PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 2, October 1988

UNIVERSALLY CATENARIAN DOMAINS OF $D+M$ TYPE

DAVID F. ANDERSON, DAVID E. DOBBS, SALAH KABBAJ AND S. B. MULAY

(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. Let T be a domain of the form $K+M$, where K is a field and M is a maximal ideal of T. Let D be a subring of K and let $R = D+M$. It is proved that if K is algebraic over D and both D and T are universally catenarian, then R is universally catenarian. The converse holds if K is the quotient field of D. As a consequence, we construct for each $n > 2$, an n-dimensional universally catenarian domain which does not belong to any previously known class of universally catenarian domains.

1. Introduction. All rings considered below are (commutative integral) domains. A ring A is said to be *catenarian* if, for each pair $P \subset Q$ of prime ideals of A, all saturated chains of primes from P to Q have a common finite length. Following [**3**], we say that A is *universally catenarian* if the polynomial rings $A[X_1, \ldots, X_n]$ are catenarian for each positive integer n. The main purpose of this note is to construct new examples of universally catenarian domains.

Any Cohen-Macaulay ring is universally catenarian. Moreover, it is known [**15**, (2.6)] that a Noetherian ring A is universally catenarian if (and only if) $A[X]$ is catenarian. Moving beyond the Noetherian context, note that each catenarian A must be locally finite-dimensional (LFD), in the sense that each prime ideal of A has finite height. It is known [**14**, **12**, p. 256, **5**] that each LFD Prüfer domain is universally catenarian. More generally, it was shown in [**4**, Theorem 1] that each LFD going-down strong S-domain (in the sense of [**12**]) must be universally catenarian. In addition, [**4**, Theorem 2] established that each LFD domain of global dimension 2 is universally catenarian. (As explained in [**4**, pp. 863-864], this assertion does not carry over to global dimension 3.) In §3, we construct for each integer $n \geq 2$, an n-dimensional universally catenarian domain which is not of any of the above types.

The constructions in §3 depend on work in §2 that studies universal catenarity for rings of the form $D+M$. Here, M is a maximal ideal of a ring $K+M$, where K is a field and D is a subring of K. Theorem 2.2 characterizes universal catenarity in case K is the quotient field of D. A useful sufficient condition is given in Corollary 2.3, and Corollary 2.4 characterizes universal catenarity for the classsical $D+M$ construction [**10**] in which $K+M$ is a valuation domain.

Received by the editors October 5, 1987. Presented to the Society, March 25, 1988 (Knoxville, Tennessee).

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 13C15, 13G05; Secondary 13B25, 13A17, 13F05, 13D05, 13E05, 13B30, 14A05, 12F05.

The third author was supported in part by NATO Collaborative Research Grant GR. 85/0035 and the University of Tennessee Science Alliance. He thanks the University of Tennessee for its warm hospitality during July–October, 1987.

2. Universal catenarity and the $D+M$ construction. We begin with a useful result that is analogous to various gluing criteria in [**5**, **4**, **1**]. It will be convenient to say that I is *S-saturated* if S is a multiplicatively closed subset of a ring A and I is an ideal of A such that $A \cap S^{-1}I = I$. Note that if I is an S-saturated proper ideal, then $I \cap S = \varnothing$.

LEMMA 2.1. *Let S be a multiplicatively closed subset of a domain A and I an S-saturated ideal of A. Let P be a prime ideal of A which contains I. Then there exists Q in* $\operatorname{Spec}(A)$ *such that $I \subset Q \subset P$ and $Q \cap S = \varnothing$.*

PROOF. We claim that IA_P is an S-saturated ideal of A_P; in other words, if $u \in A_P \cap S^{-1}IA_P$, then $u \in IA_P$. To see this, note that there exists $z \in A \setminus P$ such that $zu \in A \cap S^{-1}I$. By hypothesis, $zu \in I$. Hence, $u = (zu)z^{-1} \in IA_P$, as claimed.

