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Group Rings $R[G]$ with 4-Generated Ideals When R Is an Artinian Principal Ideal Ring

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Considerable work, part of it summarized in Sally's book [15], has been concerned with the number of generators needed for ideals in a commutative ring R. If there is a fixed bound n, valid for all ideals, on the number of generators needed, R is said to have the $n-$generator property. That means, each ideal of R is $n-$generated (i.e. can be generated by n elements). If dim $R > 1$, no such bound exists. Considerable interest has been shown in rings with the $n-$generator property. See for example [4], [11], [15] and [16].

Let G be an abelian group. The group ring associated to R and G, denoted by $R[G]$, is the ring of elements of the form $\sum_{g\in G} a_g X^g$, where $\{a_g/g \in G\}$ is a family of elements of R which are almost all zero. We refer to [5] for elementary properties of group rings. Of particular interest is the study of the question of when $R[G]$ has the $n-$generator property. This question, either in general or for specific choice of n, has received further attention by several authors. See [1],[3], [9], [10], [13], [14] and [17].

From the restriction on Krull dimension, we have $1 \geq dimR[G] = dimR + r$, where r denotes the torsion free rank of G. If $r = 0$, then G must be a finite group. If $r = 1$, then $G \cong \mathbb{Z} \oplus H$, where H is a finite abelian group and $\mathbb{Z}$ denotes the group of the integers. We will focus on the case in which R is Artinian and $r = 0$, i.e. G is a finite abelian group, since the case $r = 1$ was considered by Okon and Vicknair in [14, Theorem 5.1]. Furthermore, [1] is entirely devoted to $n = 3$. However, for $n \geq 4$ and under our assumptions, the problem of when $R[G]$ has the $n-$generator property remains open.

In this note, we consider the problem of determining when a group ring $R[G]$ has the 4−generator property, when R is an Artinian principal ideal ring and G is a finite group.

Throughtout this note rings and groups are taken to be commutative and the groups are written additively. If p is a prime integer, then the $p-$sylow subgroup of the finite abelian group G will be denoted G_p. When I is an ideal of R, we shall use $\mu(I)$ to denote the number of generators in a minimal basis for I. Finally, recall that if I is an $n-$generated ideal in a local ring, then the n generators of I may be chosen from elements of a given set of generators of I (cf. [12, (5.3), p. 14]).

PROPOSITION 1 *Assume that G is a nontrivial finite 2−group, (R, M) is an Artinian local principal ideal ring which is not a field and $2 \in M$. Then $R[G]$ has the 4−generator property if and only if*

A) (i) $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^i\mathbb{Z}$ with $i \geq 1$

(ii) when $M^2 \neq 0$, then $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

B) (i) G is a cyclic group

(ii) When $M^4 \neq 0$, then

(a) $G \cong \mathbb{Z}/2^i\mathbb{Z}$, where $1 \leq i \leq 2$, if $2 \in M^2$

(b) $G \cong \mathbb{Z}/2^i\mathbb{Z}$, where $1 \leq i \leq 3$, if $2 \in M \setminus M^2$.

Before proving this proposition we establish a lemma which will be used frequently in the sequel.

LEMMA 2 *Assume that (R, M) is a local principal ideal ring and G is a finite cyclic group. Let N be the maximal ideal of the local ring $R[G]$. Then $R[G]$ has the 4−generator property if and only if N, N^2, N^3 and N^4 are 4−generated.*

Proof. $R[G]$ is local with maximal ideal $N = (r, 1 - X^g)$, where r generates M in R and g is the generator of G. Suppose that N, N^2, N^3 and N^4 are 4−generated. We need to prove that each proper ideal I of $R[G]$ is 4−generated. By [16, Corollary 4.2.1], it suffices to consider the case where $I \not\subset N^3$. Let $x \in I \setminus N^3$,

If $x \in N^2, x = \lambda r^2 + \mu r(1 - X^g) + \delta(1 - X^g)^2$ for some $\lambda, \mu, \delta \in R[G]$. Since $x \notin N^3$, then λ or μ or δ is a unit. Therefore $N^2 = (x, r(1 - X^g), (1 - X^g)^2)$ or $N^2 = (r^2, x, (1-X^g)^2)$ or $N^2 = (r^2, r(1-X^g), x)$. Hence $\mu((N/(x))^2) = \mu(N^2/(x)) \leq 2$. By [11, Theorerm 1, 6 ⇒ 1], $R[G]/(x)$ has the 2−generator property. Then $\mu(I/(x)) \leq 2$. Therefore I is 4−generated.

If $x \notin N^2, x \in N$ because $R[G]$ is local with maximal ideal N. By [8, Theorem 159], $\mu(N/(x)) = \mu(N) - 1 = 1$. So $R[G]/(x)$ is principal then $\mu(I/(x)) = 1$, and hence $\mu(I) \leq 2$. Consequently R[G] has the 4−generator property. ◊

Proof of Proposition 1. ⇒] Assume $G \cong \mathbb{Z}/2^{t_1}\mathbb{Z} \oplus \mathbb{Z}/2^{t_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2^{t_s}\mathbb{Z}$ where $0 < t_1 \leq t_2 \leq \cdots \leq t_s$. If $R[G]$ has the 4−generator property, then the homomorphic image $(R/M)[G]$ does also. By [14, Corollary 2.2], $s \leq 3$.

We first show that the case of $s = 3$ does not hold. Indeed, if $R[\mathbb{Z}/2^{t_1}\mathbb{Z} \oplus \mathbb{Z}/2^{t_2}\mathbb{Z} \oplus \mathbb{Z}/2^{t_3}\mathbb{Z}]$ has the 4−generator property, then the homomorphic image $R[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}]$ does also. Since R is a local ring with residue field of characteristic 2, $R[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}]$ is local with maximal ideal $N := (r, 1 - X^g, 1 - X^h, 1 - X^k)$, where r generates M in R and $< g > \oplus < h > \oplus < k >= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (cf. [5, Theorem 19.1 and Corollary 19.2]). Since $| < g > | = 2$, $((1 - X^g)^2 = 2(1 - X^g) \in (r(1 - X^g))$. Likewise for $(1 - X^h)^2$ and $(1 - X^k)^2$. Hence $N^2 = (r^2, r(1 - X^g), r(1 - X^h), r(1 - X^k), (1 - X^g)(1 - X^h), (1 - X^g)(1 - X^k), (1 - X^h)(1 - X^k))$. The four generators of N^2 can be chosen from the original generators of N^2.

