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MATLIS' SEMI-REGULARITY IN TRIVIAL RING EXTENSIONS OF INTEGRAL DOMAINS

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Abstract. This paper contributes to the study of homological aspects of trivial ring extensions (also called Nagata idealizations). Namely, we investigate the transfer of the notion of (Matlis') semi-regular ring (also known as IF-ring) along with related concepts, such as coherence, to trivial ring extensions of integral domains. All along the paper, we provide new families of examples subject to semi-regularity.

1. Introduction. Throughout, all rings considered are commutative with identity and all modules are unital. A ring R is *coherent* if every finitely generated ideal of R is finitely presented. The class of coherent rings includes strictly the classes of Noetherian rings, *von Neumann regular rings* (i.e., every module is flat), valuation rings, and *semi-hereditary rings* (i.e., every finitely generated ideal is projective). During the past three decades, the concept of coherence developed towards a full-fledged topic in commutative algebra under the influence of homology; and several notions grew out of coherence (e.g., finite conductor property, quasi-coherence, v-coherence, and n-coherence). For more details on coherence see [18, 19], and for coherent-like properties see, for instance, [26, 27].

In 1982, Matlis proved that a ring R is coherent if and only if $\hom_R(M, N)$ is flat for any injective R-modules M and N [31, Theorem 1]. In 1985, he defined a ring R to be *semi-coherent* if $\hom_R(M, N)$ is a submodule of a flat R-module for any injective R-modules M and N. Then, inspired by this definition and von Neumann regularity, he defined a ring to be *semi-regular* if any module can be embedded in a flat module (or equivalently, if every injective module is flat) [32]. He then proved that semi-regularity is a local property in the class of coherent rings [32, Proposition 2.3]. Moreover, he proved that in the class of reduced rings, von Neumann regularity reduces

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to semi-regularity [32, Proposition 2.7]; and under Noetherian assumption, semi-regularity equals the self-injective property; i.e., R is quasi-Frobenius if and only if R is semi-regular and Noetherian [32, Proposition 3.4]. Beyond Noetherian settings, examples of semi-regular rings arise as factor rings of Prüfer domains over non-zero finitely generated ideals [32, Proposition 5.3]. It is worth noting, at this point, that semi-regular rings were briefly mentioned by Sabbagh (1971) in [43, Section 2] and studied in non-commutative settings by Jain (1973) in [25], Colby (1975) in [9], and Facchini & Faith (1995) in [15], among others, where they were always termed IF-rings. Also, they were extensively studied (under IF terminology) in (commutative) valuation settings by Couchot [10–12]. Finally, recall that an R-module E is *fp-injective* (or *absolutely pure*) if $\operatorname{Ext}^1_R(M, E) = 0$ for every finitely presented R-module M [17, IX-3]; and R is *self fp-injective* if it is fp-injective over itself. Also, R is semi-regular if and only if R is self fp-injective and coherent ([25, Theorem 3.10] or [9, Theorem 2]).

For a ring A and an A-module E, the *trivial ring extension* of A by E is the ring $R := A \ltimes E$ where the underlying group is $A \times E$ and multiplication is defined by $(a, e)(b, f) = (ab, af + be)$. The ring R is also sometimes called the (*Nagata*) *idealization* of E over A and denoted by $A\,(+)\,E$. This construction was first introduced, in 1962, by Nagata [33] in order to facilitate interaction between rings and their modules, and also to provide various families of examples of commutative rings containing zero-divisors. The literature abounds on trivial extensions dealing with the transfer of ring-theoretic notions in various settings (see, for instance, [1, 3, 13, 16, 20–22, 28, 29, 36–41, 44]). For more details on commutative trivial extensions (or idealizations), we refer the reader to Glaz's and Huckaba's books [18, 24], and also to Anderson & Winders relatively recent and comprehensive survey [2].

