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THE DILWORTH NUMBER OF GROUP RINGS OVER AN ARTIN LOCAL RING

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Let A be a commutative ring with identity and G be an abelian group. The group ring associated to A and G, denoted $A[G]$, is the ring of elements of the form $\sum_{g \in G} a_g x^g$, where $\{a_g \mid g \in G\}$ is a family of elements of A which are almost all zero. We refer to [4] for elementary properties of group rings. Questions about the number of generators of ideals have been much studied in commutative rings. Of particular interest is the study of the question of when $A[G]$ has the n-generator property, that is, when every ideal of $A[G]$ can be generated by n elements. This question, either in general or for specific choices of n, has been studied in several recent papers ([1], [2], [3], [6], [7], [8] and [9]).

If $A[G]$ has the n-generator property, then $\dim(A[G]) = \dim(A) + \alpha \leq 1$, where α denotes the torsionfree rank of G. If $\alpha = 0$ then G must be a finite group. If

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$\alpha = 1$ then $G \cong Z \oplus H$, where H is a finite abelian group and Z denotes the group of integers. We will focus on the case where A is an Artin ring and $\alpha = 0$.

In this note we give some further results on the Dilworth and Sperner numbers of commutative artinian group rings. Let $\mu(I)$ denote the minimal number of elements required to generate the ideal I. Using the terminology of Watanabe the Dilworth number $d(A)$ of an Artin local ring (A, m) is defined as $\max\{\mu(I) \mid I$ an ideal of $A\}$ and the Sperner number is $sp(A) = \max\{\mu(m^i) \mid i \geq 0\}$. Though papers have appeared on Dilworth and Sperner numbers, see for example [10], [12] and [13], most of the results tend to establish upper bounds for these numbers. There are only a few cases of when exact values of these numbers are known.

In Theorem 1 we will determine the Dilworth number of $A[G]$ where (A, m) is a principal Artin local ring and G is a cyclic p-group with p a prime integer and $p \in m$. In [9, Theorem 4.4] this was done for the unramified case; that is for $pA = m$.

Before stating our main theorem, we need some definitions and notation. Let (A, m) be an Artin local ring and let $p \in m$ be a prime integer. Then the group ring $A[Z/p^j Z]$ is isomorphic to $A[X]/(1 - X^{p^j})$ where X is an indeterminate. We will use x to denote the image of X in the factor ring. If $p \in m^k - m^{k+1}$ then we say the *ramification index of* p, denoted $e(p)$, is k. If $p = 0$ in A, then $e(p) = \infty$. For the group ring $R = A[Z/p^j Z]$, set $s^* = \min\{e(p)(j-t) + p^t \mid 0 \leq t \leq j\}$. In the case $p = 0$ in A, we set $e(p)(j - j) = 0$. In the case where $p \in m$, AWe will use the notation $\ell(A)$ to denote the length of A.

Theorem 1 *Let (A, m) be an Artin local ring and let $m = (z)$. Let $p \in m$ be a prime integer, let $R = A[Z/p^j Z]$. Then $d(R) = sp(R) = \min\{s^*, \ell(A)\}$.*

In the case of $e(p) = 1$, we have $s^* = j + 1$. Thus, we obtain

Corollary 2 *([9, Theorem 4.4]) Let (A, P) be an Artin local ring with $P = pP$, p a prime integer and assume A has characteristic p^i. Let $G = Z/p^j Z$. Then $sp(A[G]) = d(A[G]) = \min\{i, j + 1\}$.*

The proof of Theorem 1 involves several technical preliminary results that will establish upper and lower bounds on the Dilworth number. Our first lemma shows that it will suffice to consider numbers of generators of powers of the maximal ideal. If (A, m) is an Artin local ring with $\mu(m) = 2$, then the nth power of the maximal ideal has $n+1$ natural generators. The next lemma shows that the first power for which one of these generators becomes redundant gives a bound on the Dilworth number of the ring.

Lemma 3 *Let (A, m) be an Artin local ring with $\mu(m) \leq 2$. Then A has the n-generator property if and only if m^n is n-generated.*

Proof. If m^n is n-generated, $\mu(I) \leq \mu(I + m^{n-1})$ for each ideal I of A by [2, Lemma 4]. Thus $sp(A) \leq n$. Since $\mu(m) \leq 2$, $d(A) = sp(A)$ [5, Theorem 4.2]. Thus $d(A) \leq n$ and the converse is obvious. ☐

When A is a principal Artin local ring and G is a cyclic p-group it is possible to determine the first power of the maximal ideal of $A[G]$ for which a natural generator becomes redundant.