If $r(1 - X^g)$ is a redundant generator, then under the augmentation map $R[< g >][< h > \oplus < k >] \longrightarrow R[< g >]$, $r(1 - X^g) \in (r^2)R[< g >]$. Hence $Rr = Rr^2$, a contradiction. Likewise for $r(1 - X^h)$ and $r(1 - X^k)$.

If $(1 - X^g)(1 - X^h)$ is redundant, then applying the augmentation map $R[< g > \oplus < h >][< k >] \longrightarrow R[< g > \oplus < h >]$ and passing to the homomorphic image $R/(r)[< g > \oplus < h >]$ yields $(1 - X^g)(1 - X^h) = 0$. Hence $1 \in Rr$, a contradiction. Likewise for $(1 - X^g)(1 - X^k)$ and $(1 - X^h)(1 - X^k)$. Therefore N^2 needs more than four generators. Consequently $s \leq 2$.

A) (i) Assume $G \cong \mathbb{Z}/2^{t_1}\mathbb{Z} \oplus \mathbb{Z}/2^{t_2}\mathbb{Z}$ where $t_1 > 1$. So the homomorphic image $R[\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}]$ has the 4−generator property. Then N^2 is 4−generated, where $N = (r, 1 - X^g, 1 - X^h)$ where r generates M in R and $< g > \oplus < h >= \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

It is easy to see that $(1 - X^g)^2$ and $(1 - X^h)^2$ are required as generators of N^2. Now assume that $r(1 - X^g)$ is a redundant generator, then applying the augmentation map $R[< g > \oplus < h >] \longrightarrow R[< g >]$ and passing to the homomorphic image $R/(r^2)[< g >]$, yields $r(1 - X^g) \in ((1 - X^g)^2)R/(r^2)[< g >]$. By [1, Lemma 1.5], $r = 4\lambda$ for some $\lambda \in R/(r^2)$. This forces $Rr = Rr^2$, a contradiction. Likewise for $r(1-X^h)$. Consequently,

$$N^2 = (r(1 - X^g), r(1 - X^h), (1 - X^g)^2, (1 - X^h)^2).$$

Since $(1 - X^g)(1 - X^h) \in N^2$, then passing to the homomorphic image $R/(r)[< g > \oplus < h >]$ yields $(1 - X^g)(1 - X^h) \in ((1 - X^g)^2, (1 - X^h)^2)R/(r)[< g > \oplus < h >]$. Thus $(1 - X^g)^3(1 - X^h)^3 \in ((1 - X^g)^4, (1 - X^h)^4) = (0)$ in $R/(r)[< g > \oplus < h >]$, since $(1 - X^g)^4 = 2(1 - 2X^h + 3X^{2h})$ and $2 \in (r)$. Then $1 \in (r)$, a contradiction. Therefore N^2 needs more than four generators. Consequently, $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^i\mathbb{Z}$ where $i \geq 1$.

(ii) Assume $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^i\mathbb{Z}$ with $i > 1$ and $M^2 \neq 0$. Then $R[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}]$ has the 4−generator property. Therefore N^2 is 4−generated, where $N = (r, 1 - X^g, 1 - X^h)$, r generates M in R and g, h are the generators of $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$, respectively. Since $(1 - X^g)^2 = 2(1 - X^g)$, then

$$N^2 = (r^2, r(1 - X^g), r(1 - X^h), (1 - X^g)(1 - X^h), (1 - X^h)^2).$$

Since $M^2 \neq 0$ and $| < h > | > 2$, it is clear that r^2 and $(1 - X^h)^2$ are required as generators of N^2. Furthermore, using arguments similar to ones used above, we obtain that $r(1-X^g), r(1-X^h)$ and $(1-X^g)(1-X^h)$ also are required as generators of N^2. Then N^2 needs more than four generators, a contradiction. Consequently, $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ when $M^2 \neq 0$.

⇐] (i) Assume $M^2 = 0$ and $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^i\mathbb{Z}$ with $i > 1$. Then $R[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^i\mathbb{Z}]$ is a local ring with maximal ideal $N = (r, 1 - X^g, 1 - X^h)$, where r generates M in R and $< g > \oplus < h >= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^i\mathbb{Z}$. Since $r^2 = 0, | < g > | = 2$ and $2 \in (r)$ we get $N^2 = (r(1 - X^g), r(1 - X^h), (1 - X^g)(1 - X^h), (1 - X^h)^2)$ and $N^3 = (r(1 - X^g)(1 - X^h), r(1 - X^h)^2, (1 - X^g)(1 - X^h)^2, (1 - X^h)^3)$.

Let I be a proper ideal of $R[G]$. Since $N^3 = (1 - X^h)N^2$, [11, Lemma 2] implies that $\mu(I) \leq \mu(I + N^2)$. In order to show that I is 4–generated, we may assume $N^2 \subset I$. Let $x \in I \setminus N^2$, $x \in N$. By [8, Theorem 159], $\mu(N/(x)) = \mu(N) - 1 = 2$. Let us show that $\mu((N/(x))^2) \leq 2$. Since $\mu(N/(x)) = 2$, we have $N = (r, x, 1 - X^g)$, $N = (r, x, 1 - X^h)$ or $N = (x, 1 - X^g, 1 - X^h)$.

If $N = (r, x, 1 - X^g)$ then $N/(x) = (\overline{r}, \overline{1 - X^g})$, where bars denote images under the natural map $R[G] \longrightarrow R[G]/(x)$. Since $r^2 = 0$ then $(N/(x))^2 = \left(\overline{r(1 - X^g)}, \overline{(1 - X^g)^2}\right)$, and hence $\mu\left((N/(x))^2\right) \leq 2$. The argument for $N = (r, x, 1 - X^h)$ is similar.

If $N = (x, 1-X^g, 1-X^h)$ then $(N/(x))^2 = \left(\overline{(1 - X^g)(1 - X^h)}, \overline{2(1 - X^g)}, \overline{(1 - X^h)^2}\right)$. If $2 \in M^2 = (0)$, we're finished. Otherwise, $M = (r) = (2)$. Clearly $2 \in N$. Then $2 = \lambda x + \mu(1 - X^g) + \delta(1 - X^h)$ for some $\lambda, \mu, \delta \in R[G]$. Furthermore, we may assume that μ and δ are not invertible. So $\lambda, \delta \in N$, hence $2 = \lambda' x + \mu'(1 - X^g)(1 - X^h) + \beta'(1 - X^g)^2 + \delta'(1 - X^h)^2 = \lambda' x + \mu'(1 - X^g)(1 - X^h) + 2\beta'(1 - X^g) + \delta'(1 - X^h)^2$, for some $\lambda', \mu', \beta', \delta' \in R[G]$. Then $2(1 - \beta'(1 - X^g)) = \lambda' x + \mu'(1 - X^g)(1 - X^h) + \delta'(1 - X^h)^2$. Since $1 - \beta'(1 - X^g)$ is a unit in $R[G]$, $\overline{2} \in \left(\overline{(1 - X^g)(1 - X^h)}, \overline{(1 - X^h)^2}\right)$ and so does $\overline{2(1 - X^g)}$. Consequently, $(N/(x))^2 = \left(\overline{(1 - X^g)(1 - X^h)}, \overline{(1 - X^h)^2}\right)$, and hence $\mu\left((N/(x))^2\right) \leq 2$.