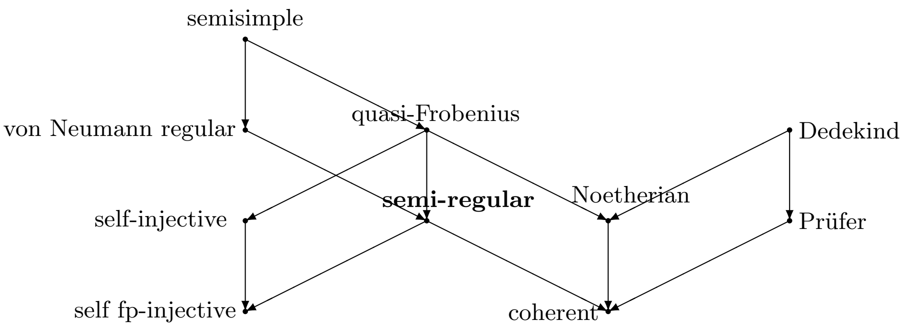

Fig. 1. A ring-theoretic perspective for semi-regularity

This paper contributes to the study of homological aspects of trivial ring extensions. Namely, we investigate the transfer of the notion of (Matlis') semi-regular ring (also known as IF-ring) along with related concepts, such as coherence, to trivial ring extensions of integral domains. All along the paper, we provide new families of examples subject to semi-regularity.

For the reader's convenience, Figure 1 displays a diagram of implications summarizing the relations among the main notions involved in this work.

2. Main result. We investigate the transfer of semi-regularity to trivial ring extensions of domains. We first state some preliminary results which will make up the proof of the main result of this paper (Theorem 2.10).

Recall that a module over a domain is *divisible* if each element of the module is divisible by every non-zero element of the domain [42]. The first lemma asserts that fp-injectivity and, a fortiori, divisibility of the module E are necessary conditions for the trivial extension $A \ltimes E$ to be semi-regular.

LEMMA 2.1. *Let A be a ring, E an A-module, and $R := A \ltimes E$. If R is self fp-injective, then E is fp-injective. In particular, if A is a domain and R is semi-regular, then E is divisible.*

Proof. Let $M := \sum_{i=1}^{n} Am_i$ be a finitely generated submodule of A^n for some positive integer n, and let $f : M \to E$ be an A-map. One can identify R^n with $A^n \ltimes E^n$ as R-modules under natural scalar multiplication. Consider the finitely generated submodule of R^n given by $N := \sum_{i=1}^{n} R(m_i, 0)$ along with the R-maps

$$N \overset{p}{\twoheadrightarrow} M \overset{f}{\to} E \overset{u}{\hookrightarrow} R$$

where p is defined by

$$p\Big(\sum_{i=1}^{n}(a_i, e_i)(m_i, 0)\Big) = \sum_{i=1}^{n} a_i m_i$$

and u is the canonical embedding. Then $g := u \circ f \circ p$ extends to R^n as $\overline{g}$, since R is self fp-injective. It follows that f extends to the A-map

$$\overline{f} : A^n \overset{i}{\hookrightarrow} R^n \overset{\overline{g}}{\to} R \overset{\pi}{\twoheadrightarrow} E$$

where i is the canonical embedding and π is the canonical surjection. Therefore, E is fp-injective [17, Theorem IX-3.1]. The second statement of the lemma is straightforward since a semi-regular ring is self fp-injective; and an fp-injective module is divisible. ∎

REMARK 2.2. The second statement of the lemma is still valid if A is an arbitrary ring (i.e., possibly with zero-divisors) and divisibility of E is taken over all non-zero-divisors of A.

The next lemma shows that divisibility of the module E controls the finitely generated ideals of the trivial extension $R := A \ltimes E$.

Lemma 2.3. *Let A be a domain, E a divisible A-module, and $R := A \ltimes E$. Then, for any finitely generated ideal $\mathcal{I}$ of R, either $\mathcal{I} = I \ltimes E$ for some non-zero finitely generated ideal I of A, or $\mathcal{I} = 0 \ltimes E'$ for some finitely generated submodule E' of E.*

Proof. First, note that if E' is a finitely generated submodule of E, then $0 \ltimes E'$ is a finitely generated ideal of R. Also, let $I := \sum_{i=1}^{n} Aa_i$ with $0 \neq a_i \in A$ for all i, and let $e \in E$. Then, by divisibility, $e = a_1 e'$ for some $e' \in E$, and hence $(0, e) = (a_1, 0)(0, e')$. It follows that

$$I \ltimes E = \sum_{i=1}^{n} (a_i, 0)R.$$

That is, $I \ltimes E$ is a finitely generated ideal of R.