Lemma 4 *Let (A, m) be a principal Artin local ring with $m = (z)$ and let $p \in m$ be a prime integer. Let $R = A[Z/p^jZ]$ and let $M = (z, 1-x)$ be the maximal ideal of R. If $s^* = e(p)(j-t) + p^t$ then the element $z^{e(p)(j-t)}(1-x)^{p^t}$ is not required as a generator of M^{s^*}.*

Proof. If $p = 0$ in R then $s^* = p^j$ and $z^{e(p)(j-j)}(1-x)^{p^j} = z^0(1-x)^{p^j} = 0$ so the result is clear in this case. Now we assume that $p \neq 0$. Then

$$
\begin{aligned}
(1-x+x)^{p^j} &= \sum_{k=0}^{p^j} \binom{p^j}{k} (1-x)^k x^{p^j-k} \\
0 &= \sum_{k=1}^{p^j} \binom{p^j}{k} (1-x)^k x^{p^j-k} \\
0 &= \sum_{k=0}^{j-1} \sum_{n=p^k}^{p^{k+1}-1} \binom{p^j}{n} (1-x)^n x^{p^j-n} + (1-x)^{p^j}
\end{aligned}
$$

Further we have that for $p^k \leq n < p^{k+1}, 0 \leq k < j$,

$$\binom{p^j}{n} = \frac{p^j}{n} \cdot \frac{p^j-1}{1} \cdot \ldots \cdot \frac{p^j-(n-1)}{n-1}.$$

Thus if $n = p^t a$ with $(a,p) = 1, \binom{p^j}{n} = up^{j-t}$ where u is a unit in A. Then $\binom{p^j}{n}(1-x)^n = p^{j-k}(1-x)^{p^k} \cdot up^{k-t}(1-x)^{n-p^k} = u_n p^{j-k}(1-x)^{p^k}$ where u_{p^k} is a unit in A and $u_n \in (1-x)R \subseteq M$ for $p^k < n < p^{k+1}$. Therefore

$$\sum_{n=p^k}^{p^{k+1}-1}(-1)^k\binom{p^j}{n}(1-x)^n x^{p^j-n} = u_k p^{j-k}(1-x)^{p^k}$$

and we have

$$0=\sum_{k=0}^{j} v_k p^{j-k}(1-x)^{p^k}$$

where v_k is a unit in R.

Now suppose $s^* = e(p)(j-t)+p^t$ where $0 \le t \le j$.. Then

$$\begin{aligned} p^{j-t}(1-x)^{p^t} &\in \left(p^j(1-x), p^{j-1}(1-x)^p, \ldots, \widehat{p^{j-t}(1-x)^{p^t}}, \ldots, (1-x)^{p^j}\right) \\ &= \left(z^{e(p)j}(1-x), z^{e(p)(j-1)}(1-x)^p, \ldots, \widehat{z^{e(p)(j-t)}(1-x)^{p^t}}, \ldots, (1-x)^{p^j}\right) \\ &\subseteq \left(\{z^{s^*-p^\alpha}(1-x)^{p^\alpha} \mid 0 \le \alpha < t\}\right) \\ &\quad + \left(\{z^{e(p)(j-\alpha)}(1-x)^{s^*-e(p)(j-\alpha)} \mid t < \alpha \le j\}\right) \end{aligned}$$

since $s^* - p^\alpha \le e(p)(j-\alpha)$ for $0 \le \alpha < t$ and $s^* - e(p)(j-\alpha) \le p^\alpha$ for $t < \alpha \le j$. This shows $p^{j-t}(1-x)^{p^t}$ and hence $z^{e(p)(j-t)}(1-x)^{p^t}$ is not required as a generator of M^{s^*}. □

We now combine Lemma 3 and Lemma 4 to get an upper bound for the Dilworth number of the group ring.

