COMMUNICATIONS IN ALGEBRA, 27(1), 387–403 (1999)

ON THE SPECTRUM OF THE GROUP RING

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Abstract

In this paper we extend the study of the prime ideal structure of group rings initiated by Gilmer (1974), Brewer-Costa-Lady (1975), and Anderson-Bouvier-Dobbs-Fontana-Kabbaj (1988). Of particular interest is the transfer from A to $A[G]$ of certain properties which are linked to the prime spectrum such as S-domain or Jaffard domain. Their study

*This author would like to thank the University of Fès for its hospitality during the preparation of part of this work. Partially supported by the " Acuerdo de Cooperación Hispano-Marroquí ".

will follow the same lines as the usual approach to the study of prime ideal structure in polynomial rings.

1 Introduction

Let A be a commutative ring, G an abelian group, and S a commutative semigroup. The *group-ring* (respectively, *semigroup-ring*) associated to A and G (resp., S), denoted by $A[G]$ (resp. $A[S]$), is the ring of elements of the form $\sum_{g\in G} a_g X^g$ (resp., $\sum_{s\in S} a_s X^s$), where $\{a_g \mid g \in G\}$ (resp., $\{a_s \mid s \in S\}$) is a family of elements of A which are almost all zero.

The aim of this paper is to extend the study of the prime ideal structure of group-rings initiated by Gilmer (1974), Brewer-Costa-Lady (1975), and Anderson-Bouvier-Dobbs-Fontana-Kabbaj (1988). Of particular interest is the transfer from A to $A[G]$ of certain properties which are linked to the prime spectrum. Their study will follow the same lines as the usual approach to the study of prime ideal structure in polynomial rings.

In the second section, we generalize to group-rings several classical results for polynomial rings and deduce from them a more general version of the theorem of Jaffard on the existence of special chains of prime ideals. The results of sections 3 and 4 sheed light on the connections between polynomial rings and group-rings. In Section 3 we study the transfer from A to $A[G]$ of the properties of being a strong S-domain, catenary, and universally catenary. Section 4 concerns the local study of the Jaffard domain property in group rings. (An integral domain A is a *Jaffard domain* if its Krull dimension is equal to its valuative dimension.) The paper concludes with some applications and some new examples in Section 5.

For a given prime ideal P in a commutative ring A we shall denote by $P[X_1, \ldots, X_n]$ the extension of P to the polynomial ring $A[X_1, \ldots, X_n]$ and $\subset$ will always mean proper containment

This paper is concerned with properties which are best studied when $A[G]$ is an integral domain. We then impose the corresponding restrictions to A and G, namely that A is an integral domain and G is a torsion-free abelian group.

We thank D. Dobbs for several suggestions and corrections to the manuscript. A sequel of this paper by S. Ameziane D. E. Dobbs and S. Kabbaj [2] has already appear and may be enlightening to readers.

2 Generalities and a generalization of Jaffard's special chain theorem

This section contains some general results on chains of prime ideals in group-rings. Many of the arguments are simplified by using the following set-up: Let A be a commutative ring and let G be a torsion-free abelian group of finite rank n. Let F be a free abelian subgroup of G such that G/F is a torsion group. Then F is free of rank n and the ring extension $A[F] \hookrightarrow A[G]$ is integral with $A[G]$ a free $A[F]$-module [6, Lemma 1]. Letting $X_1, \ldots, X_n$ be a basis for F, then $X_1, \ldots, X_n$ are indeterminates over A, and $A[F] = A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is a localization of the polynomial ring $C = A[X_1, \ldots, X_n]$.

On the other hand, if we set $Y_i = X_i + X_i^{-1}$, $i = 1, \ldots, n$, then $Y_1, \ldots, Y_n$ are also indeterminates over A. Let $B = A[Y_1, \ldots, Y_n] \subset A[F]$. Since $X_i^2 - Y_iX_i + 1 = 0$, $A[F]$ is integral over B. Moreover, the elements $1, X_1$ are linearly independent over $A[X_1 + X_1^{-1}] = A[Y_1]$, so that $A[X_1, X_1^{-1}]$ is a free $A[Y_1]$-module of rank 2. It follows by induction on n that $A[F]$ is a free B-module of rank 2^n.

Combining these extensions $B \hookrightarrow A[F]$ and $A[F] \hookrightarrow A[G]$, we get that $B \hookrightarrow A[G]$ is an integral extension with $A[G]$ a free B-module. Thus the extension is faithfully flat and Going Down (GD), Going Up (GP), and Lying Over (LO) all hold between them. It follows that $\dim A[G] = \dim A[Y_1, \ldots, Y_n]$. If P is any prime ideal of B, then $PA[G] \cap B = P$. This holds in particular for $P = pB$, where $p \in \operatorname{Spec}(A)$, so that $pA[G] \cap B = pB$. If Q is any prime ideal of $A[G]$ and $P = Q \cap B$, then $\operatorname{ht} Q = \operatorname{ht} P$, Q is maximal if and only if P is maximal, etc.

Note that if $p \in \operatorname{Spec}(A)$, then tensoring $B \hookrightarrow A[G]$ by A/p shows that $(A/p)[G] = A[G]/pA[G]$ is also a free module over and integral extension of $B/pB = (A/p)[Y_1, \ldots, Y_n]$.