By [11, Theorem 1 ($6 \Rightarrow 1$)], $R[G]/(x)$ has the 2–generator property. Then $I/(x)$ is 2–generated, and hence I is 4–generated. This completes the proof of (i).

$\Leftarrow$] (ii) Assume $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $M^2 \neq 0$. Then $R[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}]$ is a local ring with maximal ideal $N = (r, 1 - X^g, 1 - X^h)$, where r generates M in R and $< g > \oplus < h >= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since $| < g > | = | < h > | = 2$ and $2 \in (r)$, we get $N^2 = (r^2, r(1 - X^g), r(1 - X^h), (1 - X^g)(1 - X^h))$ and $N^3 = (r^3, r^2(1 - X^g), r^2(1 - X^h), r(1 - X^g)(1 - X^h))$.

Let I be a proper ideal of $R[G]$. Since $N^3 = rN^2$, [11, Lemma 2] implies that $\mu(I) \leq \mu(I + N^2)$. As before, we may assume that $N^2 \subset I$. Let $x \in I \setminus N^2$, $x \in N$. By [8, Theorem 159], $\mu(N/(x)) = \mu(N) - 1 = 2$. Thus $N = (r, x, 1 - X^g)$ or $N = (r, x, 1 - X^h)$ or $N = (x, 1 - X^g, 1 - X^h)$. It is easly seen that for the two first cases we have $\mu\left((N/(x))^2\right) \leq 2$. Now let consider the remaining case, i.e. $N = (x, 1-X^g, 1-X^h)$. Then $(N/(x))^2 = \left(\overline{(1 - X^g)(1 - X^h)}, \overline{2(1 - X^g)}, \overline{2(1 - X^h)}\right)$.

If $2 \in M^2 = (r^2)$, since $\overline{r} \in N/(x)$, then $\overline{2} = \overline{\lambda 2(1 - X^g)} + \overline{\mu(1 - X^g)(1 - X^h)} + \overline{\delta 2(1 - X^h)}$ for some λ, μ and $\delta \in R[G]$. We get, by induction,

$$\begin{aligned}(N/(x))^2 &= \left(\overline{(1 - X^g)(1 - X^h)}\right) + (N/(x))^3,\\ &= \left(\overline{(1 - X^g)(1 - X^h)}\right) + (N/(x))^4,\\ &= \left(\overline{(1 - X^g)(1 - X^h)}\right) + (N/(x))^n, \text{ for each } n \geq 3.\end{aligned}$$

Since $R[G]$ is a local Artinian ring, there exists $n_0 \in N$ such that $(N/(x))^n = 0$ for each $n \geq n_0$. Therefore $(N/(x))^2 = \left(\overline{(1 - X^g)(1 - X^h)}\right)$, and hence $\mu\left((N/(x))^2\right) \leq 2$.

If $2 \in M \setminus M^2$ then $M = (r) = (2)$. Clearly $2 \in N$. Then $2 = \lambda x + \mu(1 - X^g) + \delta(1 - X^h)$ for some $\lambda, \mu, \delta \in R[G]$. Applying arguments used above for (i), we

see that $\overline{2} \in \left(\overline{(1 - X^g)(1 - X^h)}\right)$, and so do $\overline{2(1 - X^g)}$ and $\overline{2(1 - X^h)}$. Consequently, $\mu\left((N/(x))^2\right) \leq 2$. As before, we conclude that $\mu(I/(x)) \leq 2$. Therefore I is 4–generated, as desired. This completes the proof of (ii).

B) Suppose that G is a cyclic group ($s = 1$). Let g be the generator of G. we have

$$\begin{aligned}N &= (r, 1 - X^g);\\ N^2 &= (r^2, r(1 - X^g), (1 - X^g)^2);\\ N^3 &= (r^3, r^2(1 - X^g), r(1 - X^g)^2, (1 - X^g)^3);\\ N^4 &= (r^4, r^3(1 - X^g), r^2(1 - X^g)^2, r(1 - X^g)^3, (1 - X^g)^4).\end{aligned}$$

(i) Assume $M^4 = 0$. Applying Lemma 2, we conclude that $R[G]$ has the 4–generator property.

(ii) (a) Assume $M^4 \neq 0$ and $2 \in M^2$. In order to conclude, it suffices to show that $R[\mathbb{Z}/4\mathbb{Z}]$ has the 4–generator property while $R[\mathbb{Z}/8\mathbb{Z}]$ does not. Suppose that $R[\mathbb{Z}/8\mathbb{Z}]$ has the 4–generator property. Then N^4 is 4–generated.

Since $M^4 \neq 0$ and $| < g > | > 4$, it is easily seen that r^4 and $(1 - X^g)^4$ are required as generators of N^4.

If $r^3(1 - X^g)$ is a redundant generator of N^4, then passing to the homomorphic image $(R/(r^4))[< g >]$, yields $r^3(1 - X^g) \in ((1 - X^g)^2)(R/(r^4))[< g >]$. By [1, Lemma 1.5], $r^3 = 8\lambda$ for some $\lambda \in R/(r^4)$. It follows that $r^3 = 0$ in $R/(r^4)$, a contradiction.

If $r^2(1-X^g)^2$ is redundant, then by passing to the homomorphic image $(R/(r^3))[< g >]$, we obtain that $r^2(1 - X^g)^2 = a(1 - X^g)^3$ with $a = \sum_{i=0}^{i=7} a_i X^{ig}$, where $a_i \in R/(r^3)$. After setting corresponding terms equal, we obtain the following equations :

X^o	$a_o - a_5 + 3a_6 - 3a_7 = r^2$
X^g	$-3a_o + a_1 - a_6 + 3a_7 = 0$
X^{2g}	$3a_o - 3a_1 + a_2 - a_7 = r^2$
X^{3g}	$-a_o + 3a_1 - 3a_2 + a_3 = 0$
X^{4g}	$-a_1 + 3a_2 - 3a_3 + a_4 = 0$
X^{5g}	$-a_2 + 3a_3 - 3a_4 + a_5 = 0$
X^{6g}	$-a_3 + 3a_4 - 3a_5 + a_6 = 0$
X^{7g}	$-a_4 + 3a_5 - 3a_6 + a_7 = 0$

This yields $r^2 = 0$ in $R/(r^3)$. Hence $Rr^3 = Rr^2$, a contradiction.

