacks.

The proof of our theorem requires a preliminary result, which provides a criterion for a polynomial ring over a pullback to be an AF-domain.

We first state the following.

Lemma 2.1. *Let A be an integral domain and n a positive integer. Then $A[n]$ is an AF-domain if and only if, for each prime ideal p of A,* $\operatorname{ht} p[n] + \text{t.d.}(A/p) = \text{t.d.}(A)$.

Proof. Suppose $A[n]$ is an AF-domain. So for each prime ideal p of A $\operatorname{ht} p[n] + \text{t.d.}(A[n]/p[n]) = \text{t.d.}(A) + n$, whence $\operatorname{ht} p[n] + \text{t.d.}(A/p) = \text{t.d.}(A)$. Conversely, if $Q \in \operatorname{Spec}(A[n])$ and $p = Q \cap A$, then by [21, Remark 1.b] $\operatorname{ht} Q + \text{t.d.}(A[n]/Q) = n + \operatorname{ht} p[n] + \text{t.d.}(A/p)$ since $A[n] \cong A \otimes k[n]$. Therefore, $\operatorname{ht} Q + \text{t.d.}(A[n]/Q) = n + \text{t.d.}(A) = \text{t.d.}(A[n])$. $\square$

Proposition 2.2. *Let T be an integral domain with maximal ideal M, $K = T/M$, and φ the canonical surjection. Let D be a proper subring of K and $R = \varphi^{-1}(D)$. Assume*

T and D are AF-domains. Let $r=\text{t.d.}(K)$ and $s=\text{t.d.}(D)$. Then $R[r-s]$ is an AF-domain.

Proof. Let $p\in \text{Spec}(R)$. There are two cases:

1. If $M\not\subset p$, then R_p is an AF-domain. So $\text{ht}\, p+\text{t.d.}(R/p)=\text{t.d.}(R)$. Further, by [21, Corollary 3.2] $\text{ht}\, p=\text{ht}\, p[r-s]$, whence $\text{ht}\, p[r-s]+\text{t.d.}(R/p)=\text{t.d.}(R)$.

2. If $M\subseteq p$, by Lemma 1.2, $\text{ht}\, p[r-s]=\text{ht}\, p+r-s$. Moreover $\text{t.d.}(R/p)=s+\text{ht}\, M-\text{ht}\, p$. Then $\text{ht}\, p[r-s]+\text{t.d.}(R/p)=r+\text{ht}\, M=\text{t.d.}(T)=\text{t.d.}(R)$. Consequently, $R[r-s]$ is an AF-domain by Lemma 2.1. □

We now present the main result of this section. We consider two pullbacks of k-algebras and use the same notations as in the previous sections.

Theorem 2.3. *Let T_1, T_2, D_1 and D_2 be AF-domains, with $\dim T_1=\text{ht}\, M_1$ and $\dim T_2=\text{ht}\, M_2$, then $\dim_v(R_1\otimes R_2)=\min(\dim_v R_1+t_2, \dim_v R_2+t_1)$.*

Proof. By Proposition 2.2 $R_1[r_1-s_1]$ and $R_2[r_2-s_2]$ are AF-domains. Then $R_1[r_1-s_1]\otimes R_2[r_2-s_2]$ is an AF-ring by [21, Proposition 3.1]. Consequently, by [5, Theorem 2.1] $\dim_v(R_1[r_1-s_1]\otimes R_2[r_2-s_2])=\dim(R_1[r_1-s_1]\otimes R_2[r_2-s_2])=\min(\dim R_1[r_1-s_1]+\text{t.d.}(R_2[r_2-s_2]),\ \text{t.d.}(R_1[r_1-s_1])+\dim R_2[r_2-s_2])\geq \min(d_1+\dim D_1[r_1-s_1]+r_1-s_1+t_2+r_2-s_2, d_2+\dim D_2[r_2-s_2]+r_1-s_1+t_1+r_2-s_2)=r_1-s_1+r_2-s_2+\min(d_1+d'_1+r_1-s_1+t_2, d_2+d'_2+r_2-s_2+t_1)$. It turns out that $\dim_v(R_1\otimes R_2)\geq \min(d_1+d'_1+r_1-s_1+t_2, d_2+d'_2+r_2-s_2+t_1)$. So by [1, Theorem 2.11] $\dim_v(R_1\otimes R_2)\geq \min(\dim_v R_1+t_2, t_1+\dim_v R_2)$. Therefore, by [13, Proposition 2.1] we get $\dim_v(R_1\otimes R_2)=\min(\dim_v R_1+t_2,\ \dim_v R_2+t_1)=t_1-s_1+t_2-s_2+\min(s_1+d'_2, d'_1+s_2)$. □