Remark 1 *Let (A,m) be a principal Artin local ring and let $p \in m$ be a prime integer. Let $R = A[Z/p^jZ]$. From Lemma 4, $\mu(M^{s^*}) \le s^*$, so $d(R) \le s^*$ by Lemma 3. Since $d(R) \le \ell(A)$ it follows that $d(R) \le \min\{s^*, \ell(A)\}$.*

We complete the proof of Theorem 1 by showing in Proposition 6 that M^s where $s < s^*$ (in the notation of the preceding proposition) requires $\min\{\ell(A), s+1\} = \ell(A/m^{s+1})$ generators. Note that in the proof of [9, Theorem 4.4] it was basically shown that if M is the maximal ideal of $Z/p^iZ[Z/p^jZ]$, then M^j requires $\min\{i, j+1\}$ generators. We are able to establish Proposition 6 in a slightly more general setting than we need for the proof of Theorem 1, i.e., the coefficient ring A is not assumed to be principal. Many of the technical details in the proof of Proposition 6 are the same as in the proof of [9, Proposition 4.3]. However in a few places it is necessary to sharpen the results given there. In particular, [9, Lemma 4.2] states that for $1 \le t \le j-1$, p^{j-t+1} divides $\binom{p^j+t-1}{p^j-1}$. Note that for $p=2$, $j=5$

and $t = 4$ we get that 4 divides $\binom{35}{31} = 6545 \cdot 8$ which is not the optimal result. The next lemma gives the refinement that is needed for the proof of Proposition 6.

Lemma 5 *Let t, j, p, α be integers with p prime and let $[*]$ denote the greatest integer function.*

(1) If $1 \le t \le p^j$ then $p^{j-[\log_p t]}$ divides $\binom{p^j+t-1}{p^j-1}$.

(2) If $1 \le t \le p^j$ and $\alpha \ge 0$, then $p^{j-[\log_p t]}$ divides $\binom{p^j+t+\alpha}{p^j+\alpha} - \binom{t+\alpha}{\alpha}$.

Proof. For (a) we first note that as in the proof of [9, Lemma 4.2(a)], p^{j-r} divides $\binom{p^j+t-1}{p^j-1}$ where $t = p^r n$, $(n,p) = 1$. Now $\log_p t = \log_p n + r \ge r$ so $[\log_p t] \ge r$. Therefore $p^{j-[\log_p t]}$ divides $\binom{p^j+t-1}{p^j-1}$.

For (b) we first prove the case $\alpha = 0$ by induction on t. If $t = 1$, the result is clear. For $1 \le t \le p^j$:

$$\binom{p^j+t}{p^j} - 1 = \binom{p^j+t-1}{p^j-1} + \binom{p^j+t-1}{p^j} - 1$$

Now $p^{j-[\log_p t]}$ divides $\binom{p^j+t-1}{p^j-1}$ by part (a) and $p^{j-[\log_p(t-1)]}$ divides $\binom{p^j+t-1}{p^j} - 1$ by the inductive hypothesis. Since $p^{j-[\log_p t]} \le p^{j-[\log_p(t-1)]}$ we are done. Now we proceed by induction on α. So assume $\alpha > 0$ and the result holds for smaller values of α with $1 \le t \le p^j$. Now proceed by induction on t. If $t = 1$ the result is clear. For $2 \le t \le p^j$,

$$\begin{aligned}\binom{p^j+t+\alpha}{p^j+\alpha} - \binom{t+\alpha}{\alpha} &= \binom{p^j+t+(\alpha-1)}{p^j+(\alpha-1)} - \binom{t+(\alpha-1)}{\alpha-1} \\ &\quad + \binom{p^j+(t-1)+\alpha}{p^j+\alpha} - \binom{(t-1)+\alpha}{\alpha}.\end{aligned}$$

Now $p^{j-[\log_p t]}$ divides the first difference on the right hand side and $p^{j-[\log_p(t-1)]}$ divides the second. Thus $p^{j-[\log_p t]}$ divides the left hand side. □

Proposition 6 *Let (A, m) be an Artin local ring and let $p \in m$ be a prime integer. Let $R = A[Z/p^jZ]$ and let M be the maximal ideal of R. If $0 < s < \min\{s^*, l(A)\}$ then M^s requires $\ell(A/m^{s+1})$ generators. In particular, $\ell(A/m^{s^*}) \le sp(R)$.*

Proof. In the case $p = 0$ in R, $R \cong A[x]/(x^{p^j})$ and the result is clear in this case. Now assume $p \ne 0$ in R. Writing $M = (m, 1-x)$, we will show

$$M^s = (m^s, m^{s-1}(1-x), ..., m^2(1-x)^{s-2}, m(1-x)^{s-1}, (1-x)^s)$$

requires $\ell(A/m^{s+1})$ generators. The proof is similar to that of [9, Proposition 4.3]. Thus, we will omit some of the technical details, referring the reader, at the appropriate places, to the proof in [9].