Next, let $\mathcal{I} = \sum_{i=1}^{n} (x_i, e_i)R$ with $x_i \in A$ and $e_i \in E$ for $i = 1, \ldots, n$. If $x_i = 0$ for all i, then

$$\mathcal{I} = \sum_{i=1}^{n} 0 \ltimes Ae_i = 0 \ltimes E'$$

with $E' := \sum_{i=1}^{n} Ae_i$, as desired. Next, assume the x_i's are not all null and (relabeling if necessary) let $r \in \{1, \ldots, n\}$ be such that $x_i \neq 0$ for $i \leq r$ and $x_i = 0$ for $i \geq r + 1$. We claim that $\mathcal{I} = I \ltimes E$ with $I := \sum_{i=1}^{r} Ax_i$. Indeed, for all $i \in \{1, \ldots, r\}$ and $j \in \{r + 1, \ldots, n\}$, we have

$$(x_i, e_i)R \subseteq Ax_i \ltimes (Ex_i + Ae_i) \subseteq I \ltimes E, \qquad (x_j, e_j)R = 0 \ltimes Ae_j \subseteq I \ltimes E,$$

so that $\mathcal{I} \subseteq I \ltimes E$. For the reverse inclusion, let $z := (\sum_{i=1}^{r} a_i x_i, e) \in I \ltimes E$. We can write

$$z := (a_1 x_1, e) + \sum_{i=2}^{r} (a_i x_i, 0).$$

So, it suffices to show that $(a_i x_i, e) \in (x_i, e_i)R$ for any given $e \in E$ and $i \in \{1, \ldots, r\}$. This holds if there is $e' \in E$ such that

$$e = x_i e' + a_i e_i.$$

Indeed, recall that E is divisible and suppose $e = 0$. If $a_i e_i = 0$, take $e' := 0$; and if $a_i e_i \neq 0$, then $a_i e_i = x_i e'_i$ for some $e'_i \in E$ and hence take $e' := -e'_i$. Suppose $e \neq 0$ and let $e = x_i e''_i$ for some $e''_i \in E$. If $a_i e_i = 0$, take $e' := e''_i$; and if $a_i e_i \neq 0$, take $e' := e''_i - e'_i$, proving the claim. ∎

Remark 2.4. Notice that the converse of the above lemma is always true; namely, if all finitely generated ideals of R have the two aforementioned forms, then E is divisible. For, let x be a non-zero element of A. Then $(x, 0)R = xA \ltimes xE$ is a finitely generated ideal of R with $xA \neq 0$, which forces $E = xE$.

Next, we examine the transfer of coherence to trivial extensions of domains by divisible modules. We will use Fuchs–Salce's definition of a coherent module: all finitely generated submodules are finitely presented [17, Chapter IV] (i.e., the module itself does not have to be finitely generated). In Bourbaki, such a module is called "pseudo-coherent" [7] and Wisbauer calls it "locally coherent" [45].

We first isolate the simple case when A is trivial. Namely, if $A := k$ is a field and E is a k-vector space, then a combination of [27, Theorem 2.6] and [2, Theorem 4.8] shows that *$k \ltimes E$ is coherent if and only if $k \ltimes E$ is Noetherian if and only if $\dim_k E < \infty$.* The next result handles the case when A is a non-trivial domain.

Proposition 2.5. *Let A be a domain which is not a field, E a divisible A-module, and $R := A \ltimes E$. Then R is coherent if and only if A is coherent, E is torsion coherent, and $\mathrm{Ann}_E(x)$ is finitely generated for all $x \in A$.*

Proof. Assume R is coherent. Then so are its retract A by [18, Theorem 4.1.5] and E by [18, remark following Theorem 4.4.4]. Now, assume there is a torsion-free element $e \in E$ and let $0 \neq a \in A$. Then

$$\mathrm{Ann}_R(0, e) = \mathrm{Ann}_A(e) \ltimes E = 0 \ltimes E$$

is a finitely generated ideal of R. So E is a finitely generated A-module. Let $e_1, \dots, e_n$ be a minimal generating set for E. By divisibility, we obtain $e_1 = a\sum_{i=1}^n a_i e_i$ for some $a_1, \dots, a_n \in A$. If $1 - aa_1 \neq 0$, then

$$e_1 = (1 - aa_1)\sum_{i=1}^n b_i e_i$$

for some $b_1, \dots, b_n \in A$, forcing

$$e_1 \in \sum_{i=2}^n Ae_i,$$

which is absurd. So, necessarily, $1 - aa_1 = 0$. It follows that A is a field, the desired contradiction. Hence, E is a torsion module. Finally, let $0 \neq x \in A$. Then $\mathrm{Ann}_R(x, 0) = 0 \ltimes \mathrm{Ann}_E(x)$ is finitely generated in R. So $\mathrm{Ann}_E(x)$ is a finitely generated submodule of E.