By the above comment, it follows that $IA_P \cap S = \varnothing$. Hence (cf. [**10**, Lemma 2.5]), there exists a prime ideal W of A_P such that $IA_P \subset W$ and $W \cap S = \varnothing$. Then $Q = W \cap A$ has the asserted properties. □

We next set up riding hypotheses and notation for the rest of §2. Let T be a domain of the form $K+M$, where K is a field and M is a (nonzero) maximal ideal of T. Let D be a subring of K. Let k be the quotient field of D (inside K) and let $R = D + M$.

We are interested in knowing when R is universally catenarian. The next result answers this completely in case $k = K$.

THEOREM 2.2. *Suppose that K is the quotient field of D. Then R is universally catenarian if and only if both T and D are universally catenarian.*

PROOF. The "only if" assertion follows from the fact that the class of universally catenarian domains is closed under localization and factor domains [**3**, Corollary 3.3]. The point is that $R/M \cong D$; and, if $S = D \setminus \{0\}$, then $S^{-1}R = k + M = K + M = T$.

Conversely, suppose that both T and D are universally catenarian. Hence, both are LFD. We claim that R is LFD. To see this, note first that R is the pullback of the inclusion map $D \to K$ and the canonical projection $T \to K$. Accordingly, by [**9**, Theorem 1.4], $\operatorname{Spec}(R)$ can be characterized up to homeomorphism. The order-theoretic upshot is that, as a poset, $\operatorname{Spec}(R)$ is obtained by "gluing" $\operatorname{Spec}(D)$ onto $\operatorname{Spec}(T)$ in such a way that $\{0\} \in \operatorname{Spec}(D)$ coincides with $M \in \operatorname{Spec}(T)$. In particular, R is LFD.

It follows that $A = R[X_1, \ldots, X_n]$ is also LFD for each positive integer n. To prove that A is catenarian, we consider $P_0 \subset \cdots \subset P_s = P$, any saturated chain of $s+1$ distinct primes in A.

Suppose $M[X_1, \ldots, X_n] \subset P_0$. It will suffice to show $\operatorname{ht}(P/P_0) = s$. Note that

$$P_0/M[X_1, \ldots, X_n] \subset \cdots \subset P_s/M[X_1, \ldots, X_n]$$

is a saturated chain of distinct primes in the catenarian domain $A/M[X_1, \ldots, X_n] \cong D[X_1, \ldots, X_n]$. Now, it is easy to see that if $J_1 \subset J_2$ are primes of a catenarian domain, then $\operatorname{ht}(J_2) - \operatorname{ht}(J_1) = \operatorname{ht}(J_2/J_1)$. Thus,

$$\operatorname{ht}(P/M[X_1, \ldots, X_n]) - \operatorname{ht}(P_0/M[X_1, \ldots, X_n]) = \operatorname{ht}(P/P_0).$$

Since

$$\mathrm{ht}(P/M[X_1,\ldots,X_n]) = \mathrm{ht}(P_0/M[X_1,\ldots,X_n]) + s,$$

we conclude that $\mathrm{ht}(P/P_0) = s$, as required in this case.

Suppose next that $M[X_1,\ldots,X_n] \not\subset P_0$. It will suffice to show $\mathrm{ht}(P) - \mathrm{ht}(P_0) = s$. Let $S = D \setminus \{0\}$. We claim that P_0 is S-saturated. Indeed, consider $u \in A \cap S^{-1}P_0$; we shall show that $u \in P_0$. Choose $z \in S$ such that $zu \in P_0$. As z is a unit in T, we have $zM = M$, whence $uM[X_1,\ldots,X_n] = zuM[X_1,\ldots,X_n]$ is contained in P_0. Since P_0 is prime, it follows that $u \in P_0$, as claimed.