Let N be such that $m^N \neq 0$ and $m^{N+1} = 0$ and for $0 \leq i \leq N$ let $\{m_{i1}, ..., m_{in(i)}\}$ be a minimal set of generators for m^i. If M^s can be generated by fewer elements then, since R is local, one of the generators, say $m_{(s-t)1}(1-x)^t$, can be written in terms of the others. If $t = s$, then we have the contradiction $1 \in m$. Thus we may assume that $t < s$. We may pass to the ring $A/m^{s-t+1}[Z/p^jZ]$ obtaining

$$\begin{aligned} m_{(s-t)1}(1-x)^t &= \sum_{\alpha=2}^{n(s-t)} f_{t\alpha} m_{(s-t)\alpha}(1-x)^t \\ &+ \sum_{n=t+1}^{s} \sum_{\alpha=1}^{n(s-n)} f_{n\alpha} m_{(s-n)\alpha}(1-x)^n \end{aligned}$$

where $f_{t\alpha}$, $f_{n\alpha} \in R$. The first summand in the above expression can be written $\sum_{i=0}^{p^j-1} b_i x^i$ where $b_i \in J = \left(m_{(s-t)2}, ..., m_{(s-t)n(s-t)}\right) A$. For $t < n \leq s$, let $f_{n\alpha} = \sum_{i=0}^{p^j-1} a_{n\alpha i} x^i$ where $a_{n\alpha i} \in A$. Then, for $t < n \leq s$,

$$\begin{aligned} f_{n\alpha}(1-x)^n &= \sum_{i=0}^{p^j-1} a_{n\alpha i} x^i \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^r \\ &= \sum_{i=0}^{p^j-1} \left(\sum_{r=0}^{n} (-1)^r \binom{n}{r} a_{n\alpha p^j - r + i} \right) x^i \end{aligned}$$

After making these substitutions we have

$$\begin{aligned} m_{(s-t)1}(1-x)^t &= \sum_{i=0}^{p^j-1} b_i x^i \\ &+ \sum_{n=t+1}^{s} \sum_{\alpha=1}^{n(s-n)} \sum_{i=0}^{p^j-1} \sum_{r=0}^{n} (-1)^r \binom{n}{r} m_{(s-n)\alpha} a_{n\alpha p^j - r + i} x^i \end{aligned}$$

Equating coefficients we get the system of equations (see [9, p. 821]):

$$m_{(s-t)1}\tbinom{t}{0} = b_0 + \sum_{n=t+1}^{s} \sum_{\alpha=1}^{n(s-n)} m_{(s-n)\alpha} \left(\sum_{r=0}^{n} (-1)^r \binom{n}{r} a_{n\alpha p^j - r} \right)$$

$$
\begin{aligned}
-m_{(s-t)1}\tbinom{t}{1} &= b_1 + \sum_{n=t+1}^{s}\sum_{\alpha=1}^{n(s-n)} m_{(s-n)\alpha}\left(\sum_{r=0}^{n}(-1)^r\binom{n}{r}a_{n\alpha p^j-r+1}\right)\\
(-1)^t m_{(s-t)1} &= b_t + \sum_{n=t+1}^{s}\sum_{\alpha=1}^{n(s-n)} m_{(s-n)\alpha}\left(\sum_{r=0}^{n}(-1)^r\binom{n}{r}a_{n\alpha p^j-r+t}\right)\\
0 &= b_{t+1} + \sum_{n=t+1}^{s}\sum_{\alpha=1}^{n(s-n)} m_{(s-n)\alpha}\left(\sum_{r=0}^{n}(-1)^r\binom{n}{r}a_{n\alpha p^j-r+t+1}\right)\\
&\vdots\\
0 &= b_{p^j-1} + \sum_{n=t+1}^{s}\sum_{\alpha=1}^{n(s-n)} m_{(s-n)\alpha}\left(\sum_{r=0}^{n}(-1)^r\binom{n}{r}a_{n\alpha p^j-r+p^j-1}\right)
\end{aligned}
$$