Proposition 2.1 *Let A be an integral domain, and let G be a torsion-free abelian group of finite rank n. Then*

(a) If $n = 1$, $\operatorname{ht} p \leq \operatorname{ht} pA[G] \leq 2\operatorname{ht} p$ for all $p \in \operatorname{Spec}(A)$.

(b) For all $P \in \operatorname{Spec}(A[G])$, $\operatorname{ht} P = \operatorname{ht} pA[G] + \operatorname{ht}(P/pA[G])$, where $p = P \cap A$.

(c) For all $P \in \operatorname{Spec}(A[G])$, $\operatorname{ht} P \leq \operatorname{ht} pA[G] + n$, where $p = P \cap A$.

Lemma 2.2 *Let A be an integral domain, and let G be a torsion-free abelian group of rank 1. Then there do not exist prime ideals $Q_1 \subset Q_2 \subset Q_3$ in $A[G]$ such that $Q_1 \cap A = Q_2 \cap A = Q_3 \cap A$.*

Proof. Let F be a rank one free group such that G/F is a torsion group. Then $A[G]$ is integral over $A[F]$ by [6, Lemma 1(b)], and $A[F] = A[X, X^{-1}]$ is integral over $A[X + X^{-1}]$, where X is an indeterminate over A.

Suppose that $Q_1 \subset Q_2 \subset Q_3$ are prime ideals of $A[G]$ such that $Q_1 \cap A = Q_2 \cap A = Q_3 \cap A = p$. Letting $Q_i' = Q_i \cap A[X + X^{-1}]$ for $i = 1, 2, 3$, we have that $Q_1' \subset Q_2' \subset Q_3'$ are distinct because $A[X + X^{-1}]$ is a polynomial ring.

Proof of Proposition 2.1 (a) First note that for each $p \in \operatorname{Spec}(A)$, $pA[G]$ is a prime ideal of $A[G]$ by [9, Proposition 2.9] and that $pA[G] \cap A = p$, since $A[G]$ is a free A-module. It follows that $\operatorname{ht} p \leq \operatorname{ht} pA[G]$. (This also holds if G is of rank $n \geq 1$.)

Let $m = \operatorname{ht} pA[G]$. Then there exist prime ideals $P_1, \ldots, P_m$ of $A[G]$ such that

$$(0) \subset P_1 \subset \cdots \subset P_m = pA[G].$$

Letting $p_i = P_i \cap A$ for $i = 1, \ldots, m$, we have $(0) \subseteq p_1 \subseteq \cdots \subseteq p_m$ and Lemma 2.2 implies that $m \leq 2 \operatorname{ht} p$.

(b) Since G is of rank n, there is a free abelian subgroup F of G such that $\operatorname{rank}(F) = n$ and G/F is a torsion-group. Using the set-up described above, let $B = A[Y_1, \ldots, Y_n]$, where $Y_i = X_i + X_i^{-1}$, and $X_1, \ldots, X_n$ is a basis for F. Then $B \hookrightarrow A[G]$ satisfies going down. If $P \in \operatorname{Spec}(A[G])$, let $P' = P \cap B$ and $p = P \cap A = P' \cap A$. Then $\operatorname{ht} P = \operatorname{ht} P'$ and since $pB = pA[G] \cap B$, $\operatorname{ht} pA[G] = \operatorname{ht} pB$. Applying the same arguments to $A/p[G] = A[G]/pA[G]$ and B/pB, we also have $\operatorname{ht}(P/pA[G]) = \operatorname{ht}(P'/pB)$. We are thus reduced to showing that $\operatorname{ht} P' = \operatorname{ht} pB + \operatorname{ht}(P'/pB)$. But this is true by [7, Theorem 1], since B is a polynomial ring over A.

(c) We now have $\operatorname{ht} P = \operatorname{ht} pA[G] + \operatorname{ht}(P/pA[G])$ and $\operatorname{ht}(P/pA[G]) = \operatorname{ht}(P'/pB)$,
where $P' = P \cap B$ as above. Now $\operatorname{ht}(P'/pB) = \operatorname{ht}(P'B_p/pB_p)$. But $B_p/pB_p = (A_p/pA_p)[Y_1, \ldots, Y_n]$ has dimension n, so $\operatorname{ht}(P'/pB) \leq n$. It follows that $\operatorname{ht} P \leq \operatorname{ht} pA[G] + n$.

Remark 2.3 R. Matsuda in [19, Theorem 7.3] established the same result as in (b) by a totally different (and stronger) way.

Corollary 2.4 *Let A be an integral domain, G a torsion-free abelian group of rank n, and Q a maximal ideal of $A[G]$.*

(a) *If $q = Q \cap A$, then* $\operatorname{ht} Q = \operatorname{ht} qA[G] + n$.

(b) *If* $\operatorname{ht} Q = \dim A[G] < \infty$, *then $q = Q \cap A$ is a maximal ideal of A.*

Proof. (a) Using the same set-up as above, since Q is maximal in $A[G]$, $Q' = Q \cap B$ is maximal in B. Then $Q'B_q/qB_q$ is maximal in $K[Y_1, \dots, Y_n]$, where $K = A_q/qA_q$ is a field. It follows that $\operatorname{ht}(Q/qA[G]) = \operatorname{ht}(Q'/pB) = \operatorname{ht}(Q'B_q/qB_q) = n$. By (2.1 (b)), $\operatorname{ht} Q = \operatorname{ht} qA[G] + n$.