Since P_0 is S-saturated, $P_0 \cap S = \varnothing$. There exists an integer r such that $0 \le r \le s$, $P_r \cap S = \varnothing$ and, if $r < s$, then $P_{r+1} \cap S \ne \varnothing$. Defining

$$I = P_r + M[X_1,\ldots,X_n],$$

we have

$$S^{-1}I = S^{-1}P_r + S^{-1}M[X_1,\ldots,X_n] = S^{-1}P_r + M[X_1,\ldots,X_n].$$

Thus,

$$A \cap S^{-1}I = (A \cap S^{-1}P_r) + M[X_1,\ldots,X_n] = P_r + M[X_1,\ldots,X_n] = I;$$

that is, I is an S-saturated ideal of A.

Suppose, for the moment, that $r < s$. Pick $d \in P_{r+1} \cap S$ and observe that

$$M = d(d^{-1}M) \subset dM \subset P_{r+1}.$$

It follows that P_{r+1} contains I. Hence, Lemma 2.1 may be applied, yielding $Q \in \mathrm{Spec}(A)$ such that $I \subset Q \subset P_{r+1}$ and $Q \cap S = \varnothing$. As $Q \ne P_{r+1}$ and $\mathrm{ht}(P_{r+1}/P_r) = 1$, we have $Q = P_r$. In particular, P_r contains $M[X_1,\ldots,X_n]$. Viewing the chain induced in the catenarian domain $A/M[X_1,\ldots,X_n] \cong D[X_1,\ldots,X_n]$, we conclude that

$$\mathrm{ht}(P/M[X_1,\ldots,X_n]) - \mathrm{ht}(P_r/M[X_1,\ldots,X_n]) = s - r.$$

We claim that $\mathrm{ht}(P) - \mathrm{ht}(P_r) = s - r$. By the above comments, it suffices to show

$$\mathrm{ht}(P) - \mathrm{ht}(P_r) = \mathrm{ht}(P/M[X_1,\ldots,X_n]) - \mathrm{ht}(P_r/M[X_1,\ldots,X_n]).$$

Setting $p = P \cap R$ and $p_r = P_r \cap R$, we infer from [**10**, Theorem 30.18] that

$$\mathrm{ht}(P) = \mathrm{ht}(p[X_1,\ldots,X_n]) + \mathrm{ht}(P/p[X_1,\ldots,X_n])$$

and

$$\mathrm{ht}(P_r) = \mathrm{ht}(p_r[X_1,\ldots,X_n]) + \mathrm{ht}(P_r/p_r[X_1,\ldots,X_n]).$$

Now, viewing R as the pullback of $D \to K$ and $T \to K$, we see via [**1**, Corollary 2.12] that R is a locally Jaffard domain, in the sense of [**1**]. It follows that $\mathrm{ht}(p) = \mathrm{ht}(p[X_1,\ldots,X_n])$, with a similar assertion for $\mathrm{ht}(p_r)$. Hence, by the three previously displayed equations, the claim will follow if we show

$$\begin{aligned}\mathrm{ht}(p) - \mathrm{ht}(p_r) &+ \mathrm{ht}(P/p[X_1,\ldots,X_n]) - \mathrm{ht}(P_r/p_r[X_1,\ldots,X_n]) \\ &= \mathrm{ht}(P/M[X_1,\ldots,X_n]) - \mathrm{ht}(P_r/M[X_1,\ldots,X_n]).\end{aligned}$$

Now, since $A/M[X_1,\ldots,X_n] \cong D[X_1,\ldots,X_n]$ is catenarian,

$$\begin{aligned}\mathrm{ht}(P/p[X_1,\ldots,X_n]) &= \mathrm{ht}((P/M[X_1,\ldots,X_n])/(p[X_1,\ldots,X_n]/M[X_1,\ldots,X_n])) \\ &= \mathrm{ht}(P/M[X_1,\ldots,X_n]) - \mathrm{ht}(p[X_1,\ldots,X_n]/M[X_1,\ldots,X_n]).\end{aligned}$$

A similar rewriting of $\mathrm{ht}(P_r/p_r[X_1,\ldots,X_n])$ is possible.