Conversely, we first show that the intersection of any two finitely generated ideals of R is finitely generated. Let I_1 and I_2 be non-zero finitely generated ideals of A, and let E_1 and E_2 be finitely generated submodules of E. Since A is a coherent domain, $I_1 \cap I_2$ is a non-zero finitely generated ideal of A. By Lemma 2.3,

$$(I_1 \ltimes E) \cap (I_2 \ltimes E) = (I_1 \cap I_2) \ltimes E$$

is a finitely generated ideal of R. Further, obviously,

$$(I_1 \ltimes E) \cap (0 \ltimes E_1) = 0 \ltimes E_1$$

is finitely generated. Moreover, since E is coherent, $E_1 \cap E_2$ is a finitely generated submodule of E [17, (D), p. 128]. Hence,

$$(0 \ltimes E_1) \cap (0 \ltimes E_2) = 0 \ltimes (E_1 \cap E_2)$$

is a finitely generated ideal of R. In view of Lemma 2.3, we are done. By [18, Theorem 2.3.2(7)], it remains to show that $\mathrm{Ann}_R(x, e)$ is finitely generated for any $(x, e) \in R$. Indeed, if $x \neq 0$, then

$$\mathrm{Ann}_R(x, e) = 0 \ltimes \mathrm{Ann}_E(x)$$

is finitely generated in R (since by hypothesis $\mathrm{Ann}_E(x)$ is finitely generated). Next, assume $x = 0$. In view of the exact sequence

$$0 \to \mathrm{Ann}_A(e) \to A \to Ae \to 0,$$

since E is torsion coherent, $\mathrm{Ann}_A(e)$ is a non-zero finitely generated ideal of A. By Lemma 2.3,

$$\mathrm{Ann}_R(0, e) = \mathrm{Ann}_A(e) \ltimes E$$

is a finitely generated ideal of R, completing the proof of the proposition. ■

In the above result, the assumption that $\mathrm{Ann}_E(x)$ is finitely generated for all $x \in A$ is not superfluous in the presence of the other assumptions, as shown by the next example. Throughout, for a domain A, $Q(A)$ will denote its quotient field.

Example 2.6. Let A be a coherent domain which is not a field (e.g., any non-trivial Prüfer domain) and $E := \bigoplus_{n \geq 0} E_n$ with $E_n := Q(A)/A$. Then E is a divisible coherent A-module [17, (C), p. 37 & (B), p. 128], and clearly E is torsion. However, the condition "$\mathrm{Ann}_E(x)$ is finitely generated for all $x \in A$" does not hold. Indeed, let x be any non-zero non-unit element of A. Then one can easily check that

$$\mathrm{Ann}_E(x) = \bigoplus_{n \geq 0} \overline{(1/x)},$$

which is not finitely generated.

In order to proceed further, we need to extend, to A-modules, Matlis' double annihilator condition in a ring A; i.e., $\mathrm{Ann}_A(\mathrm{Ann}_A(I)) = I$ for each finitely generated ideal I of A [32, Section 4, Definition].

Definition 2.7. Let A be a ring. An A-module E is said to satisfy the *double annihilator condition* (for short, DAC) if the following two assertions hold:

(DAC1) $\mathrm{Ann}_A(\mathrm{Ann}_E(I)) = I$ for every finitely generated ideal I of A.
(DAC2) $\mathrm{Ann}_E(\mathrm{Ann}_A(E')) = E'$ for every finitely generated submodule E' of E.

Obviously, this definition coincides with Matlis' double annihilator condition when $E = A$. Moreover, all these conditions are unrelated in general, as shown by the following basic examples.

EXAMPLE 2.8. Let A be a ring and E a non-zero A-module.