Now multiply the kth equation by $\binom{t+k}{k}$ and sum from $k=0$ to $k=p^j-1$. On the left we get:

$$
m_{(s-t)1}\sum_{k=0}^{t}(-1)^k\binom{t}{k}\binom{t+k}{k} = (-1)^t m_{(s-t)1}
$$

by [9, Lemma 4.1(c)]. To compute the sum on the right, we fix n and α and write a_i for $a_{n\alpha i}$. Then we have an element of J and a sum of terms of the form $m_{(s-n)\alpha}\sum_{k=0}^{t}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{t+k}{k}a_{p^j-r+k}$ where (see [9, p. 822])

$$
\begin{aligned}
\sum_{k=0}^{t}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{t+k}{k}a_{p^j-r+k} &= \sum_{\beta=0}^{p^j-1}\sum_{p^j-r+k=\beta}(-1)^r\binom{n}{r}\binom{t+k}{k}a_{p^j-r+k}\\
= \sum_{k=0}^{n}&(-1)^k\binom{n}{k}\binom{t+k}{k}a_0 + \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{t+1+k}{1+k}a_1\\
&+\ldots+\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{t+p^j-n-1+k}{p^j-n-1+k}a_{p^j-n-1}\\
&+\left[\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\binom{t+p^j-n+k}{p^j-n+k} + (-1)^n\binom{t}{0}\binom{n}{n}\right]a_{p^j-n}\\
&+\left[\sum_{k=0}^{n-2}(-1)^k\tbinom{n}{k}\tbinom{t+p^j-n+1+k}{p^j-n+1+k} + \sum_{k=n-1}^{n}(-1)^k\tbinom{t+k-(n-1)}{k-(n-1)}\tbinom{n}{k}\right]a_{p^j-n+1}\\
&+\ldots+\left[\binom{n}{0}\binom{t+p^j-1}{p^j-1} + \sum_{k=1}^{n}(-1)^k\binom{t+k-1}{k-1}\binom{n}{k}\right]a_{p^j-1}
\end{aligned}
$$

As in [9, pp. 823-824] the coefficient of $a_{p^j-k_0}$ is zero for $n+1\le k_0\le p^j$ and for $1\le k_0\le n$ is:

$$
\sum_{k=k_0}^{n}(-1)^k\binom{n}{k}\left(\tbinom{t+p^j-k_0+k}{p^j-k_0+k} - \tbinom{t+k-k_0}{k-k_0}\right).
$$

Then, by Lemma 5, each non-zero term in the above sum is divisible by $p^{j-[\log_p t]}$. So

$$\begin{aligned} m_{(s-t)1} &\in (m_{(s-t)2},\ldots,m_{(s-t)n(s-t)})\,A + m^{s-n}p^{j-[\log_p t]}A \\ &= J + m^{s-n}p^{j-[\log_p t]}A \subseteq J + m^{s-n+e(p)(j-[\log_p t])}. \end{aligned}$$

By our choice of s we have $n \le s < e(p)(j-[\log_p t]) + p^{[\log_p t]}$. Thus, we have

$$\begin{aligned} s-t &< e(p)(j-[\log_p t]) + p^{[\log_p t]} - t \\ &\le e(p)(j-[\log_p t]) \\ &\le s-n+e(p)(j-[\log_p t]). \end{aligned}$$

Hence, we have a contradiction. Thus M^s requires $\ell(A/m^{s+1})$ generators. ◻

Our main result now follows easily.

Proof of Theorem 1. From Proposition 6 and Remark 1, we have $\min\{s^*,\ell(A)\} = \ell(A/m^{s^*}) \le sp(R) \le d(R) \le \min\{s^*,\ell(A)\}$. ◻

Example 1 *Let (A,m) be an Artin local ring with $m=(z)$ and $p \in m$ a prime integer. We will now list the Dilworth number for $R = A[Z/p^jZ]$ for $\ell(A) \le 5$. Note that for $\ell(A) \le 2, d(R) = \ell(A)$. We will list $\ell(A), e(p), j$ and the elements of the set $S = \{e(p)j+1, e(p)(j-1)+p, \ldots, p^j\}$.*