(b) If q is not maximal in A, then there exists $p \in \operatorname{Max}(A)$ such that $q \subset p$, and $\operatorname{ht} qA[G] < \operatorname{ht} pA[G]$. Choose $M \in \operatorname{Max}(A[G])$ such that $pA[G] \subset M$. (Recall that $(A/p)[G]$ is not a field.) Then $\dim A[G] \geq \operatorname{ht} M = \operatorname{ht} pA[G] + n > \operatorname{ht} qA[G] + n = \operatorname{ht} Q = \dim A[G]$, which is absurd.

Corollary 2.5 *Let A be an integral domain of finite dimension, and let G be a torsion-free abelian group of rank n. Then* $\dim A[G] = \max\{\dim A_M[G] \mid M \in \operatorname{Max}(A)\} = n + \max\{\operatorname{ht} MA[G] \mid M \in \operatorname{Max}(A)\}$.

Proof. It is clear from (2.4) that $\dim A_M[G] = n + \operatorname{ht} MA[G]$ for each maximal idal M of A. Since $\dim A[G] \geq \max\{\dim A_M[G] \mid M \in \operatorname{Max}(A)\}$, it suffices to prove the existence of an $M \in \operatorname{Max}(A)$ such that $\dim A[G] = n + \operatorname{ht} MA[G]$. But $A[G]$ is finite-dimensional, and so we may choose $Q \in \operatorname{Max}(A[G])$ such that $\operatorname{ht} Q = \dim A[G]$. Then by (2.4 (b)), $M = Q \cap A$ is the desired maximal ideal.

Recall that a ring A is said to be equicodimensional if all its minimal primes have the same coheight.

Corollary 2.6 *Let A be an integral domain, and G a torsion-free abelian group of rank n. Then $A[G]$ is equicodimensional if and only if the polynomial ring $A[X_1, \dots, X_n]$ is equicodimensional.*

Proof. Letting $B = A[Y_1, \dots, Y_n]$ as in the set-up above, $B \hookrightarrow A[G]$ is a faithfully flat integral extension. It follows that $A[G]$ is equicodimensional if and only if B is.

One says that a ring A satisfies the dimension inequality if for every extension B of A, (A, B) satisfies the dimension inequality.

Corollary 2.7 *Let A be an integral domain, and let $X_1, \ldots, X_n, Y$ be indeterminates over A. The following assertions are equivalent:*

(i) *A satisfies the dimension inequality.*

(ii) *The pair $(A, A[X_1, \ldots, X_n])$ satisfies the dimension formula.*

(iii) $\operatorname{ht} P = \operatorname{ht} P[Y]$, $\forall n \in \mathbb{N}$ *and all* $P \in \operatorname{Spec}(A[X_1, \ldots, X_n])$.

(iv) $\operatorname{ht} p = \operatorname{ht} p[X_1, \ldots, X_n]$, *for all* $p \in \operatorname{Spec}(A)$ *and all* $n \in \mathbb{N}$.

(v) $\operatorname{ht} p = \operatorname{ht} pA[G]$, *for all* $p \in \operatorname{Spec}(A)$, $\forall n \in \mathbb{N}$, *and for every torsion-free abelian group G of rank n.*

(vi) $\operatorname{ht} P = \operatorname{ht} PA[X_1, \ldots, X_n][H]$, $\forall n \in \mathbb{N}$, *for all* $P \in \operatorname{Spec}(A[X_1, \ldots, X_n])$ *and for any torsion-free abelian group H of rank* 1.

(vii) $\operatorname{ht} P = \operatorname{ht} P[Y]$, $\forall n \in \mathbb{N}$, *for all G of rank n and for all* $P \in \operatorname{Spec}(A[G])$.

(viii) $\operatorname{ht} P = \operatorname{ht} PA[G][H]$, $\forall n \in \mathbb{N}$, *for all G of rank n, for all* $P \in \operatorname{Spec}(A[G])$ *and for all H of rank 1.*

Proof. That (i), (ii), (iii), and (iv) are equivalent is [16, Lemma 1.4].

By the proof of (2.1 (b)), $\operatorname{ht} pA[G] = \operatorname{ht} p[X_1, \ldots, X_n]$, whence (v) $\Leftrightarrow$ (iv). That (v) $\Rightarrow$ (vi) is trivial. To see that (vi) $\Rightarrow$ (iii) note that for $P \in \operatorname{Spec}(A[X_1, \ldots, X_n])$, $\operatorname{ht} PA[X_1, \ldots, X_n][H] = \operatorname{ht} P[Y]$ as in the proof of (2.1 (b)).

For $P \in \operatorname{Spec}(A[G])$, $\operatorname{ht} PA[G][H] = \operatorname{ht} P[Y]$ again, so we get (vii) $\Leftrightarrow$ (viii).

To see that (iii) $\Rightarrow$ (vii), let $P \in \operatorname{Spec}(A[G])$, and let $P' = P \cap B$, where $B = A[Y_1, \ldots, Y_n]$ as in the basic set-up above. Then $\operatorname{ht} P = \operatorname{ht} P'$. By (iii), $\operatorname{ht} P' = \operatorname{ht} P'[Y]$. Now $B[Y] \hookrightarrow A[G][Y]$ is integral and a free-module extension and $P'[Y] = P[Y] \cap B[Y]$, so $\operatorname{ht} P'[Y] = \operatorname{ht} P[Y]$. We conclude that $\operatorname{ht} P = \operatorname{ht} P[Y]$ as required. Reversing the argument proves (vii) $\Rightarrow$ (iii), and completes the proof of the theorem.