3. Some applications and examples

We may now state a stability result. It asserts that, under mild assumptions on transcendence degrees, tensor products of pullbacks issued from AF-domains preserve Jaffard rings.

Theorem 3.1. *If T_1, T_2, D_1 and D_2 are AF-domains, M_1 is the unique maximal ideal of T_1 with $\text{ht}\, M_1=\dim T_1$ and M_2 is the unique maximal ideal of T_2 with $\dim T_2=\text{ht}\, M_2$, then $R_1\otimes R_2$ is a Jaffard ring if and only if either $r_1-s_1\leq t_2$ and $r_2-s_2\leq s_1$ or $r_1-s_1\leq s_2$ and $r_2-s_2\leq t_1$.*

Proof. Suppose $r_1-s_1\leq t_2$ and $r_2-s_2\leq s_1$. Then $\alpha_1=t_1-s_1+t_2-s_2+\min(s_1+d'_2, d'_1+s_2)=\min(\dim_v R_1+t_2, t_1+\dim_v R_2)$. By Theorems 1.9 and 2.3 $\alpha_1\leq \dim(R_1\otimes R_2)\leq \dim_v(R_1\otimes R_2)=\min(\dim_v R_1+t_2, t_1+\dim_v R_2)=\alpha_1$. Hence $R_1\otimes R_2$ is a Jaffard ring. Likewise for $r_1-s_1\leq s_2$ and $r_2-s_2\leq t_1$. Conversely, since $R_1/M_1\cong D_1$ is an AF-domain, by [21, Theorem 3.7] $\dim((R_1/M_1)\otimes R_2)=\text{D}(s_1, d'_1, R_2)=\max\{\Delta(s_1, d'_1, p_2)\,|\, p_2\in \text{Spec}(R_2)\}$. If $M_2\subseteq p_2$, by the proof of Lemma 1.8 it follows that $\Delta(s_1, d'_1, p_2)=d_2+$