$l(A)$	j	$e(p)$	S	$d(R)$
3	1	1	$2, p$	2
3	1	$e(p) \ge 2$	$e(p)+1, p$	$\min\{3,p\}$
3	$j \ge 2$	$e(p) \ge 1$	$e(p)j+1,\ldots,p^j$	3
4	1	$1 \le e(p) \le 3$	$e(p)+1, p$	$\min\{e(p)+1,p\}$
4	1	$e(p) \ge 4$	$e(p)+1, p$	$\min\{4,p\}$
4	2	1	$3, 1+p, p^2$	3
4	2	$e(p) \ge 2$	$e(p)2+1, e(p)+p, p^2$	4
4	$j \ge 3$	$e(p) \ge 1$	$e(p)j+1,\ldots,p^j$	4
5	1	$1 \le e(p) \le 4$	$e(p)+1, p$	$\min\{e(p)+1,p\}$
5	1	$e(p) \ge 5$	$e(p)+1, p$	$\min\{5,p\}$
5	2	1	$3, 1+p, p^2$	3
5	2	2	$5, 2+p, p^2$	$\min\{5,2+p\}$
5	2	$e(p) \ge 3$	$e(p)2+1, e(p)+p, p^2$	$\min\{5,p^2\}$
5	3	1	$4, 2+p, 1+p^2, p^3$	4
5	3	$e(p) \ge 2$	$e(p)3+1,\ldots,p^3$	5
5	$j \ge 4$	$e(p) \ge 1$	$e(p)j+1,\ldots,p^j$	5

The problem of determining the Dilworth number of a group ring when the group is not cyclic appears to be very difficult. We conclude with a few results in this case. Since the Dilworth number of $A[G \oplus H]$ is bounded by $\min\{\ell(A[G]), \ell(A[H])\}$ it will be useful to know more about the length of a group ring. If r is a non-zero element in an Artin local ring then the *nilpotency index* of r is the largest positive integer n such that $r^n \neq 0$.

Proposition 7 *Let (A, m) be an Artin local ring with $p \in m$, a prime integer. If $p^i = 0$ while $p^{i-1} \neq 0$ and $G = p^j Z$, then the nilpotency index of $(1 - x)$ in $A[G] \cong A[X]/(1 - X^n)$ is $|G| + (i-1)\phi(|G|) - 1$ where ϕ denotes the Euler ϕ-function.*

Proof. Since the map $(Z/p^iZ)[G] \to A[G]$ is an injection this follows from [9, Theorem 4.6]. □

Remark 2

(1) *Let $A \subseteq B$ be rings and assume that for each maximal ideal P of B the natural map $A/(P \cap A) \to B/P$ is an isomorphism. Then $\ell_A(M) = \ell_B(M)$ for each B-module M.*

(2) *Let A be an Artin ring and let p be a prime integer belonging to each maximal ideal of A. If G is a finite abelian p-group then $\ell_{A[G]}(A[G]) = \ell_A(A) \cdot |G|$.*

Proof.

(1) If $M = M_0 \supseteq M_1 \supseteq ... \supseteq M_n$ is a composition series of M as a B-module, the M_i are also A-modules, and for each of the quotients we have $M_i/M_{i+1} \cong B/P \cong A/(P \cap A)$ for some maximal ideal of B. Thus the above sequence is also a composition series of A-modules.

(2) For any ring R and any R-module M we have

$$\ell_R(M) = \sum_{P \text{ maximal}} \ell_{R_P}(M_P).$$

Therefore we may assume A is local. Let m be the maximal ideal of A. Then $M = (m, I(G))$ is the only maximal ideal of $A[G]$, where $I(G)$ is the augmentation ideal. (2) now follows from (1). □

The next corollary gives a stronger version of [11, Theorem 5.2] for the case of an abelian group. This result also proves one direction of [9, Corollary 2.8] using different techniques.