Let A be a commutative ring, $n \geq 1$ an integer, and $X_1, \ldots, X_n$ indeterminates over A. Let G be a torsion-free abelian group of rank n. A chain $(0) \subset Q_1 \subset \cdots \subset Q_m$ of prime ideals of $A[X_1, \ldots, X_n]$ is called a *special chain* if for each $i = 1, \ldots, m$ there is a $j \in \{0, \ldots, i\}$ such that $(Q_i \cap A)[X_1, \ldots, X_n] = Q_j$. P. Jaffard proved in [15] that for every prime ideal P of $A[X_1, \ldots, X_n]$ there is a special chain realizing the height of P. By analogy, we say that a chain $(0) \subset Q_1 \subset \cdots \subset Q_m$ of prime ideals of $A[G]$ is *special* if for each $i = 1, \ldots, m$ there is a $j \in \{0, \ldots, i\}$ such that $(Q_i \cap A)A[G] = Q_j$. We now give the analogue of Jaffard's special chain theorem for group-rings.

Theorem 2.8 *Let A be an integral domain and G a torsion-free abelian group of rank n. Given a prime ideal P of $A[G]$ of finite height, there exists a special chain in $A[G]$ of length* $\operatorname{ht} P$ *terminating in P.*

Proof. Let $P \in \operatorname{Spec} A[G])$ and let $p = P \cap A$. The proof will be by induction on $s = \operatorname{ht} p$.

If $s = 0$, the result is trivial. So we suppose the theorem true for prime ideals Q of $A[G]$ with $\operatorname{ht}(Q \cap A) \leq s - 1$. Let $r = \operatorname{ht} pA[G]$, and let $(0) \subset P_1 \subset \cdots \subset P_{r-1} \subset P_r = pA[G]$ be a saturated chain of primes in $A[G]$ of length r. Then $\operatorname{ht} P_{r-1} = r - 1$ and if $p_{r-1} = P_{r-1} \cap A$, $\operatorname{ht} p_{r-1} \leq s - 1$. By the induction hypothesis, there is a special chain $(0) \subset P_1' \subset \cdots \subset P_{r-2}' \subset P_{r-1}$ of length $r - 1$ in $A[G]$. Since $\operatorname{ht} P = \operatorname{ht} pA[G] + \operatorname{ht} P/pA[G]$ by (2.1 (b)) we have $(0) \subset P_1' \subset \cdots \subset P_{r-2}' \subset P_{r-1} \subset P_r = pA[G] \subset P_{r+1} \subset \cdots \subset P_m = P$, where $m = \operatorname{ht} P$. This is a special chain in $A[G]$.

3 Some results of transfer

With the object of constructing new classes of universal strong S-domains and universally catenary rings, we pursue the study of the transfer of these notions to group-rings.

Theorem 3.1 *Let A be an integral domain, and let G be a torsion-free abelian group of rank $n \geq 1$. Then $A[G]$ is an S-domain.*

Proof. If F is a free subgroup of G with G/F torsion, then $A[F] \hookrightarrow A[G]$ is a free-module extension and satisfies GD. By [18, Theorem 4.9] it suffices to show that $A[F]$ is an S-domain. But $A[F]$ is a localization of a polynomial ring in $\operatorname{rank} G$ indeterminates over A. If $\operatorname{rank} G = n < \infty$ and $n \geq 1$, then $A[X_1, \ldots, X_n]$, and hence $A[F]$, is an S-domain by [11, Proposition 2.1]. If rank $G = \infty$, $A[X_1, X_2, \ldots]$ is an S-domain by [10, Corollary 2.13], and hence $A[F]$ is an S-domain in this case also.

Proposition 3.2 *Let A be an integral domain and let G be a torsion free abelian group of finite rank n. Then*

(a) $A[G]$ is strong S-domain implies that $A[X_1, \ldots, X_n]$ is strong S-domain.

(b) $A[G]$ is catenary implies that $A[X_1, \ldots, X_n]$ is catenary.

Proof. Let B as in the basic set-up. Then $B \hookrightarrow A[G]$ is integral and has GD, being $B \simeq A[X_1, \ldots, X_n]$. For (a) it suffices to apply [18, Theorem 4.6].

(b) Assume that $A[G]$ is catenary. Let $P_1 \subset P_2$ be consecutive prime ideals in B. Since $B \hookrightarrow A[G]$ satisfies GU there exist $Q_1 \subset Q_2$ prime ideals in $A[G]$ such that $P_i = Q_i \cap B$ for $i = 1, 2$. [18, Lemma 4.1] implies then that $Q_1 \subset Q_2$ are consecutive prime ideals in $A[G]$. By the assumption, $\operatorname{ht} Q_2 = \operatorname{ht} Q_1 + 1$. Therefore, $\operatorname{ht} P_2 = \operatorname{ht} P_1 + 1$ since $B \hookrightarrow A[G]$ is integral and satisfies GD, which implies $\operatorname{ht} P_i = \operatorname{ht} Q_i$ for $i = 1, 2$. Furthermore, $A[G]$ locally of finite dimension implies B is. Consequently, B is catenary.

Theorem 3.3 *Let A be an integral noetherian ring. The following are equivalent:*

(i) A is universally catenary.

(ii) $A[G]$ is catenary for all finite rank torsion-free abelian groups G.