Moreover, $p[X_1,\ldots,X_n]/M[X_1,\ldots,X_n]$ may be viewed as $(p/M)[X_1,\ldots,X_n]$ in the (universally catenarian, hence stably strong $S-$) domain $D[X_1,\ldots,X_n]$. Thus, $\mathrm{ht}(p[X_1,\ldots,X_n]/M[X_1,\ldots,X_n]) = \mathrm{ht}(p/M)$; a similar assertion holds with p_r replacing p. Hence, the claim will follow if we show

$$\begin{aligned}\mathrm{ht}(p) - \mathrm{ht}(p_r) &+ (\mathrm{ht}(P/M[X_1,\ldots,X_n]) - \mathrm{ht}(p/M)) \\ &- (\mathrm{ht}(P_r/M[X_1,\ldots,X_n]) - \mathrm{ht}(p_r/M)) \\ &= \mathrm{ht}(P/M[X_1,\ldots,X_n]) - \mathrm{ht}(P_r/M[X_1,\ldots,X_n]).\end{aligned}$$

This, in turn, will follow if $\mathrm{ht}(p/M) = \mathrm{ht}(p) - \mathrm{ht}(M)$ and $\mathrm{ht}(p_r/M) = \mathrm{ht}(p_r) - \mathrm{ht}(M)$. But *these* equations do hold. The reason is that R is catenarian. Since T and D are catenarian, *this* fact follows from the earlier order-theoretic description of $\mathrm{Spec}(R)$. The upshot is that we have proved the claim, $\mathrm{ht}(P) - \mathrm{ht}(P_r) = s - r$, in case $r < s$. Since this equation reduces to $0 = 0$ in case $r = s$, the claim has been established.

Finally, consider the saturated chain $S^{-1}P_0 \subset \cdots \subset S^{-1}P_r$ of distinct primes in the catenarian domain $S^{-1}A = (S^{-1}R)[X_1,\ldots,X_n] = T[X_1,\ldots,X_n]$. We have

$$\mathrm{ht}(P_r) - \mathrm{ht}(P_0) = \mathrm{ht}(S^{-1}P_r) - \mathrm{ht}(S^{-1}P_0) = \mathrm{ht}(S^{-1}P_r/S^{-1}P_0) = r.$$

Hence,

$$\mathrm{ht}(P) - \mathrm{ht}(P_0) = (\mathrm{ht}(P_r) - \mathrm{ht}(P_0)) + (\mathrm{ht}(P) - \mathrm{ht}(P_r)) = r + (s - r) = s. \quad \square$$

COROLLARY 2.3. *If T and D are both universally catenarian and if K is algebraic over D, then R is universally catenarian.*

PROOF. It will suffice to show that $k + M$ is universally catenarian. Indeed, since M is a maximal ideal of $k+M$ and D is universally catenarian, the conclusion will then follow from the "if" assertion in Theorem 2.2. Thus, we may assume that $D = k$ is a field. By hypothesis, K is algebraic over k, and so T is integral over R. Hence, to show that R is universally catenarian, [**3**, Theorem 6.1] shows that it suffices to prove $\mathrm{ht}(q_1) = \mathrm{ht}(q_2)$ whenever q_1, $q_2 \in \mathrm{Spec}(T)$ satisfy $q_1 \cap R = q_2 \cap R$. However, this holds (indeed, $q_1 = q_2$) since pullback considerations, using [**9**, Theorem 1.4] as in the proof of Theorem 2.2, yield that $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is a homeomorphism. $\square$

COROLLARY 2.4. *Suppose that T is a finite-dimensional valuation domain which is not a field and that D is finite-dimensional. Then R is universally catenarian if and only if D is universally catenarian and K is algebraic over D.*