(1) Assume $A := K$ is a field. Then E satisfies (DAC1). Moreover, E satisfies (DAC2) if and only if $\dim_K(E) = 1$. Indeed, the first statement is straightforward, and the second holds as $\mathrm{Ann}_E(\mathrm{Ann}_K(e)) = E$ for any non-zero $e \in E$.
(2) Assume $(A, \mathfrak{m})$ is local and $E := A/\mathfrak{m}$. Then E satisfies (DAC2). Moreover, E satisfies (DAC1) if and only if $l(\mathfrak{m}) = 1$. Indeed, the first statement is clear since E has no non-zero proper submodules. The second statement holds since $\mathrm{Ann}_A(\mathrm{Ann}_E(x)) = \mathfrak{m}$ for any $x \in \mathfrak{m}$.
(3) Assume A satisfies Matlis' double annihilator condition (e.g., is semi-regular) and E has a torsion-free element. Then E satisfies (DAC) if and only if $E \cong A$. This is so because $\mathrm{Ann}_E(\mathrm{Ann}_A(e)) = E$ for any given torsion-free element $e \in E$.

We also need the next lemma which characterizes the double annihilator condition in a trivial ring extension via the (DAC) property of its divisible module.

LEMMA 2.9. *Let A be a domain, E a divisible A-module, and $R := A \ltimes E$. Then R satisfies Matlis' double annihilator condition if and only if E satisfies* (DAC).

Proof. First, notice that $\mathrm{Ann}_A(\mathrm{Ann}_E(0)) = \mathrm{Ann}_A(E) = 0$, since $aE = E$ when $0 \neq a \in A$. Now, by Lemma 2.3, the finitely generated ideals of R have the forms $I \ltimes E$ or $0 \ltimes E'$, where I is a non-zero finitely generated ideal of A and E' is a finitely generated submodule of E. Moreover, one can easily check that

$$\mathrm{Ann}_R(I \ltimes E) = 0 \ltimes \mathrm{Ann}_E(I), \qquad \mathrm{Ann}_R(0 \ltimes E') = \mathrm{Ann}_A(E') \ltimes E.$$

It follows that

$$\mathrm{Ann}_R(\mathrm{Ann}_R(I \ltimes E)) = \mathrm{Ann}_A(\mathrm{Ann}_E(I)) \ltimes E,$$
$$\mathrm{Ann}_R(\mathrm{Ann}_R(0 \ltimes E')) = 0 \ltimes \mathrm{Ann}_E(\mathrm{Ann}_A(E')),$$

leading to the conclusion. ∎

Finally, we are ready to state the main theorem of this section on the transfer of semi-regularity to trivial ring extensions.

THEOREM 2.10. *Let A be a domain and E an A-module. Then $R := A \ltimes E$ is semi-regular if and only if either A is a field with $E \cong A$, or A is a coherent domain, E is a divisible* (*resp., fp-injective*) *torsion coherent module which satisfies* (DAC), *and $\mathrm{Ann}_E(x)$ is finitely generated for all $x \in A$.*

Proof. Let us first isolate the simple case when A is trivial. Namely, let $A := k$ be a field and E a non-zero k-vector space. Then, by Example 2.8(1), $\dim_k E = 1$ if and only if $k \ltimes E$ satisfies (DAC) if and only if $k \ltimes E$ is semi-regular. Now, assume that A is a domain which is not a field, and combine Lemma 2.1, Proposition 2.5, and Lemma 2.9 with Matlis' result that a ring is semi-regular if and only if it is coherent and satisfies the double annihilator condition (on finitely generated ideals) [32, Proposition 4.1]. ∎

At this point, recall that a non-zero fractional ideal I of a domain A is *disorial* if $I = I_v := (I^{-1})^{-1}$. A domain is called divisorial if all its non-zero (fractional) ideals are divisorial. Divisorial domains have been studied by, among others, Bass [4] and Matlis [30] for the Noetherian case, Heinzer [23] for the integrally closed case, Bastida–Gilmer [5] in the transfer to $D + M$ constructions, and Bazzoni [6] in more general settings. It is worth recalling that a domain in which all finitely generated ideals are divisorial is not necessarily divisorial [6, Example 2.11]. Finally, recall that a domain A is *totally divisorial* if every overring of A is a divisorial domain; and A is *stable* if every non-zero ideal of A is projective over its ring of endomorphisms [17, 35]. A domain A is totally divisorial if and only if A is a stable divisorial domain [35, Theorem 3.12].