(iii) $A[G]$ is universally catenary for all finite rank torsion-free abelian groups G.

(iv) $A[G]$ is catenary for some non-trivial trosion-free abelian group G of finite rank.

(v) $A[\mathbb{Z}]$ is catenary.

Proof. (i) $\Rightarrow$ (ii) Let $Q_1 \subset Q_2$ be consecutive prime ideals in $A[G]$. Set $P_i = Q_i \cap B$ for $i = 1, 2$, where B is as in the basic set-up, and let $S = B \setminus P_2$. $B_{P_2} \hookrightarrow S^{-1}A[G]$ is an integral extension, and B_{P_2} is a local noetherian domain. Since A is universally catenary, $B_{P_2}[X]$ is catenary, and [21, remark A.11] implies that B_{P_2} is a GB domain. Consequently, $\operatorname{ht} P_2/P_1 = \operatorname{ht} S^{-1}P_2/S^{-1}P_1 =$

1 ($\text{ht}\, S^{-1}Q_2/S^{-1}Q_1 = \text{ht}\, Q_2/Q_1 = 1$, $S^{-1}Q_i \cap B_{P_2} = S^{-1}P_i$ for $i = 1,2$, and B_{P_2} is a GB-domain), and $\text{ht}\, P_2 = \text{ht}\, P_1 + 1$ because of B is catenary. On the other hand, $\text{ht}\, Q_i = \text{ht}\, P_i$ for $i = 1,2$, thus $\text{ht}\, Q_2 = \text{ht}\, Q_1 + 1$. Taking into account that B is locally of finite dimension (A is universally catenary), we finally get $A[G]$ is a catenary domain.

(ii) $\Rightarrow$ (i) comes from (3.2), while the equivalence with (iii) results from the isomorphism $A[G][X_1,\ldots,X_n] \simeq A[X_1,\ldots,X_n][G]$.

The implications (ii) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (v) are evident.

(iv) $\Rightarrow$ (i) Since $A[G]$ is catenary for some non-trivial torsion-free group of finite rank, $A[X_1,\ldots,X_n]$ is catenary by (3.2), where $n = \text{rank}\, G$ ($n \geq 1$). Then by [22] A is universally catenary because of A is noetherian and $A[X]$ is catenary.

(v) $\Rightarrow$ (i) $A[\mathbb{Z}]$ catenary implies that $A[X]$ is catenary (3.2). Since A is noetherian and $A[X]$ is catenary, [22] implies A is universally catenary.

Proposition 3.4 *Let A be an integral domain of dimension 1 and let G be a torsion-free abelian group of rank 1.*

(a) $A[G]$ is strong S-domain if and only if $A[X]$ is strong S-domain.

(b) If A is noetherian, $A[G]$ is a universal strong S-domain.

Proof. (a) ($\Rightarrow$) comes from (3.2 (a)). ($\Leftarrow$) Since $\dim A[G] = \dim A[X] = 2$ ([13, Corollary 1]) and $A[G]$ is S-domain by (3.1), [18, Theorem 4.5] implies that $A[G]$ is strong S-domain.

(b) It suffices to apply the set-up and [18, Proposition 4.20].

Example 3.5 Let K be a field and $n \in \mathbb{N}$. Then the rings $K[\mathbb{Q} \oplus \mathbb{Z}^n]$ and $\mathbb{Z}[\mathbb{Q} \oplus \mathbb{Z}^n]$ are universally catenary.

Definition 3.6 Let A be a commutative ring.

(a) A satisfies the *first chain condition* (f.c.c.) if every maximal chain of prime ideals of A has length equal to the Krull dimension of A.

(b) A satisfies the *second chain condition* (s.c.c.) if for every prime ideal P of A and every integral domain B integral over A/P, every maximal chain of prime ideals of B has length $\dim A$.

Corollary 3.7 *Let A be an integral noetherian domain which is universally catenary, and let G be a trosion-free abelian group of rank n. Then $A[G]$*

satisfies the f.c.c. if and only if $A[X_1,\ldots,X_n]$ *satisfies the f.c.c..*

Proof. This follows from (2.6) and (3.3) since the f.c.c. is equivalent to being both catenary and equicodimensional.

Example 3.8 If K is a field, $K[\mathbb{Q}]$ and $\mathbb{Z}[\mathbb{Q}]$ satisfy the f.c.c..

Proposition 3.9 *Let* A *be an integral domain and* G *a torsion-free abelian group of rank* n*. Then* $A[G]$ *satisfies the s.c.c. if and only if* $A[X_1,\ldots,X_n]$ *does.*

Proof. If $A[G]$ satisfies the s.c.c. then so does B because $A[G]$ is integral over B [21, (1.3.4)]. Conversely, any integral domain integral over $A[G]$ is integral over B and $\dim A[G] = \dim B$, so if B satisfies the s.c.c. $A[G]$ does, too.

Example 3.10 For any field K, $K[\mathbb{Q}\oplus\mathbb{Z}^n]$ satisfies the s.c.c., for all $n\in\mathbb{N}$.

4 Local study of the Jaffard condition for group-rings

The notion of a Jaffard domain is not stable under localization and one says that A is *locally Jaffard* if A_p is a Jaffard domain for each prime ideal p of A [1]. In fact, locally Jaffard domains are none other than the domains satisfying the dimension inequality [16]. By the same token, Jaffard domains are not stable under quotient and so P.-J. Cahen in [8] introduced the following two concepts: We say that A is *residually Jaffard* if A/p is Jaffard for every prime ideal p of A. We call A *totally Jaffard* if A_p is residually Jaffard for each prime ideal p of A (equivalently, if A/p is locally Jaffard for each prime ideal p of A).