$\min(s_1, r_2 - s_2) + \min(s_1 + \operatorname{ht} q_2, d_1' + s_2)$ where q_2 is the unique prime ideal of D_2 such that $p_2 = \varphi_2^{-1}(q_2)$. If $M_2 \not\subset p_2$, since R_{2p_2} is an AF-domain, then $\Delta(s_1, d_1', p_2) = \operatorname{ht} p_2 [s_1] + \min(s_1, d_1' + \text{t.d.}(R_2/p_2)) = \operatorname{ht} p_2 + \min(s_1, d_1' + \text{t.d.}(R_2/p_2) = \min(s_1 + \operatorname{ht} p_2, d_1' + \text{t.d.}(R_2/p_2) + \operatorname{ht} p_2) = \min(s_1 + \operatorname{ht} p_2, d_1' + t_2)$. In conclusion, since $\operatorname{ht} p_2 \leq d_2 - 1$ being M_2 the unique maximal ideal of T_2 with $\dim T_2 = \operatorname{ht} M_2$, we get $\dim((R_1/M_1) \otimes R_2) = \max\{d_2 + \min(s_1, r_2 - s_2) + \min(s_1 + d_2', d_1' + s_2),\ \min(s_1 + d_2 - 1, d_1' + t_2)\}$. Similarly, $\dim(R_1 \otimes (R_2/M_2)) = \max\{d_1 + \min(s_2, r_1 - s_1) + \min(s_1 + d_2', d_1' + s_2), \min(s_2 + d_1 - 1, d_2' + t_1)\}$. Moreover by Theorem 2.3 $\dim_v(R_1 \otimes R_2) = \min(\dim_v R_1 + t_2, \dim_v R_2 + t_1) = t_1 - s_1 + t_2 - s_2 + \min(s_1 + d_2', d_1' + s_2)$. Let us assume $s_1 + d_2' \leq d_1' + s_2$. Necessarily, $s_1 + d_2 \leq t_2 + d_1'$. Applying Corollary 1.3, we obtain $\operatorname{ht} M_1[t_2] + \dim((R_1/M_1) \otimes R_2) = d_1 + \min(t_2, r_1 - s_1) + d_2 + \min(s_1, r_2 - s_2) + s_1 + d_2'$. On the other hand, $d_1 + \min(s_2, r_1 - s_1) + s_1 + d_2' = \min(s_2 + d_1, t_1 - s_1) + s_1 + d_2' = \min(d_2' + t_1, s_2 + d_1 + s_1 + d_2') \geq \min(s_2 + d_1 - 1, d_2' + t_1)$. Therefore, $\operatorname{ht} M_2[t_1] + \dim(R_1 \otimes (R_2/M_2)) = d_2 + \min(t_1, r_2 - s_2) + d_1 + \min(s_2, r_1 - s_1) + s_1 + d_2'$. Consequently, $\dim(R_1 \otimes R_2) = \max\{d_1 + \min(t_2, r_1 - s_1) + d_2 + \min(s_1, r_2 - s_2) + s_1 + d_2', d_2 + \min(t_1, r_2 - s_2) + d_1 + \min(s_2, r_1 - s_1) + s_1 + d_2'\}$ and $\dim_v(R_1 \otimes R_2) = t_1 + t_2 - s_2 + d_2' = d_1 + r_1 + d_2 + r_2 - s_2 + d_2'$. Since $R_1 \otimes R_2$ is a Jaffard ring, then either $d_1 + \min(t_2, r_1 - s_1) + d_2 + \min(s_1, r_2 - s_2) + s_1 + d_2' = d_1 + r_1 + d_2 + r_2 - s_2 + d_2'$ or $d_2 + \min(t_1, r_2 - s_2) + d_1 + \min(s_2, r_1 - s_1) + s_1 + d_2' = d_1 + r_1 + d_2 + r_2 - s_2 + d_2'$. Hence, either $r_1 - s_1 \leq t_2$ and $r_2 - s_2 \leq s_1$ or $r_1 - s_1 \leq s_2$ and $r_2 - s_2 \leq t_1$. Similar arguments run for $d_1' + s_2 \leq s_1 + d_2'$, completing the proof. □

Our next result states, under weak assumptions, a formula similar to that of Theorem 1.9. It establishes a satisfactory analogue of [4, Theorem 5.4] (also [1, Proposition 2.7, 9, Corollary 1]) for tensor products of pullbacks issued from AF-domains.

Theorem 3.2. *Assume T_1 and T_2 are AF-domains, with $\dim T_1 = \operatorname{ht} M_1$ and $\dim T_2 = \operatorname{ht} M_2$. Suppose that either $\text{t.d.}(D_1) \leq \text{t.d.}(K_2 : D_2)$ or $\text{t.d.}(D_2) \leq \text{t.d.}(K_1 : D_1)$. Then $\dim(R_1 \otimes R_2) = \max\{\operatorname{ht} M_1[t_2] + \dim(D_1 \otimes R_2), \operatorname{ht} M_2[t_1] + \dim(R_1 \otimes D_2)\}$.*

Here, since none of D_i is supposed to be an AF-domain ($i = 1, 2$), the "$\dim(D_i \otimes R_j) = D(s_i, d_i', R_j)$" assertion is no longer valid in general ($(i,j) = (1,2), (2,1)$). Neither is the "$\dim(D_1 \otimes D_2) = \min(s_1 + d_2', d_1' + s_2)$" assertion. Put $\alpha_3' = \min(d_1 + t_2, t_1 + d_2) + \dim(D_1 \otimes D_2)$.