PROOF. Since T is an LFD Prüfer domain, T is universally catenarian, by results recalled in §1. The "if" assertion is therefore a special case of Corollary 2.3. Conversely, suppose that R is universally catenarian. Then so is $R/M \cong D$, by [**3**, Corollary 3.3]. By universal catenarity, [**3**, Corollary 3.3] shows that the valuative dimension of R (resp., D; resp., T) coincides with its (Krull) dimension. Viewing R as the pullback of $D \to K$ and $T \to K$, we may thus infer from [**1**, Theorem 2.6] that $\dim(R) = \dim(D) + \dim(T) + \mathrm{t.d.}(K/k)$. However, it is well known (cf. [**10**, Exercise 12(4), pp. 202–203]) that $\dim(R) = \dim(D) + \dim(T)$. Hence, $\mathrm{t.d.}(K/k) = 0$; that is, K is algebraic over D. $\square$

We do not know if the converse of Corollary 2.3 is valid. We shall close this section with some remarks in this regard.

REMARK 2.5. (a) Suppose that R is universally catenarian. Then, by [**3**, Corollary 3.3], so are D and $k+M$; and the valuative dimension of R (resp., D) coincides with its dimension. Now, if R is also assumed finite-dimensional and if T is quasilocal but not a field, various pullback results [**1**, Lemma 2.1(d) and Theorem 2.6(a)] yield that K is algebraic over D (that is, over k).

(b) If either $k+M$ or T is catenarian, then so is the other. This follows from the order-isomorphism $\mathrm{Spec}(T) \to \mathrm{Spec}(k+M)$: see [**9**, Theorem 1.4].

(c) Assume that $D = k$, K/k is algebraic, and $A = R[X_1, \dots, X_n]$, where R is universally catenarian. With $S = k[X_1, \dots, X_n] \setminus \{0\}$, we have that $S^{-1}A = k(X_1, \dots, X_n) + S^{-1}M[X_1, \dots, X_n]$ is universally catenarian. Since K is algebraic over k, $B = T[X_1, \dots, X_n]$ satisfies $S^{-1}B = K(X_1, \dots, X_n) + S^{-1}M[X_1, \dots, X_n]$. By applying (b), we infer that $S^{-1}B$ is catenarian. It follows, via [**9**, Theorem 1.4] as in the proof of Theorem 2.2, that $K[X_1, \dots, X_n] + S^{-1}M[X_1, \dots, X_n]$ is catenarian. For this reason, we suggest that this ring's relation to $T[X_1, \dots, X_n]$ merits closer attention.

We are also led to raise the following question. Let $A = E \oplus P$ be a domain, where E is a subring and $P \in \mathrm{Spec}(A)$. Let S denote $E \setminus \{0\}$ and assume that $B = E + S^{-1}P$ is catenarian. Under what conditions is A catenarian?

Note that some conditions need to be imposed, for A need not be catenarian in general. For instance, take E to be a catenarian domain such that $E[X]$ is not catenarian, as in [**13**, Example 2, p. 203]. Put $P = XE[X]$ and let L denote the quotient field of E. Then, since $L[X]$ and E are catenarian, one may use [**9**, Theorem 1.4] as in the proof of Theorem 2.2 to conclude that $B = E + S^{-1}P = E + XL[X]$ is catenarian. However, in this example, $A = E + P = E[X]$ is not catenarian.

(d) Suppose that R is universally catenarian. In studying whether T must be universally catenarian, we may assume that $D = k$ is a field (since, by (a), $k+M$ is universally catenarian). We claim that if the field extension K/k is (algebraic) purely inseparable, then T is universally catenarian.