In this section we study the transfer of these notions to group-rings.

Theorem 4.1 *Let* A *be an integral domain of finite dimension, and let* G *be a torsion-free abelian group of rank* n*.*

(a) $A[G]$ *is locally Jaffard if and only if* $A[X_1,\ldots,X_n]$ *is.*

(b) $A[G]$ *is residually Jaffard if and only if* $A[X_1,\ldots,X_n]$ *is.*

(c) $A[G]$ *totally Jaffard implies* $A[X_1,\ldots,X_n]$ *is.*

Lemma 4.2 *Let* A *be an integral domain of finite dimension.*

(a) *A is locally Jaffard if and only if $S^{-1}A$ is a Jaffard domain for every multiplicative system S of A.*

(b) *A is totally Jaffard if and only if $S^{-1}A$ is residually Jaffard for each multiplicative system S of A.*

Proof. (a) [5, Proposition 2.5(a)].

(b) ($\Leftarrow$) Trivial. ($\Rightarrow$) If $S^{-1}A$ is not residually Jaffard for some multiplicatively closed set S, there is a $Q \in \operatorname{Spec}(A)$ such that $Q \cap S = \emptyset$ and $S^{-1}A/S^{-1}Q$ is not Jaffard. Consequently, there is a valuation ring V such that $S^{-1}A/S^{-1}Q \subseteq V \subsetneq \operatorname{Frac}(A/Q)$ and $\dim S^{-1}A/S^{-1}Q < \dim V$. Let M be the maximal ideal of V. Then $M \cap A/Q = P/Q$, where $P \in \operatorname{Spec}(A)$, $P \cap S = \emptyset$, and $Q \subset P$. Now $\dim A_P/QA_P = \operatorname{ht} P/Q = \operatorname{ht} S^{-1}P/S^{-1}Q \leq \dim S^{-1}A/S^{-1}Q < \dim V$ and $A_P/QA_P \subset V \subsetneq \operatorname{Frac}(A_P/QA_P)$. Thus, A_P/QA_P is not a Jaffard domain, contradicting the assumption that A is totally Jaffard.

Lemma 4.3 *Let $A \hookrightarrow B$ be an integral extension of integral domains with* $\dim A < \infty$.

(a) *B is a Jaffard domain if and only if A is Jaffard.*

(b) *B is residually Jaffard if and only if A is residually Jaffard.*

(c) *B is locally Jaffard implies A is locally Jaffard.*

(d) *B is totally Jaffard implies that A is totally Jaffard.*

Proof. (a) See [1].

(b) For each $Q \in \operatorname{Spec}(B)$, B/Q is integral over $A/Q \cap A$. And, for each $q \in \operatorname{Spec}(A)$ there is a $Q \in \operatorname{Spec}(B)$ with $q = Q \cap A$. Then (a) applied to $A/q \hookrightarrow B/Q$ proves (b).

(c) Let $p \in \operatorname{Spec}(A)$ and let $S = A \setminus p$. Then $S^{-1}B$ is integral over A_p and $S^{-1}B$ is Jaffard by (4.2(a)), so A_p is Jaffard by (a).

(d) Let $p \in \operatorname{Spec}(A)$, and let $S = A \setminus p$. Since $S^{-1}B$ is integral over A_p, and $S^{-1}B$ is residually Jaffard by (4.2(b)), A_p is residually Jaffard by part (b). This shows that A is totally Jaffard.

Remark 4.4 Recall that the converse of (c) is false, in general, [3, Example III-4.18].

Proof of Theorem 4.1 Let $B = A[Y_1, \ldots, Y_n]$ be as in the basic set-up, so that $B \simeq A[X_1, \ldots, X_n]$ and $B \hookrightarrow A[G]$ is integral with $A[G]$ a free B-module.

(a) ($\Rightarrow$) It follows from (4.3(c)) that B is locally Jaffard.

($\Leftarrow$) By [16, Lemma 1.4] (or 2.7), it suffices to prove that for each $m \in \mathbb{N}$ and for each $P \in \operatorname{Spec}(A[G][X_1, \ldots, X_m])$, $\operatorname{ht} P = \operatorname{ht} P[Y]$. From the isomorphisms $A[G][X_1, \ldots, X_n] \simeq A[X_1, \ldots, X_n][G]$ and $A[G][X_1, \ldots, X_m][Y] \simeq A[X_1, \ldots X_m][Y][G]$, we can suppose that $P \in \operatorname{Spec}(A[X_1, \ldots, X_m][G])$.

Let $A' = A[X_1, \ldots X_m]$, $p = P \cap A'$, and $p' = P[Y] \cap A'[Y] = (P \cap A')[Y] = p[Y]$. By (2.1(b)), $\operatorname{ht} P = \operatorname{ht} pA'[G] + \operatorname{ht} P/pA'[G]$, $\operatorname{ht} P[Y] = \operatorname{ht} pA'[Y][G] + \operatorname{ht} P[Y]/pA'[Y][G]$. On the other hand, $A[Y_1, \ldots, Y_n]$ satisfies the dimension inequality [16], hence $A'[Y_1, \ldots, Y_n]$ too [8, Proposition 1 i)], and $\operatorname{ht} pA'[G] = \operatorname{ht} p[Y_1, \ldots, Y_n]$, $\operatorname{ht} pA'[Y][G] = \operatorname{ht} p[Y][Y_1, \ldots, Y_n]$ (see the proof of (2.1(b)).