If $r(1-X^g)^3$ is redundant, then by passing to the homomorphic image $(R/(r^2))[< g >]$, we obtain that $r(1 - X^g)^3 \in ((1 - X^g)^4)(R/(r^2))[< g >]$. Since $2 \in M^2 = (r^2)$, $r(1-X^g)^7 = 0$ in $(R/(r^2))[< g >]$. This forces $Rr = Rr^2$, a contradiction. Consequently, N^4 needs more than four generators, contradicting the fact that N^4 is 4–generated.

Now let us show that $R[\mathbb{Z}/4\mathbb{Z}]$ has the 4–generator property. If $2 \in M^2 \setminus M^3$ then $M^2 = (r^2) = (2)$. We have

$$\begin{aligned}1 &= (1 - X^g + X^g)^4\\ &= 1 + 4(1 - X^g)X^{3g} + 6(1 - X^g)^2X^{2g} + 4(1 - X^g)^3X^g + (1 - X^g)^4\end{aligned}$$

Then $2(1-X^g)^2 \in (4(1-X^g),(1-X^g)^4) \subset (r^4,(1-X^g)^4)$. Therefore $r^2(1-X^g)^2 \in (r^4,(1-X^g)^4)$. Consequently, N^4 is 4−generated. If $2 \in M^3$, we get

$$\begin{aligned}(1-X^g)^4 &= 1-4X^g+6X^{2g}-4X^{3g}+X^{4g}\\ &= 2-4X^g+6X^{2g}-4X^{3g}\\ &= 2-2X^g-2X^g+2X^{2g}+4X^{2g}-4X^{3g}\\ &= 2(1-X^g)-2X^g(1-X^g)+4X^{2g}(1-X^g)\\ &= 2(1-X^g)(1-X^g+2X^{2g}).\end{aligned}$$

Then $(1-X^g)^4 \in (2(1-X^g)) \subset (r^3(1-X^g))$. Hence N^4 is 4−generated. Lemma 2 completes the proof.

b) Assume $M^4 \neq 0$ and $2 \in M \setminus M^2$. It suffices to prove that $R[\mathbb{Z}/8\mathbb{Z}]$ has the 4−generator property while $R[\mathbb{Z}/16\mathbb{Z}]$ does not. Clearly $M=(r)=(2)$ and

$$N^4=(16,8(1-X^g),4(1-X^g)^2,2(1-X^g)^3,(1-X^g)^4).$$

Assume $< g >= \mathbb{Z}/8\mathbb{Z}$. We have

$$\begin{aligned}1 &= (1-X^g+X^g)^8\\ &= \sum_{i=o}^{i=8}\binom{8}{i}(1-X^g)^iX^{(8-i)g}\\ &= 1+8(1-X^g)X^{7g}+28(1-X^g)^2X^{6g}+56(1-X^g)^3X^{5g}\\ &\quad +(1-X^g)^4\left(\sum_{i=4}^{i=8}\binom{8}{i}(1-X^g)^{(i-4)}X^{(8-i)g}\right).\end{aligned}$$

Then $8(1-X^g) \in (4(1-X^g)^2,(1-X^g)^4)$, and hence N^4 is 4−generated. Thus $R[\mathbb{Z}/8\mathbb{Z}]$ has the 4−generator property.

Assume $< g >= \mathbb{Z}/16\mathbb{Z}$. Let prove that N^4 is not 4−generated. It is clear that 16 and $(1-X^g)^4$ are required as generators of N^4.

If $8(1-X^g)$ is redundant, then passing to the homomorphic image $(R/(16))[< g >]$, yields $8(1-X^g) \in ((1-X^g)^2)(R/(16))[< g >]$. By [1, Lemma 1.5], $8=16\lambda$ for some λ in $R/(16)$. Hence $8=0$ in $R/(16)$, a contradiction.

If $2(1-X^g)^3$ is redundant, then by passing to the homomorphic image $(R/(4))[< g >]$, we obtain that $2(1-X^g)^3 = a(1-X^g)^4$ with $a=\sum_{i=0}^{i=15} a_iX^{ig}$, where $a_i \in R/(4)$.

After setting corresponding terms equal, we obtain among other equations the following :

X^o	$a_o+a_{12}+2a_{14}$	$=$	2
X^{2g}	$2a_o+a_2+a_{14}$	$=$	2
X^{4g}	$a_o+2a_2+a_4$	$=$	0
X^{6g}	$a_2+2a_4+a_6$	$=$	0
X^{8g}	$a_4+2a_6+a_8$	$=$	0
X^{10g}	$a_6+2a_8+a_{10}$	$=$	0
X^{12g}	$a_8+2a_{10}+a_{12}$	$=$	0
X^{14g}	$a_{10}+2a_{12}+a_{14}$	$=$	$0.$

After resolving this system, we obtain $2=0$ in $R/(4)$, then $2 \in M^2$, a contradiction.

If $4(1-X^g)^2$ is redundant, then passing to the homomorphic image $(R/(8))[< g >]$, yields $4(1-X^g)^2 = a(1-X^g)^3)$ where $a \in (R/(8))[< g >]$. As before, we obtain a system of 16 linear equations in 16 unknowns. After resolving this system, we obtain $4=0$ in $R/(8)$, a contradiction ($M^4 \neq 0$).

It follows that N^4 needs more than four generators. Hence $R[\mathbb{Z}/16\mathbb{Z}]$ does not have the 4−generator property. This completes the proof of Proposition 1. $\diamondsuit$

PROPOSITION 3 *Assume that G is a nontrivial finite 3−group, (R,M) is an Artinian local principal ideal ring which is not a field and $3 \in M$. Then $R[G]$ has the 4−generator property if and only if*

A) $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $3 \in M \setminus M^2$ and $M^2 = 0$.

B) (i) G is a cyclic group

(ii) When $M^4 \neq 0$, then

(a) $G \cong \mathbb{Z}/3\mathbb{Z}$, if $3 \in M^2$.

(b) $G \cong \mathbb{Z}/3^i\mathbb{Z}$, where $1 \leq i \leq 3$, if $3 \in M \setminus M^2$.