For a proof, note first that T $(= K + M)$ is integral over R $(= k + M)$. Indeed, since K/k is purely inseparable and $KM \subset M$, each element of T has a power in R. This property is inherited by the extension $R[X_1, \dots, X_n] \subset T[X_1, \dots, X_n]$. Hence, for all $n \geq 1$, $\mathrm{Spec}(R[X_1, \dots, X_n])$ and $\mathrm{Spec}(T[X_1, \dots, X_n])$ are homeomorphic, and therefore order-isomorphic. (This can also be seen by showing that T is the weak normalization of R in T and using the fact that weak normalization is a universal homeomorphism [**2**, Teorema 1].) In particular, $T[X_1, \dots, X_n]$ is catenarian, proving the claim.

(e) Suppose that (T, M) is quasilocal and finite-dimensional. Under these conditions, we may use (a) and (d) to reduce the converse of Corollary 2.3 to the following question. If $k+M$ is universally catenarian and the field extension K/k is separable, is $T = K + M$ necessarily universally catenarian?

3. Examples. In this section, we apply Corollary 2.4 to construct the new family of universally catenarian domains promised in §1.

EXAMPLE 3.1. For each integer $n > 2$, there exists an n-dimensional non-Noetherian universally catenarian domain R_n such that $\mathrm{gl.\,dim}(R_n) > 2$ and R_n

is neither a going-down strong S-domain nor a polynomial ring over a universally catenarian domain.

PROOF. Let k be a field and take n indeterminates $X_1, \dots, X_{n-1}$, Y over k. Consider the discrete (rank 1) valuation ring $V_n = k(X_1, \dots, X_{n-1})[Y]_{(Y)} = K_n + M_n$, where $K_n = k(X_1, \dots, X_{n-1})$ and $M_n = YV_n$. Put $R_n = D_n + M_n$, where $D_n = k[X_1, \dots, X_{n-1}]$. Then, by well-known properties of the classical $D+M$ construction, $\dim(R_n) = \dim(D_n) + \dim(V_n) = (n-1) + 1 = n$ (cf. [**10**, Exercise 12(4), pp. 202–203]); and R_n is not Noetherian (cf. [**10**, Exercise 8(3), pp. 270–271]). As D_n is universally catenarian (because k is), Corollary 2.4 yields that R_n is also universally catenarian.

Note that $\operatorname{gl.dim}(V_n) = 1$ and $\operatorname{gl.dim}(D_n) = n-1$. Hence, by [**7**, Proposition 2.1(1)], we have that $\operatorname{gl.dim}(R_n) = n-1$ if $\operatorname{p.d.}_{D_n}(K_n) < n-1$, and $\operatorname{gl.dim}(R_n) = n$ if $\operatorname{p.d.}_{D_n}(K_n) = n-1$. Thus, $\operatorname{gl.dim}(R_n) > 2$ if $n > 3$. For the case $n = 3$, we must choose k more carefully: take k to be any uncountable field, for instance $\mathbf{R}$. Then [**11**, Theorem 2] assures that $\operatorname{p.d.}_{D_3}(K_3) = 2$, and so the above consequence of [**7**] yields $\operatorname{gl.dim}(R_3) = 3$. Hence, for all $n > 2$, $\operatorname{gl.dim}(R_n) > 2$.

In $\operatorname{Spec}(R_n)$, $X_1D_n + M_n$ and $X_2D_n + M_n$ are incomparable prime ideals contained in the maximal ideal $(X_1, \dots, X_{n-1}) + M_n$. Thus R_n is not treed; hence by [**6**, Theorem 2.2], R_n is not a going-down (strong $S-$) domain. Finally, R_n is not a polynomial ring because it has a unique height 1 prime ideal, namely M_n. □

REMARK 3.2. (a) Note that, for the above construction, it is necessary to assume $n > 2$. Indeed, any one-dimensional domain is a going-down domain; and, by [**8**, Corollary], so is any two-dimensional domain that is constructed via the classical $D+M$ construction.

(b) It remains to discuss the possibility of "new" examples for dimensions 1 and 2. In dimension 1, [**3**, Corollary 6.3] or [**4**, Theorem 1] tells the whole story: a one-dimensional domain is universally catenarian if and only if it is a strong S-domain.