Corollary 8 *Let F be a field of characteristic p where p is a prime integer. Let $n_1 \leq n_2 \leq ... \leq n_r$ be positive integers. Then $d(F[Z/p^{n_1}Z \oplus ... \oplus Z/p^{n_r}Z]) = p^{\sum_{i=1}^{r-1} n_i}$ if $p^{n_r} > \sum_{i=1}^{r-1}(p^{n_i} - 1)$.*

Proof. Let $A = F[Z/p^{n_1}Z \oplus ... \oplus Z/p^{n_{r-1}}Z]$ and consider $A[Z/p^{n_r}Z]$. In the notation of Theorem 1, $s^* = p^{n_r}$. Thus

$$\ell(A/m^{p^{n_r}}) \leq d\left(F[Z/p^{n_1}Z \oplus ... \oplus Z/p^{n_r}Z]\right) \leq \ell(A).$$

Now $m^{p^{n_r}} = 0$ if and only if $p^{n_r} > \sum_{i=1}^{r-1}(p^{n_i} - 1)$. □

The next corollary shows that if $p > n$, G is a finite p-group and $A[G]$ has the n-generator property then G must be cyclic.

Corollary 9 *Let (A, m) be an Artin local ring and let $p \in m$ be a prime integer. Let $G = Z/p^{n_1}Z \oplus ... \oplus Z/p^{n_r}Z, n_1 \leq n_2 \leq ... \leq n_r$ be a $p-$group. Let $n > 0$. If $p^{n_i} > n$ for some $1 \leq i \leq r-1$ then $A[G]$ does not have the n-generator property.*

Proof. This is immediate from the preceding corollary. □

The next example gives some bounds on the Dilworth number when the coefficient ring is not a field and the group is not cyclic.

Example 2 *Let p, i, j be a integers with p prime and $i \leq j$. Then*

$$\ell\left((Z/p^2Z)[Z/p^iZ]/(p, 1-x)^{j+1}\right) \leq d\left((Z/p^2Z)[Z/p^iZ \oplus Z/p^jZ]\right) \leq 2p^i.$$

Further if $j \geq p^i + \phi(p^i) - 1$ then $d\left((Z/p^2Z)[Z/p^iZ \oplus Z/p^jZ]\right) = 2p^i$.

Proof. We note that $s^* = \min\{j - t + p^t \mid 0 \leq t \leq j\} = j+1$ and by Proposition 6, $\ell\left((Z/p^2Z)[Z/p^iZ]/(p, 1-x)^{s^*}\right) \leq d\left((Z/p^2Z)[Z/p^iZ \oplus Z/p^jZ]\right)$.

The proof of Lemma 4 shows that $(1-x)^{p^i} \in (p)$. Therefore $(p, 1-x)^{s^*} = (p(1-x)^j, (1-x)^{j+1}) = ((1-x)^{j+1})$ if $j \geq p^i$. If $j \geq p^i + \phi(p^i) - 1$ then $(p, 1-x)^{s^*} = 0$

by Proposition 7. Therefore

$$\begin{aligned}\ell\left((Z/p^2)Z[Z/p^iZ]\right) &\leq d\left((Z/p^2Z)[Z/p^iZ \oplus Z/p^jZ]\right)\\ &\leq \ell\left((Z/p^2Z)[Z/p^iZ]\right).\end{aligned}$$

By Remark 2 we are done. □

If (A,m) is a local ring and $p \in m$ is a prime integer, we will determine which group rings have the n-generator property for $n \leq p$. We first prove $d(A[G]) \geq d(A) + d((A/m)[G])$ where G is a finite p-group. Note that if $A = Z/4Z$ and $G = Z/2Z$ then $d(A[G]) = 2 = d(A) + d(Z/2Z[Z/2Z])$ so this bound cannot be improved in the general case.

Proposition 10 *Let (A,m) be an Artin local ring, let $p \in m$ be a prime integer and let G be a non-trivial finite p-group. Then $d(A[G]) \geq d(A) + d((A/m)[G])$.*

Proof. Let d be such that $\mu(M^d) = d((A/m)[G])$ where $M = (1 - x^{g_1}, ..., 1 - x^{g_s})$ is the maximal ideal of $(A/m)[G]$ and let I be an ideal of A such that $\mu(I) = d(A)$. Consider the ideal $J = I + (1 - x^{g_1}, ..., 1 - x^{g_s})^d$ and suppose J can be generated by fewer than $\mu(I) + \mu(M^d)$ elements. Since $A[G]$ is local, one of the generators can be written in terms of the others. If one of the minimal generators of I can be expressed in terms of the others we apply the augmentation map to obtain a contradiction. If $J = \left(r_1, ..., r_{d(A)}\right) + (f_1, ..., f_k)$ where $f_i \in M^d$ and $k < d((A/m)[G])$ then, passing to $(A/m)[G]$ we see that M^d is generated by $k < d((A/m)[G])$ elements, a contradiction. □

Proposition 11 *Let (A,m) be a principal Artin local ring, let $p \in m$ be a prime integer and let G be a non-trivial finite p-group. Then $A[G]$ has the n-generator property for $n \leq p$ if and only if*

(1) If A is a field then

(a) G is cyclic or $G \cong Z/pZ \oplus Z/p^jZ$ if $p = n$

(b) G is cyclic if $p > n$.