As an application of Theorem 2.10, the next corollary will provide new families of examples subject to semi-regularity. If I and J are (fractional) ideals of a domain A, let

$$(I : J) = \{x \in Q(A) \mid xJ \subseteq I\}, \qquad (I :_A J) = \{a \in A \mid aJ \subseteq I\}.$$

COROLLARY 2.11. *Let A be a coherent domain which is not a field and I a non-zero finitely generated fractional ideal of A. Then:*

(1) *$A \ltimes (Q(A)/I)$ is semi-regular if and only if $(I : (I : J)) = J$ for each non-zero finitely generated (fractional) ideal J of A.*
(2) *In particular, $A \ltimes (Q(A)/A)$ is semi-regular if and only if each non-zero finitely generated (fractional) ideal of A is divisorial.*

Proof. (1) First, notice that $Q(A)$ is a coherent A-module since it is torsion-free [17, IV-2, Lemma 2.5]. Further, given any exact sequence $0 \to M' \to M \to M'' \to 0$ of modules over a coherent ring, if any two of M', M, M'' are finitely presented, then so is the third [17, IV-2, Exercise 2.5]. It follows that $E := Q(A)/I$ is coherent, with I regarded as a finitely generated submodule of $Q(A)$. Moreover, E is clearly a divisible torsion module, and $\mathrm{Ann}_E(x) = \overline{(1/x)I}$ for any non-zero $x \in A$. Therefore, by Theorem 2.10, $A \ltimes E$ is semi-regular if and only if E satisfies (DAC). So, we just need to prove the following claim.

CLAIM. *$Q(A)/I$ satisfies (DAC) if and only if $(I : (I : J)) = J$ for each non-zero finitely generated (fractional) ideal J of A.*

Indeed, assume $(I : (I : J)) = J$ for each non-zero finitely generated (fractional) ideal J of A. Note first that for $J := A$, we get

$$A = (I : (I : A)) = (I : I).$$

Next, let $\overline{J}$ be a non-zero finitely generated submodule of E; that is, J is a non-zero finitely generated fractional ideal of A containing I. Then $(I : J) \subseteq (I : I) = A$, and hence

$$\mathrm{Ann}_A(\overline{J}) = A \cap (I : J) = (I :_A J) = (I : J).$$

Moreover, let K be a non-zero finitely generated ideal of A. Then

$$\mathrm{Ann}_E(K) = \overline{(I : K)}.$$

Therefore, since $KI \subseteq I$, we obtain

$$\mathrm{Ann}_A(\mathrm{Ann}_E(K)) = \mathrm{Ann}_A\big(\overline{(I : K)}\big) = (I : (I : K)) = K$$

and

$$\mathrm{Ann}_E(\mathrm{Ann}_A(\overline{J})) = \overline{(I : (I :_A J))} = \overline{(I : (I : J))} = \overline{J}$$

proving the "if" assertion.

Conversely, assume that E satisfies (DAC), and let $0 \neq a \in A$ be such that $aI \subseteq A$. Since $Q(A)/aI \cong Q(A)/I$ as A-modules and $(aI : (aI : J)) = (I : (I : J))$ for each J, we may assume without loss of generality that I is an (integral) ideal of A. Then (DAC2), applied to $J := A$, yields

$$\overline{A} = \mathrm{Ann}_E(\mathrm{Ann}_A(\overline{A})) = \overline{(I : (I :_A A))} = \overline{(I : I)},$$

so that $A = (I : I)$. Now, let J be a non-zero finitely generated ideal of A. Then, via the basic fact $I \subseteq (I : J)$, (DAC1) yields

$$J = \mathrm{Ann}_A(\mathrm{Ann}_E(J)) = \mathrm{Ann}_A\big(\overline{(I : J)}\big) = (I :_A (I : J)) = (I : (I : J)),$$

completing the proof of (1).

(2) Straightforward via (1) with $I := A$ and the fact $(A : (A : J)) = J_v$. ∎

The above proof reveals that $A \ltimes (Q(A)/I)$ is semi-regular if and only if $Q(A)/I$ satisfies (DAC). So, let A be a coherent domain which is not a field and I a non-zero finitely generated fractional ideal of A. By Lemma 2.1, if $Q(A)/I$ satisfies (DAC), then it is fp-injective. We do not know if the converse holds in general.