LEMMA 4 *Let (R,M) be a local ring such that M^n is n−generated, where n is a positive integer. Then for each ideal I of R, $\mu(I) \leq \mu(I+M^{n-1})$.*

Proof. We may assume that R has an infinite residue field (see [15, p.10]). Since M^n is n−generated, then [15, Theorem 2.3, p.36] implies that $M^n = yM^{n-1}$ for some $y \in M$. By [11, Lemma 2], $\mu(I) \leq \mu(I+M^{n-1})$ for each ideal I of R. $\diamondsuit$

LEMMA 5 *Let (R,M) be a local ring such that M^2 is 3−generated, I a proper ideal of R and $x \in I \setminus M^3$ such that $x \in M^2$. Then $\mu(I/(x)) \leq \mu(M/(x))$.*

Proof. M^2 is 3−generated and $x \in M^2 \setminus M^3$ implies that $\mu((M/(x))^2) = \mu(M^2/(x)) \leq \mu(M^2)-1 = 2$. By applying Lemma 4 to $R/(x)$, We get $\mu(I/(x)) \leq \mu(I/(x)+M/(x)) = \mu(M/(x))$. $\diamondsuit$

Proof of Proposition 3. By hypothesis, $G \cong \mathbb{Z}/3^{t_1}\mathbb{Z} \oplus \mathbb{Z}/3^{t_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/3^{t_s}\mathbb{Z}$ where $0 < t_1 \leq t_2 \leq \cdots \leq t_s$. Suppose that $R[G]$ has the 4−generator property, then the homomorphic image $(R/M)[G]$ does also. By [14, Corollary 2.2], $s \leq 2$.

A) ⇒] If $s=2$, [14, Proposition 2.1(a)] implies that $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^i\mathbb{Z}$ with $i \geq 1$.

Assume $3 \in M^2$. Let $N=(r,1-X^g,1-X^h)$, where r generates M in R and $< g > \oplus < h >= \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. We have

$$N^2 = (r^2, r(1-X^g), r(1-X^h), (1-X^g)(1-X^h), (1-X^g)^2, (1-X^h)^2).$$

Using arguments similar to ones used above it is easy to check that $r(1-X^g), r(1-X^h), (1-X^g)(1-X^h), (1-X^g)^2$ and $(1-X^h)^2$ are required as generators of N^2. Thus $R[\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}]$ does not have the 4−generator property, a contradiction. Consequently, $3 \in M \setminus M^2$ and hence $M=(r)=(3)$.

Now assume $M^2 = (9) \neq 0$. Let $N=(3,1-X^g,1-X^h)$ be the maximal ideal of $R[\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}]$. Consider the ideal $I=(9)+N^3$. Then

$$I = (9, 3(1-X^g), 3(1-X^h), (1-X^g)^2(1-X^h), (1-X^g)(1-X^h)^2).$$

It is easily seen that all these elements are required as generators of I. Thus $R[\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}]$ does not have the 4–generator property, a contradiction. Consequently, $M^2 = 0$.

We claim that $R[\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3^2\mathbb{Z}]$ does not have the four generator property. Let N be its maximal ideal and g, h the generators of $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/3^2\mathbb{Z}$, respectively. Then we have

$$N^2 = (3(1-X^h), (1-X^g)(1-X^h), (1-X^g)^2, (1-X^h)^2)$$
$$N^3 = (3(1-X^g), 3(1-X^h)^2, (1-X^g)^2(1-X^h), (1-X^g)(1-X^h)^2, (1-X^h)^3).$$

If $3(1-X^h)^2$ is a redundant generator of N^3, then by applying the augmentation map $R[<h>][<g>] \longrightarrow R[<h>]$, we get $3(1-X^h)^2 \in (1-X^h)^3R[<h>]$. By [1, Lemma 1.7], $3 = 9\lambda$ for some $\lambda \in R$. Then $M = (3) = (0)$, a contradiction. The arguments for $3(1-X^g), (1-X^g)^2(1-X^h), (1-X^g)(1-X^h)^2$ and $(1-X^h)^3$ are similar to ones used above. Hence $\mu(N^3) > 4$.

$\Leftarrow$] Assume $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $3 \in M \setminus M^2$ and $M^2 = 0$. Let us show that $R[G]$ has the 4–generator property. Let N be the maximal ideal of $R[G]$ and $<g> \oplus <h> = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. We have

$$\begin{aligned} N &= (3, 1-X^g, 1-X^h) \\ N^2 &= ((1-X^g)^2, (1-X^h)^2, (1-X^g)(1-X^h)) \\ N^3 &= (3(1-X^g), 3(1-X^h), (1-X^g)^2(1-X^h), (1-X^g)(1-X^h)^2) \\ N^4 &= (3(1-X^g)^2, 3(1-X^h)^2, 3(1-X^g)(1-X^h), (1-X^g)^2(1-X^h)^2). \end{aligned}$$

Let I be a proper ideal of $R[G]$, we need to prove that I is 4–generated. Applying Lemma 4 to N^4, yields $\mu(I) \leq \mu(I + N^3)$. Since N^3 is 4–generated, we may assume $N^3 \subset I$. Let $x \in I \setminus N^3$. If $x \in N^2$, since N^2 is 3–generated, Lemma 5 implies the desired conclusion. If $x \notin N^2$, by [8,Theorem 159], it follows that $N = (3, x, 1-X^g)$ or $N = (3, x, 1-X^h)$ or $N = (x, 1-X^g, 1-X^h)$. If $N = (3, x, 1-X^g)$ then $N/(x) = (\overline{3}, \overline{1-X^g})$ and $(N/(x))^2 = (\overline{(1-X^g)^2})$, where bars denote images under the natural map $R[G] \rightarrow R[G]/(x)$. As in the proof of Lemma 2, we conclude via part (6) of [11, Theorem 1]. Likewise for $N = (3, x, 1-X^h)$.

If $N = (x, 1-X^g, 1-X^h)$, then $\left(\frac{N}{(x)}\right)^3 = \frac{N^3+(x)}{(x)} \subseteq \frac{I}{(x)}$. We consider separately two cases :

If $\left(\frac{N}{(x)}\right)^3 \subset \frac{I}{(x)}$, choose $z \in I$ such that $\overline{z} \in \frac{I}{(x)} \setminus \left(\frac{N}{(x)}\right)^3$. Assume $\overline{z} \in \left(\frac{N}{(x)}\right)^2$.

Since $\left(\frac{N}{(x)}\right)^2$ is 3–generated, Lemma 5 yields

$$\begin{aligned} \mu\left(\frac{I}{(x,z)}\right) &= \mu\left(\frac{I/(x)}{(\overline{z})}\right) \\ &\leq \mu\left(\frac{N/(x)}{(\overline{z})}\right) \\ &\leq \mu\left(\frac{N}{(x)}\right) \\ &\leq 2. \end{aligned}$$

Therefore I is 4–generated. Now assume $\overline{z} \notin \left(\frac{N}{(x)}\right)^2$. Then

$$\begin{aligned} \mu\left(\frac{N}{(x,z)}\right) &= \mu\left(\frac{N/(x)}{(\overline{z})}\right) \\ &= \mu\left(\frac{N}{(x)}\right) - 1 \\ &= 2-1 \\ &= 1. \end{aligned}$$

Thus $\frac{R[G]}{(x,z)}$ is a principal ideal ring, and hence $\frac{I}{(x,z)}$ is principal. Consequently, I is 4–generated.