Furthermore, $\operatorname{ht} p[Y_1, \ldots, Y_n] = \operatorname{ht} p[Y][Y_1, \ldots, Y_n]$ by [16, Lemma 1.4]. Consequently, $\operatorname{ht} pA'[G] = \operatorname{ht} pA'[Y][G]$. Likewise we have $\operatorname{ht} P/pA'[G] = \operatorname{ht} P[Y]/pA'[Y][G]$, since $P/pA'[G] \in \operatorname{Spec}((A'/p)[G])$ and $P[Y]/pA'[Y][G] \in \operatorname{Spec}(A'/p[G][Y])$.

Let $T = A'/p \setminus \{0\}$. Then, $P/pA'[G] \cap T = \emptyset$ and $P[Y]/pA'[G][Y] \cap T = \emptyset$. Thus $\operatorname{ht} P/pA'[G] = \operatorname{ht} T^{-1}(P/pA'[G])$ and $\operatorname{ht} P[Y]/pA'[G][Y] = \operatorname{ht} T^{-1}(P[Y]/pA'[G][Y]) = \operatorname{ht} T^{-1}(P/pA'[G])[Y]$.

On the other hand, $T^{-1}(P/pA'[G]) \in \operatorname{Spec}(K[G])$, where $K = \operatorname{Frac}(A'/p)$. Since $K[G]$ is universally catenary (3.3), it is universal strong S-domain. Consequently, $\operatorname{ht} T^{-1}(P/pA'[G]) = \operatorname{ht} T^{-1}(P/pA'[G])[Y]$. Therefore $\operatorname{ht} P/pA'[G] = \operatorname{ht} P[Y]/pA'[G][Y] = \operatorname{ht} P[Y]/p[Y]A'[G]$, and $\operatorname{ht} P = \operatorname{ht} P[Y]$ for each $m \in \mathbb{N}$ and each $P \in \operatorname{Spec}(A[G][X_1, \ldots, X_m])$. That is, $A[G]$ is locally Jaffard.

(b) This is just an application of (4.3(b)) to $B \hookrightarrow A[G]$.

(c) Comes from (4.3(d)) applied to $B \hookrightarrow A[G]$.

5 Applications and construction of new examples

In this section we construct new classes of locally Jaffard domains, and universally catenary rings, different from ones previously known.

Proposition 5.1 *(a) For all $n \geq 2$ there exists a factorial, non-noetherian, universally catenary domain of dimension n and characteristic $p \geq 0$.*

(b) *For all $n \geq 2$ there exists a local, factorial, non-noetherian, universally catenary domain of dimension n and characteristic $p > 0$.*

(c) *For all $n \geq 2$ there exists a local, factorial, non-noetherian, universally catenary domain of dimension n and characteristic 0.*

Proof. (a) We use the construction in [5]. Let P be the Pontryagin group. Thus, P is a torsion-free abelian group of rank 2 in which every rank 1 subgroup is cyclic. For $n \geq 2$, let $G_n = P \oplus \mathbb{Z}^{n-2}$, and let K be a field of characteristic $p \geq 0$. The ring $K[G_n]$ is the desired example. It is universally catenary by (3.3). The other properties are established in [5].

(b) We use the construction in [13]. Let $A = K[G_n]_M$, where K is a field of characteristic $p > 0$ and M is the maximal ideal of $K[G_n]$ generated by $\{1 - X^g \mid g \in G_n\}$, the "augmentation ideal". That A is factorial and non-noetherian it is shown in [13, Theorem 4]. Since $K[G_n]$ is universally catenary by (3.3), A is universally catenary.

(c) We use the construction in [6]. For $n \geq 3$, let $r = n - 1$. There is a group L_r of rank r such that $\mathbb{Z}[L_r]$ is factorial, equicodimensional, of dimension n, characteristic 0, and which has a maximal ideal M such that $\mathbb{Z}[L_r]_M$ is non-noetherian [6]. The ring $\mathbb{Z}[L_r]$ is universally catenary by 3.3, so $\mathbb{Z}[L_r]_M$ is the desired example.

For $n = 2$, we reproduce the construction in [6, pp. 306–307]. Let p be a prime integer, $\mathbb{Z}[\frac{1}{p}] = \{a/p^n \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$, $G = \mathbb{Z}[\frac{1}{p}] \oplus \mathbb{Z}[\frac{1}{p}] \oplus \mathbb{Z}[\frac{1}{p}]$, $H = \mathbb{Z}[\frac{1}{p}] \oplus \mathbb{Z}[\frac{1}{p}]$ a subgroup of G, and $G_n = H \oplus p^{-n}\mathbb{Z}$. It is clear that $G = \cup_{n \geq 0} G_n$ and that $\mathbb{Z}[G] = \cup_{n \geq 0}\mathbb{Z}[G_n] = \cup_{n \geq 0}\mathbb{Z}[H][p^{-n}\mathbb{Z}]$. ($\mathbb{Z}[G_n] = \mathbb{Z}[H][p^{-n}\mathbb{Z}]$ by [14, Theorem 7.1].)