A von Neumann regular ring is a reduced semi-regular ring [32, Proposition 2.7]. Matlis noticed that "(von Neumann) regular rings and quasi-Frobenius rings are seen to have a common denominator of definition—they are both extreme examples of semi-regular rings." Next, we provide various examples of semi-regular trivial ring extensions which are neither von Neumann regular (being non-reduced) nor quasi-Frobenius (being non-Noetherian).

Example 2.12. Let A be a coherent domain which is not a field and let $R := A \ltimes (Q(A)/A)$. Note that R is not Noetherian since $Q(A)/A$ is not finitely generated.

(1) Assume A is integrally closed. Then

$$R \text{ is semi-regular } \Leftrightarrow \ A \text{ is Prüfer.}$$

Indeed, combine Corollary 2.11 with the fact that every invertible ideal is divisorial and Krull's result that an integrally closed domain in which all non-zero finitely generated ideals are divisorial is Prüfer (cf. [23, proof of Theorem 5.1]). For an example, take A to be any non-trivial Prüfer domain (e.g., $A := \mathbb{Z} + X\,\mathbb{Q}[X]$).

(2) If A is a divisorial domain, then R is semi-regular by Corollary 2.11. For an example, take A to be any pseudo-valuation domain issued from a valuation domain (V, M) with M finitely generated and $[V/M : k] = 2$. Then A is a (non-integrally-closed) divisorial domain [5, Theorem 2.1 & Corollary 4.4], which is coherent ([14, Theorem 3] or [8, Theorem 3]).

(3) Next, we provide a non-integrally-closed non-divisorial domain A in which every finitely generated ideal is divisorial; and hence R is semi-regular by Corollary 2.11. Indeed, let D be a non-integrally-closed pseudo-valuation domain which is divisorial and coherent (e.g., take D to be the domain A of (2) above) and let K be its quotient field. By [34, Theorem 2.6], D is not stable and hence not totally divisorial by [35, Theorem 3.12]. Let V be a valuation domain of the form $K + M$ and let $A := D + M$. Then A is a non-integrally-closed non-divisorial domain [5, Theorem 2.1 & Corollary 4.4] which is coherent ([14, Theorem 3] or [8, Theorem 3]). Moreover, since D is divisorial, every finitely generated ideal of A is divisorial by [5, Theorems 2.1(k) & 4.3].

Other examples stem from Prüfer domains via Corollary 2.11. For instance, for any Prüfer domain A and non-zero finitely generated (fractional) ideal I of A, the trivial ring extension $A \ltimes (Q(A)/I)$ is semi-regular. Indeed, let J be a non-zero finitely generated ideal of A. Then the basic facts $(IJ^{-1})J \subseteq I$ and $J(I : J) \subseteq I$ yield $(I : J) = IJ^{-1}$. It follows that $(I : (I : J)) = (I : IJ^{-1}) = I(IJ^{-1})^{-1} = J_v = J$, as desired.

Observe that for an example of a module E which is not of the form $Q(A)/I$, one may appeal to non-standard uniserial modules. From [17, X-3], a uniserial module over a valuation domain with quotient field Q is *standard* if it is isomorphic to J/I for some ideals $0 \subseteq I \subseteq J \subseteq Q$. A uniserial module is *non-standard* if it is not isomorphic to such a quotient. In this connection, recall that torsion-free uniserial modules are necessarily standard. Next, by [17, Example VII-4.1 & Theorem X-4.5 & following comment], let A be a valuation domain for which there exists a divisible non-standard uniserial module E whose non-zero elements have principal annihilators. Then the trivial ring extension $R := A \ltimes E$ is a chained ring that is not a homomorphic image of a valuation domain [17, Theorem X-6.4]. Moreover, by [10, Theorem 10], R is semi-regular: Indeed, let $0 \neq e$ be a non-zero torsion element of E with $\mathrm{Ann}_A(e) = aA$ for some $0 \neq a \in A$. Since E is divisible, it is easily seen that $\mathrm{Ann}_R(0, e) = \mathrm{Ann}_A(e) \ltimes E = aA \ltimes E = (a, 0)R$, as desired.

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