If $\left(\frac{N}{(x)}\right)^3 = \frac{N^3+(x)}{(x)} = \frac{I}{(x)}$, then $I = N^3 + (x)$. More precisely,

$$I = (x, 3(1-X^g), 3(1-X^h), (1-X^g)^2(1-X^h), (1-X^g)(1-X^h)^2)$$

$x \in N \setminus N^2$, then $x = 3a + b(1-X^g) + c(1-X^h)$ for some $a, b, c \in R[G]$. Moreover, we may assume that b and c are not units of $R[G]$. Hence there exist $a', b', c', d' \in R[G]$ such that $x = 3a' + b'(1-X^g)^2 + c'(1-X^h)^2 + d'(1-X^g)(1-X^h)$. Clearly, since $x \notin N^2$, a' is a unit. If $b' \in N$, then $3(1-X^g) \in (x, (1-X^g)^2(1-X^h), (1-X^g)(1-X^h)^2)$ since $x(1-X^g) = (a' - b'X^g)3(1-X^g) + c'(1-X^g)(1-X^h)^2 + d'(1-X^g)^2(1-X^h)$. If b' is a unit, then $(1-X^g)^2(1-X^h) \in (x, 3(1-X^h), (1-X^g)(1-X^h)^2)$ since $x(1-X^h) = (a' - c'X^h)3(1-X^h) + b'(1-X^g)^2(1-X^h) + d'(1-X^g)(1-X^h)^2$. In either case, I is 4–generated. Consequently, $R[\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}]$ has the 4–generator property, as we wished to show.

B) Assume that G is a cyclic group ($s = 1$). Let g be the generator of G. To show that $R[G]$ has the 4–generator property, by Lemma 2, it suffices to prove that N, N^2, N^3 and N^4 are 4–generated, where N denotes the maximal ideal of $R[G]$. We have

$$\begin{aligned} N &= (r, 1-X^g) \\ N^2 &= (r^2, r(1-X^g), (1-X^g)^2) \\ N^3 &= (r^3, r^2(1-X^g), r(1-X^g)^2, (1-X^g)^3) \\ N^4 &= (r^4, r^3(1-X^g), r^2(1-X^g)^2, r(1-X^g)^3, (1-X^g)^4). \end{aligned}$$

(i) it is clear that N, N^2, N^3 and N^4 are 4–generated when $M^4 = 0$. Then $R[G]$ has the 4–generator property.

(ii) (a) Assume $M^4 \neq 0$ and $3 \in M^2$. In order to conclude, it suffices to prove that $R[\mathbb{Z}/3\mathbb{Z}]$ has the 4–generator property while $R[\mathbb{Z}/9\mathbb{Z}]$ does not.

Assume $G = \mathbb{Z}/3\mathbb{Z}$. Since $| <g> | = 3$, $r(1-X^g)^3 = -3rX^g(1-X^g) \in (r^3(1-X^g))$. Hence N^4 is 4–generated. It follows that $R[\mathbb{Z}/3\mathbb{Z}]$ has the 4–generator property, as asserted.

Assume $G = \mathbb{Z}/9\mathbb{Z}$. Since $M^4 \neq 0$ and $| <g> | > 4$, it is clear that r^4 and $(1-X^g)^4$ are required as generators of N^4. If $r^3(1-X^g)$ is redundant, then passing to the homomorphic image $(R/(r^4))[< g >]$ and applying [1, Lemma 1.5], yields $r^3 = 0$ in $R/(r^4)$, a contradiction. If $r(1-X^g)^3$ is redundant, then by passing to the homomorphic image $(R/(r^2))[< g >]$, we get $r(1-X^g)^3 \in ((1-X^g)^4)(R/(r^2))[< g >]$, then $r(1-X^g)^8 = 0$ in $(R/(r^2))[< g >]$. So $r = 0$ in $R/(r^2)$, a contradiction. If $r^2(1-X^g)^2$ is redundant, then passing to the homomorphic image $(R/(r^3))[< g >]$ and applying, [1, Lemma 1.7] yields $r^2 = 0$ in $R/(r^3)$, a contradiction. In conclusion, N^4 needs more than four generators, and hence $R[\mathbb{Z}/9\mathbb{Z}]$ does not have the 4-generator property.

b) Assume $M^4 \neq 0$ and $3 \in M \setminus M^2$. Let us prove that $R[\mathbb{Z}/27\mathbb{Z}]$ has the 4-generator property while $R[\mathbb{Z}/81\mathbb{Z}]$ does not.

Assume $G = \mathbb{Z}/27\mathbb{Z}$. Clearly, $N = (3, 1-X^g)$ and $N^4 = (81, 27(1-X^g), 9(1-X^g)^2, 3(1-X^g)^3, (1-X^g)^4)$. We have

$$\begin{aligned}
1 &= (1-X^g+X^g)^{27}\\
&= \sum_{i=o}^{i=27}\binom{27}{i}(1-X^g)^i X^{(27-i)g}\\
&= 1+27(1-X^g)X^{26g}+(27\times 13)(1-X^g)^2X^{25g}+(9\times 13\times 25)(1-X^g)^3X^{24g}\\
&\quad +(1-X^g)^4\left(\sum_{i=4}^{i=27}\binom{27}{i}(1-X^g)^{(i-4)}X^{(27-i)g}\right),
\end{aligned}$$

Then $27(1-X^g) \in (9(1-X^g)^2, 3(1-X^g)^3, (1-X^g)^4)$. Therefore N^4 is 4–generated. Lemma 2 allows us to conclude.