(2) If A is not a field then G is cyclic and, if $\ell(A) > n$, then $\min\{e(p)j+1, p^j\} \leq n$. In particular, if $n < p$ then $e(p)j \leq n - 1$.

Proof. We first assume $A[G]$ has the n-generator property and A is a field. We first note the $(p-1)st$ power of the maximal ideal of $A[Z/pZ \oplus Z/pZ \oplus Z/pZ]$ requires more than p generators. Thus G has at most two cyclic summands. If $G \cong Z/p^2Z \oplus Z/p^jZ$ with $j \geq 2$ then $d(A[G]) = p^2$ by Corollary 8. Thus G is cyclic or $G \cong Z/pZ \oplus Z/p^jZ$. If $p > n$ then G is cyclic by Corollary 9. Therefore (1) holds.

Now assume $A[G]$ has the n-generator property and A is not a field. Then since $d((A/m^2)[Z/pZ \oplus Z/pZ]) \geq d(A/m^2) + d((A/m)[Z/pZ \oplus Z/pZ]) \geq 1 + p$, G is cyclic. So assume $G = Z/p^jZ$. By Theorem 1, $\min\{s^*, \ell(A)\} \leq n$. If $\ell(A) > n$ then we must have $\min\{e(p)(j-s) + p^s \mid 0 \leq s \leq j\} = s^* \leq n \leq p$. The only elements of this set which can be less than or equal to p are $e(p)j + 1$ or p^j. So we must have $\min\{e(p)j + 1, p^j\} \leq n$.

The converse is immediate from Corollary 8 and [4, Theorem 19.14] when A is a field and Theorem 1 when A is not a field. □

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References

[1] Ameziane Hassani, S., M. Fontana and S. Kabbaj, Group Rings R[G] with 3-generated ideals when R is Artinian, Comm. in Algebra 24(4)(1996), 1253-1280.

[2] Ameziane Hassani, S., and S. Kabbaj, Group Rings R[G] with 4-generated ideals when R is an Artinian principal ideal ring, Lect. Notes Dekker 185(1997) 1-14.

[3] Ameziane Hassani, S. and S. Kabbaj, Group Rings R[G] with 4-generated ideals when R is an Artinian ring with the 2-generator property, manuscript.

[4] Gilmer, R., Commutative Semigroup Rings, University of Chicago Press, Chicago 1984.

[5] Ikeda, H., Results on Dilworth and Rees numbers of Artinian local rings, Japan J. Math., 22 (147) (1996), 147-158.

[6] Matsuda, R., Torsion free abelian semigroup rings V, Bull. Fac. Sci., Ibaraki Univ. 11(1979), 1-37.

[7] Matsuda, R., n-Generator property of a polynomial ring, Bull. Fac. Sci., Ibaraki Univ., Series A Math. 16 (1984), 17-23.

[8] Okon, J., and P. Vicknair, Group rings with n-generated ideals, Comm. in Algebra, 20(1992), 189-217.

[9] Okon, J., D. Rush and J. Vicknair, Numbers of generators of ideals in zero-dimensional group rings, Comm. in Algebra., 25(3), 803-831 (1997).

[10] Sekiguchi, H., The upper bound of the Dilworth number and the Rees number of Noetherian local rings with a Hilbert function, Advances in Math., 124 (1996), 197-206.

[11] Shalev, A., Dimension subgroups, nilpotency indices, and the number of generators of ideals in p-group algebras, J. Algebra, 129 (1990), 412-438.

[12] Watanabe, J., The Dilworth number of Artinian rings and finite posets with rank function, Commutative Algebra and Combinatorics, Advanced Studies in Pure Mathematics, Vol. 11, 303-312, Kinokuniya Co./North-Holland, Amterdam 1987.

[13] Watanabe, J., The Dilworth number of Artin Gorenstein Rings, Advances in Math., 76(1989), 194-199.

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