Proof. The proof runs parallel with the treatment of Theorem 1.9. An appropriate modification of its proof yields $\dim(R_1 \otimes R_2) \leq \max\{\operatorname{ht} M_1[t_2] + \dim((R_1/M_1) \otimes R_2), \operatorname{ht} M_2[t_1] + \dim(R_1 \otimes (R_2/M_2)), \alpha_3'\}$. Now there is no loss of generality in assuming that $\text{t.d.}(D_1) \leq \text{t.d.}(K_2 : D_2)$ (That is, $s_1 \leq r_2 - s_2$). By Lemma 1.1 and Corollary 1.5 $\operatorname{ht}(M_1 \otimes R_2) + \operatorname{ht}(D_1 \otimes M_2) = \operatorname{ht} M_1[t_2] + \operatorname{ht} M_2[s_1] = \operatorname{ht} M_1 + \min(t_2, r_1 - s_1) + \operatorname{ht} M_2 + \min(s_1, r_2 - s_2) = \min(d_1 + t_2 + d_2 + s_1, t_1 + d_2) \geq \min(d_1 + t_2, t_1 + d_2)$. Clearly, $\alpha_3' = \min(d_1 + t_2, t_1 + d_2) + \dim(D_1 \otimes D_2) \leq \operatorname{ht}(M_1 \otimes R_2) + \operatorname{ht}(D_1 \otimes M_2) + \dim(D_1 \otimes D_2) \leq \operatorname{ht}(M_1 \otimes R_2) + \dim(D_1 \otimes R_2)$. □

We now move to the significant special case in which $R_1 = R_2$.

Corollary 3.3. *Let T be an AF-domain with maximal ideal M with* $\operatorname{ht} M = \dim T = d$, *$K = T/M$, and φ the canonical surjection. Let D be a proper subring of K and $R = \varphi^{-1}(D)$. Assume D is a Jaffard domain. Then* $\dim(R \otimes R) = \operatorname{ht} M[t] + \dim(D \otimes R)$, *where* $t = \text{t.d.}(T)$. *If moreover* $\text{t.d.}(K:D) \le \text{t.d.}(D)$, *then* $\dim(R \otimes R) = \dim_v(R \otimes R) = t + \dim_v R$.

Proof. If $\text{t.d.}(D) \le \text{t.d.}(K:D)$, the result is immediate by Theorem 3.2. Assume $\text{t.d.}(K:D) \le \text{t.d.}(D)$. Then $\dim(R \otimes R) \ge \operatorname{ht}(M \otimes R) + \operatorname{ht}(D \otimes M) + \dim(D \otimes D) \ge \operatorname{ht} M[t] + \operatorname{ht} M[s] + \dim D + \text{t.d.}(D) = d + \min(t, \text{t.d.}(K:D)) + d + \min(s, \text{t.d.}(K:D)) + \dim D + \text{t.d.}(D) = \min(t + d, t - \text{t.d.}(D)) + d + \text{t.d.}(K:D) + \dim D + s = t - s + t + d' = t + \dim_v R \ge \dim_v(R \otimes R)$. This completes the proof. □

The following example illustrates the fact that in Theorm 1.9 and Corollary 3.3 the "$\dim T_i = \operatorname{ht} M_i (i = 1, 2)$" hypothesis cannot be deleted.