Assume $G = \mathbb{Z}/81\mathbb{Z}$. Using techniques similar to ones used above, one can easily check that $81, 27(1-X^g), 9(1-X^g)^2$ and $(1-X^g)^4$ are required as generators of N^4. Moreover, if $3(1-X^g)^3$ is a redundant generator, then passing to the homomorphic image $(R/(9))[< g >]$, we get $3(1-X^g)^3 = a(1-X^g)^4$ with $a = \sum_{i=o}^{i=80} a_i X^{ig}$, where $a_i \in R/(9)$. Thus setting corresponding terms equal, we obtain a system of 81 linear equations in 81 unknowns. After resolving this system (with the use of a computer), we obtain $1 = 0$ in $R/(9)$, a contradiction. Consequently, $R[\mathbb{Z}/81\mathbb{Z}]$ does not have the 4-generator property, as desired. This completes the proof of Proposition 3. ◇

PROPOSITION 6 *Let (R, M) be a local Artinian principal ideal ring which is not a field, p a prime integer such that $p > 3$ and $p \in M$. Let G be a nontrivial finite p–group. Then $R[G]$ has the 4–generator property if and only if*

(i) G is a cyclic group

(ii) If $M^4 \neq 0$, then $p \notin M^4$ and

(a) $G \cong \mathbb{Z}/p\mathbb{Z}$, if $p \in M^2$

(b) $G \cong \mathbb{Z}/p^i\mathbb{Z}$, where $1 \leq i \leq 3$, if $p \in M \setminus M^2$.

Proof. If $R[G]$ has the 4–generator property, by [14, Proposition 3.5], G is a cyclic group, and if in addition $M^4 \neq 0$ then $G \cong \mathbb{Z}/p^i\mathbb{Z}$ with $i \leq 3$.

Let g be the generator of G and $N = (r, 1-X^g)$ the maximal ideal of $R[G]$. As before, to show that $R[G]$ has the 4–generator property, by Lemma 2 it suffices to prove that N^4 is 4–generated. We have

$$N^4 = (r^4, r^3(1-X^g), r^2(1-X^g)^2, r(1-X^g)^3, (1-X^g)^4).$$

(i) Clearly, if $M^4 = 0$ then N^4 is 4–generated.

(ii) If $M^4 \neq 0$, let us show that $R[\mathbb{Z}/p\mathbb{Z}]$ does not have the 4–generator property when $p \in M^4$. Indeed, it is straightforward to check that r^4 and $(1-X^g)^4$ are required as generators of N^4. If $r^3(1-X^g)$ is redundant, then passing to the homomorphic image $(R/(r^4))[< g >]$ and applying [1, Lemma 1.5], yields $r^3 = \lambda p$ for some $\lambda \in R/(r^4)$. Since $p \in M^4$, $r^3 = 0$ in $R/(r^4)$, a contradiction. If $r^2(1-X^g)^2$ is redundant, then passing to the homomorphic image $(R/(r^3))[< g >]$ and applying [1, Lemma 1.7], yields $r^2 = \lambda p$ for some $\lambda \in R/(r^3)$. Since $p \in M^4$, $r^2 = 0$ in $R/(r^3)$, a contradiction. Finally, If $r(1-X^g)^3$ is redundant, then by passing to the homomorphic image $(R/(r^2))[< g >]$, we get $r(1-X^g)^3 \in ((1-X^g)^4)(R/(r^2))[< g >]$. So $r(1-X^g)^{p-1} \in ((1-X^g)^p) \subset p(R/(r^2))[< g >]$. Since $p \in M^4$, $r(1-X^g)^{p-1} = 0$ in $(R/(r^2))[< g >]$. Therefore $M = (r) = 0$, a contradiction. Thus N^4 needs more than four generators.

a) Suppose $p \in M^2$. Let show that $R[\mathbb{Z}/p\mathbb{Z}]$ has the 4–generator property while $R[\mathbb{Z}/p^2\mathbb{Z}]$ does not. Indeed, assume $G = \mathbb{Z}/p\mathbb{Z}$. Then

$$\begin{aligned}
1 &= (1-X^g+X^g)^p\\
&= \sum_{i=o}^{i=p}\binom{p}{i}(1-X^g)^i X^{(p-i)g}\\
&= 1+p(1-X^g)X^{(p-1)g}+\frac{p(p-1)}{2}(1-X^g)^2X^{(p-2)g}\\
&\quad +\frac{p(p-1)(p-2)}{6}(1-X^g)^3X^{(p-3)g}\\
&\quad +(1-X^g)^4\left(\sum_{i=4}^{i=p}\binom{p}{i}(1-X^g)^{(i-4)}X^{(p-i)g}\right).
\end{aligned}$$

Hence, since $p > 3$, $p(1-X^g) \in (p(1-X^g)^2, (1-X^g)^4)$. If $p \in M^3$, then $M^3 = (r^3) = (p)$. Therefore $r^3(1-X^g) \in (r^2(1-X^g)^2, (1-X^g)^4)$. Otherwise, if $p \in M^2 \setminus M^3$, $M^2 = (r^2) = (p)$. So $r^2(1-X^g) \in (r^2(1-X^g)^2, (1-X^g)^4)$, and hence $r^2(1-X^g)^2 \in (r^2(1-X^g)^3, (1-X^g)^5) \subset (r(1-X^g)^3, (1-X^g)^4)$. Therefore N^4 is 4-generated. Consequently, $R[\mathbb{Z}/p\mathbb{Z}]$ has the 4-generator property, as desired.

Now assume $G = \mathbb{Z}/p^2\mathbb{Z}$. As before, and using the fact that $p \in M^2$, one can easily see that $r^4, r^3(1-X^g), r^2(1-X^g)^2, r(1-X^g)^3$ and $(1-X^g)^4$ are required as generators of N^4. Then $R[\mathbb{Z}/p^2\mathbb{Z}]$ does not have the 4-generator property.

b) Suppose $p \in M \setminus M^2$. It remains to show that $R[\mathbb{Z}/p^3\mathbb{Z}]$ has the 4–generator property. Clearly, $M = (r) = (p)$ and

$$N^4 = (p^4, p^3(1-X^g), p^2(1-X^g)^2, p(1-X^g)^3, (1-X^g)^4).$$

We have

$$
\begin{aligned}
1 = & \ (1-X^g+X^g)^{p^3} \\
= & \sum_{i=o}^{i=p^3}\binom{p^3}{i}(1-X^g)^i X^{(p^3-i)g} \\
= & \ 1+\binom{p^3}{1}(1-X^g)X^{(p^3-1)g}+\binom{p^3}{2}(1-X^g)^2X^{(p^3-2)g} \\
& +\binom{p^3}{3}(1-X^g)^3X^{(p^3-3)g}+(1-X^g)^4\left(\sum_{i=4}^{i=p^3}\binom{p^3}{i}(1-X^g)^{(i-4)}X^{(p^3-i)g}\right).
\end{aligned}
$$

It is straightforward that $p^3(1-X^g) \in (p^3(1-X^g)^2, p^3(1-X^g)^3, (1-X^g)^4) \subset (p^2(1-X^g)^2, p(1-X^g)^3, (1-X^g)^4)$. Hence N^4 is 4–generated. This completes the proof of Proposition 6. ◊

The previous propositions were steps to state the following theorem.