Let $\delta_1, \delta_2, \ldots$ be a sequence of elements of $\mathbb{Z}[H] \setminus \{0\}$ such that $cl(\delta_n) \neq 0$ in $\mathbb{Z}/p\mathbb{Z}[H]$, where $cl(\delta_n)$ denotes the class of δ_n modulo $p\mathbb{Z}[H]$ and $cl(\delta_n)^p = cl(\delta_{n-1})$ for all $n \geq 2$. (The sequence $\delta_n = X^{(p^{-n},0)} + X^{(0,p^{-n})}$ will do.) Then $P_n = (p, \delta_n + Y^{p^{-n}})$ is a prime ideal of $\mathbb{Z}[H][Y^{p^{-n}}, Y^{-p^{-n}}]$ which we identify with $\mathbb{Z}[H][p^{-n}\mathbb{Z}]$, and $P = \cup_{n \geq 0} P_n$ is a prime ideal of $\mathbb{Z}[G]$.

Letting $A = [G]_p$, it is shown in [6, pp. 306–307] that A is factorial, non-noetherian, local of dimension 2, and characteristic 0. In addition, A is universally catenary since $\mathbb{Z}[G]$ is by (3.3).

Example 5.2 Set $V = \mathbb{Q}(\sqrt{2})[[X]] = \mathbb{Q}(\sqrt{2}) + X\mathbb{Q}(\sqrt{2})[[X]]$, and $R = \mathbb{Q} + X\mathbb{Q}(\sqrt{2})[[X]]$. Then by [2, Corollary 2.3] and (3.3), the ring $R[\mathbb{Q}]$ is universally catenary.

We now consider a construction of M. Nagata [20]. Let K be a field. Let $W = K[X,Y]_{(X-1,Y)}$, and let V be a noetherian valuation ring of $K(X,Y)$ containing $K[X,Y]$ with maximal ideal P such that $P \cap K[X,Y] = (X,Y)$ and $V/P = K$. Set $T = V \cap W$. Then T is semi-local with two maximal ideals M, N.

Example 5.3 Let $A = K + M \cap N$. Then

(a) $A[\mathbb{Q}]$ is non-noetherian.

(b) $A[\mathbb{Q}]$ is locally and residually Jaffard.

(c) $A[\mathbb{Q}]$ is not catenary.

Proof. (a) $A[\mathbb{Q}]$ is not noetherian because $\mathbb{Q}$ is not a finitely generated group [12, Theorem 20.7].

(b) A is noetherian [20] and so is universal strong S-domain. Then $A[X]$ is locally and residually Jaffard. Therefore $A[\mathbb{Q}]$ is locally and residually Jaffard (4.1, (a), (b)).

(c) Nagata showed in [20] that A is not universally catenary. Since A is noetherian, $A[X]$ cannot be catenary. (For by [22], that would make A universally catenary.) Then (3.2) shows that $A[\mathbb{Q}]$ is not catenary.

Example 5.4 Let $V = K(X)[Y]_{(Y)} = K(X) + YK(X)[Y]_{(Y)}$, and $R = K + YK(X)[Y]_{(Y)}$. Then

(a) $R[\mathbb{Q}]$ is locally Jaffard.

(b) $R[\mathbb{Q}]$ is not residually Jaffard.

(c) $R[\mathbb{Q}]$ is not a strong S-domain.

(d) $R[\mathbb{Q}]$ is not catenary.

Proof. (a) We have $\dim R = \dim K + \dim V = 1$ and $\dim_v R = \dim_v K + \dim_v V + \text{tr.deg}_k K(X) = 2$ by [5, Proposition 2.1]. Then [8, Proposition 1(ii)] implies that $R[X]$ is locally Jaffard and (4.1(a)) shows that $R[\mathbb{Q}]$ is locally Jaffard.

(b) We have $\omega\mathbb{Q} \in \operatorname{Spec}(R[\mathbb{Q}])$ [9, Proposition 1] and $R[\mathbb{Q}]/\omega\mathbb{Q} \simeq R$, which is not Jaffard since $\dim_v R > \dim R$. Thus, $R[\mathbb{Q}]$ is not residually Jaffard.

(c) If $R[\mathbb{Q}]$ were a strong S-domain, then (3.2(a)) would show that $R[X]$ is a strong S-domain. From this would follow that R is a strong S-domain, which is absurd since $K(X)/K$ is transcendental [18, Theorem 5.1].

(d) If $R[\mathbb{Q}]$ is catenary, then so is $R[X]$ by (3.2(b)), and hence R is a strong S-domain by [4, Lemma 2.3]. This is the same contradiction as in the proof of (c).

Example 5.5 Let $T = \mathbb{Q}(i)[X]$ and let $R = \mathbb{Z} + X\mathbb{Q}(i)[X]$. For each torsion-free abelian group G with finite rank, $R[G]$ is residually and locally Jaffard.

Proof. By [17, Corollaire 1.8], R is universal strong S-domain. Then for each n, $R[X_1, \ldots, X_n]$ is locally and residually Jaffard. By (4.1(a), (b)) we then have that $R[G]$ is locally and residually Jaffard for each finite rank torsion-free abelian group.

We end this paper with the following conjectures:

Conjecture 5.6 $A[X_1, \ldots, X_n]$ is strong S-domain implies that $A[G]$ is strong S-domain.

Conjecture 5.7 $A[X_1, \ldots, X_n]$ is catenary implies that $A[G]$ is catenary.

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Received: June 1997

Revised: January 1998