As for dimension 2, let (V, M) be a non-Noetherian one-dimensional valuation domain. Then $A = V[X]_{(M,X)}$ is a two-dimensional non-Noetherian universally catenarian domain, of global dimension at least 3, such that A is neither a going-down (strong $S-$) domain nor a polynomial ring (over a universally catenarian domain).

We shall make only three comments by way of proof, with the rest of the verification left to the reader. If $\{I_i\}$ is a strictly ascending chain of ideals in V, then $\{I_iA\}$ is also strictly ascending, and so A is non-Noetherian. If a and b are nonassociated elements of M in V, then a calculation shows $(X+a)(X+b)^{-1} \notin A$, and so A is not a valuation domain. Since A is integrally closed and has valuative dimension 2, it follows from [**6**, Proposition 2.7] that A is not a going-down domain.

(c) The construction in (b) generalizes to give another new family of examples, as follows. Let $n \geq 2$ and (V, M) be an $(n-1)$-dimensional valuation domain. Assume that V_P is not Noetherian, where P is the unique height 1 prime of V. Then $A = V[X]_{(M,X)}$ is an n-dimensional non-Noetherian universally catenarian domain, of global dimension at least 3, such that A is neither a going-down domain nor a polynomial ring.

The proof is nearly the same as in (b), with the following exception. To see that $\operatorname{gl.dim}(A) > 2$, note that $B = V[X]_{(P,X)} = V_P[X]_{(P_P,X)}$ is a localization of A and $\operatorname{gl.dim}(B) > 2$ by (b).

References

1. D. F. Anderson, A. Bouvier, D. E. Dobbs, M. Fontana and S. Kabbaj, *On Jaffard domains,* Exposition. Math. **6** (1988), 145–175.
2. A. Andreotti and E. Bombieri, *Sugli omeomorfismi delle varietà algebriche*, Ann. Scuola Norm. Sup. Pisa **23** (1969), 431–450.
3. A. Bouvier, D. E. Dobbs and M. Fontana, *Universally catenarian integral domains*, Adv. in Math. (to appear).
4. ____, *Two sufficient conditions for universal catenarity*, Comm. Algebra **15** (1987), 861–872.
5. A. Bouvier and M. Fontana, *The catenarian property of polynomial rings over a Prüfer domain*, Sém. Algèbre P. Dubreil et M. Malliavin, Lecture Notes in Math., vol. 1146, Springer-Verlag, Berlin and New York, 1985, pp. 340–354.
6. D. E. Dobbs, *On going-down for simple overrings.* II, Comm. Algebra **1** (1974), 439–458.
7. ____, *On the global dimensions of $D + M$*, Canad. Math. Bull. **18** (1975), 657–660.
8. D. E. Dobbs and I. J. Papick, *On going-down for simple overrings.* III, Proc. Amer. Math. Soc. **54** (1976), 35–38.
9. M. Fontana, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl. **123** (1980), 331–355.
10. R. Gilmer, *Multiplicative ideal theory*, Dekker, New York, 1972.
11. I. Kaplansky, *The homological dimension of a quotient field*, Nagoya J. Math. **27** (1966), 139–142.
12. S. Malik and J. L. Mott, *Strong S-domains*, J. Pure Appl. Algebra **28** (1983), 249–264.
13. M. Nagata, *Local rings*, Interscience, New York, 1962.
14. ____, *Finitely generated rings over a valuation domain*, J. Math. Kyoto Univ. **5** (1966), 163–169.
15. L. J. Ratliff, Jr., *On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals.* II, Amer. J. Math. **92** (1970), 99–144.

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1300 (Current address of D. F. Anderson, D. E. Dobbs and S. B. Mulay)

Département de Mathématiques, Université de Lyon I, 69622 Villeurbanne Cedex, France (Current address of Salah Kabbaj)