THEOREM *Let R be an Artinian principal ideal ring and G a nontrivial finite abelian group. Then $R[G]$ has the 4–generator property if and only if $R = R_1 \oplus R_2 \oplus \cdots \oplus R_s$ where, for each j, (R_j, M_j) is a local Artinian principal ideal ring subject to :*

(I) *Assume R_j is a field of characteristic $p \neq 0$.*

(α) when $p=2$, then G_p is a homomorphic image of $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2^i\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2^i\mathbb{Z}$ where $i \geq 0$

(β) when $p=3$, then G_p is a homomorphic image of $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3^i\mathbb{Z}$ where $i \geq 0$

(γ) when $p>3$, then G_p is a cyclic group.

(II) *Assume (R_j, M_j) is a principal ideal ring which is not a field and p a prime integer such that p divides Ord(G) and $p \in M_j$*

(α) Assume $p=2$,

A) (i) $G_p \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2^i\mathbb{Z}$ with $i \geq 1$

(ii) when $M_j^2 \neq 0$, then $G_p \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.

B) (i) G_p is a cyclic group

(ii) When $M_j^4 \neq 0$, then

(a) $G_p \cong \mathbb{Z}/2^i\mathbb{Z}$, where $1 \leq i \leq 2$, if $2 \in M_j^2$

(b) $G_p \cong \mathbb{Z}/2^i\mathbb{Z}$, where $1 \leq i \leq 3$, if $2 \in M_j \setminus M_j^2$.

(β) Assume $p=3$,

A) $G_p \cong \mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$, $3 \in M_j \setminus M_j^2$ and $M_j^2 = 0$.

B) (i) G_p is a cyclic group

(ii) When $M_j^4 \neq 0$, then

(a) $G_p \cong \mathbb{Z}/3\mathbb{Z}$, if $3 \in M_j^2$

(b) $G_p \cong \mathbb{Z}/3^i\mathbb{Z}$, where $1 \leq i \leq 3$, if $3 \in M_j \setminus M_j^2$.

(γ) Assume $p>3$,

(i) G_p is a cyclic group

(ii) If $M_j^4 \neq 0$, then $p \notin M_j^4$ and

(a) $G_p \cong \mathbb{Z}/p\mathbb{Z}$, if $p \in M_j^2$

(b) $G_p \cong \mathbb{Z}/p^i\mathbb{Z}$, where $1 \leq i \leq 3$, if $p \in M_j \setminus M_j^2$.

Proof. If R is an Artinian principal ideal ring, then $R = R_1 \oplus \cdots \oplus R_s$, where each (R_j, M_j) is a local Artinian principal ideal ring (cf. [7, Vol.II, Theorem 7.15]). It is easy to see that $R[G]$ has the $n-$generator property if and only if each $R_j[G]$ has the n-generator property.

(I) If R_j is a field, it suffices to apply [1, Remark 1.2 (1)] and [14, Example 2.6].

(II) Assume that R_j is not a field. It is stated in [5, Theorem 19.15] that when the order of G is a unit of R_j and R_j is a principal ideal ring then so is $R_j[G]$. Therefore, we may suppose, without loss of generality, that the order of G is not a unit of R_j. For simplicity, let us denote (R_j, M_j) by (R, M). So Ord$(G) = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s} \in M$, where each p_i is a prime integer. Hence, there exists $p \in \{p_1, p_2, \cdots, p_s\}$ such that $p \in M$. Whence p is the characteristic of R/M . Let $G = G_p \oplus H$, where H is a finite group and p does not divide Ord(H). Clearly, the order of H is a unit of R.

($\Rightarrow$) If $R[G]$ has the 4–generator property, then its homomorphic image $R[G_p]$ does as well. To conclude, it suffices to apply Propositions 1, 3 and 6.

($\Leftarrow$) For the case $G = G_p$, it suffices to apply Propositions 1, 3 and 6. For the general case, $R[G] = R[H][G_p]$. We notice that $R[H]$ is an Artinian ring [5, Theorem 20.7]. By [5, Theorem 19.15], $R[H]$ is a principal ideal ring, and hence $R[H] = A_1 \oplus \cdots \oplus A_q$ where each (A_i, N_i) is a local Artinian principal ideal ring, $1 \leq i \leq q$. Furthermore, $MR[H]$ is equal to the nilradical of $R[H]$ by [5, Corollary 9.18], and for $k \geq 2$, $M^k = 0$ implies that $N_i^k = 0$, for each i (see the proof of [1, Theorem 1]). Consequently, for each i, $A_i[G_p]$ has the 4–generator property by Propositions 1, 3, 6 and [14, Example 2.6]. Hence $R[G]$ has the 4–generator property. ◊

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Quotients of Unit Groups of Commutative Rings

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For a commutative ring R with identity, let $T(R)$ denote its total quotient ring and $U(R)$ its group of units. For an extension of commutative rings $R \subseteq S$ we can form $U(S)/U(R)$, the quotient of the unit groups. In the case where R is an integral domain with quotient field K, then $U(K)/U(R) = K^*/U(R)$ is the *group of divisibility of* R and is denoted by $G(R)$. Here $K^* = K - \{0\}$ is the multiplicative group of K. We will be particularly interested in the following two questions.

(1) When is $U(S)/U(R)$ finite or finitely generated?

(2) When does $U(S)/U(R)$ finite or finitely generated imply that S is a finitely generated R-module?

First, suppose that $K = R \subseteq S = F$ are both fields. Brandis' Theorem [4] or [8, Theorem 4.3.11] answers both questions.

BRANDIS' THEOREM. Let $K \subseteq F$ be a field extension. Then F^*/K^* is finitely generated if and only if (1) $K = F$ or (2) K is finite and $[F : K] < \infty$.

Actually, a stronger result due to L. Avramov and Davis and Maroscia [6] is true. Let $K \subseteq F$ be a field extension and let $r_0(F^*/K^*) = \dim_{\mathbb{Q}}((F^*/K^*) \otimes \mathbb{Q})$ be the torsion-free rank of F^*/K^*. Then the following statements are equivalent: (a) $r_0(F^*/K^*) < \infty$, (b) $r_0(F^*/K^*) = 0$, (c) $\operatorname{char} K = p > 0$ and either F is algebraic over $\mathbb{Z}_p$ or F is purely inseparable over K. For a simpler proof of this result and for a discussion of the group F^*/K^*, the reader is referred to [5].

Here, in the extreme case where $R \subseteq S$ are both fields, $U(S)/U(R)$ is finitely generated if and only if it is finite, and in this case S is a finitely generated R-module.