Example 3.4. Let K be an algebraic extension field of k, $T = S^{-1}K[X, Y]$, where $S = K[X, Y] - ((X) \cup (X - 1, Y))$ and $M = S^{-1}(X)$. Consider the following pullback

$$\begin{array}{ccc} R & \longrightarrow & k(Y) \\ \downarrow & & \downarrow \\ S^{-1}K[X,Y] & \longrightarrow & K(Y) \end{array}$$

Since $S^{-1}K[X, Y]$ is an AF-domain and the extension $k(Y) \subset K(Y)$ is algebraic, by [13] R is an AF-domain, so that $\dim(R \otimes R) = \dim R + \text{t.d.}(R) = 2 + 2 = 4$ by [21, Corollary 4.2]. However, $\operatorname{ht} M[2] = \operatorname{ht} M = 1$ and $\dim(k(Y) \otimes R) = \min(2, 1 + 2) = 2$. Hence, $\operatorname{ht} M + \dim(k(Y) \otimes R) = 3$. □

Theorem 1.9 allows one, via [13], to compute (Krull) dimensions of tensor products of two k-algebras for a large class of (not necessarily AF-domains) k-algebras. The next example illustrate this fact.

Example 3.5. Consider the following pullbacks

$$\begin{array}{ccc} R_1 & \longrightarrow & k(X) \\ \downarrow & & \downarrow \\ k(X,Y)[Z]_{(Z)} & \longrightarrow & k(X,Y) \end{array} \qquad \begin{array}{ccc} R_2 & \longrightarrow & k \\ \downarrow & & \downarrow \\ k(X)[Z]_{(Z)} & \longrightarrow & k(X) \end{array}$$

Clearly, $\dim R_1 = \dim R_2 = 1$ and $\dim_v R_1 = \dim_v R_2 = 2$. Therefore, none of R_1 and R_2 is an AF-domain. By Theorem 1.9, we have $\dim(R_1 \otimes R_2) = 4$. Finally, note that Wadsworth's formula fails since $\min\{\dim R_1 + \text{t.d.}(R_2), \dim R_2 + \text{t.d.}(R_1)\} = 3$.

The next example shows that a combination of Theorems 1.9 and 3.2 allows one to compute $\dim(R_1 \otimes R_2)$ for more general k-algebras.

Example 3.6. Consider the pullback

$$\begin{array}{ccc} R_1 & \longrightarrow & k \\ \downarrow & & \downarrow \\ k(X)[Y]_{(Y)} & \longrightarrow & k(X) \end{array}$$

R_1 is a one-dimensional pseudo-valuation domain with $\dim_v R_1 = 2$. Clearly, R_1 is not an AF-domain. By Theorem 1.9 $\dim(R_1 \otimes R_1) = 3$. Consider now the pullback

$$\begin{array}{ccc} R_2 & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ k(X,Y,Z)[T]_{(T)} & \longrightarrow & k(X,Y,Z) \end{array}$$

We have $\dim R_2 = 2$ and $\dim_v R_2 = 4$. The second pullback does not satisfy conditions of Theorem 1.9. Applying Theorem 3.2, we get $\dim(R_1 \otimes R_2) = \max\{\mathrm{ht}\, M_1[4] + \dim(k \otimes R_2),\ \mathrm{ht}\, M_2[2] + \dim(R_1 \otimes R_1)\} = \max\{2+2, 2+3\} = 5$. □

The next example shows that Corollary 3.3 enables us to construct an example of an integral domain R which is not an AF-domain while $R \otimes R$ is a Jaffard ring.

Example 3.7. Consider the pullback

$$\begin{array}{ccc} R & \longrightarrow & k(X) \\ \downarrow & & \downarrow \\ k(X,Y)[Z]_{(Z)} & \longrightarrow & k(X,Y) \end{array}$$

$\dim R = 1$ and $\dim_v R = 2$. Then R is not an AF-domain. By Corollary 3.3 $\dim(R \otimes R) = \dim_v(R \otimes R) = 5$ since $\mathrm{t.d.}(k(X,Y) : k(X)) < \mathrm{t.d.}(R)$. □

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