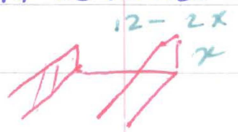
4.7 Optimization Problems

minimizing or maximizing function over finite or infinite interval

Ex 1. A long rectangular sheet of metal 12 in. wide is to be made into a rain gutter by turning up two sides so that they are perpendicular to the sheet. How many inches should be turned up in order to give the gutter its greatest capacity



Ans. x = number of inches turned up on each side

width of the base is $= 12-2x$

The capacity is greatest when the area of the rectangle has its greatest value $f(x) = x(12-2x) = $ area.

$0 \le x \le 6$ closed interval

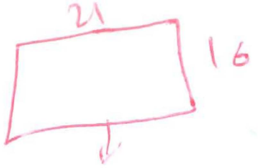
$f'(x) = 4(3-x)$ critical point $x=3$ $f'' = -4 < 0$

the point $x=3$ is max point $f(0)=0$, $f(6)=0$ $f(3)=18$

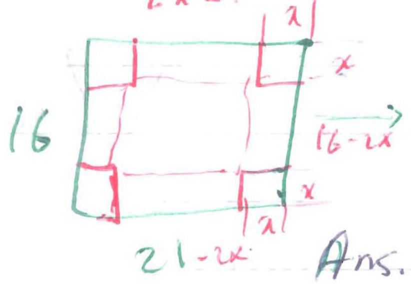
$\Rightarrow$ Absolute max $x=3$ It follows that 3 inches should be turned up to achieve max. capacity.

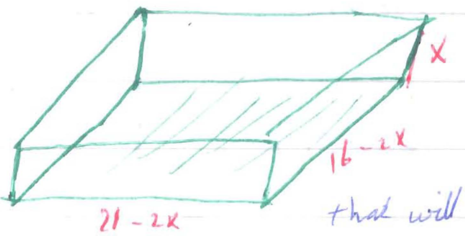
Guidelines for solving optimization problem.

- Read the problem several times.
- Sketch diagrams and label quantities and unknowns.
- Find a formula for the quantity to be maximized or minimized
- Determine which variable is to be max or min and express this variable as a function of one variable.
- Find the interval of possible values and critical points.
- Find Abs max or min.



Ex 2.





open box

Find the size of corner square that will produce a box with largest volume

Ans. ① read the problem ② sketch ③ $V = w \cdot l \cdot h$

④ $V = x(21-2x)(16-2x)$ ⑤ $0 \le 2x \le 16$ $\quad 0 \le x \le 8$

critical points $\quad V = 2(168x - 37x^2 + 2x^3)$

$$V' = 2(168 - 74x + 6x^2) = 4(3x-28)(x-3)$$

$\frac{28}{3}$ is outside $\quad$ take $\boxed{x=3}$

with $x = 0$ or $x = 8$ $\quad V = 0$ $\quad$ then $x = 3$ $\quad V = 450$ is Abs max

Ex 3.

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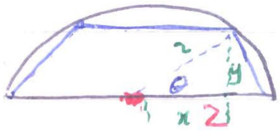


Find the max. area of the trapezoid?

$x^2 + y^2 = 4$

Ans. $\quad y = 2\sin\theta \qquad x = 2\cos\theta$

$\frac{\text{below base} + \text{above base}}{2}$ $\quad A = \left(\frac{4+2x}{2}\right) \times y = (2 + 2\cos\theta)\, 2\sin\theta$

$A = 4\sin\theta(1+\cos\theta) \qquad 0 \le \theta \le \frac{\pi}{2}$

$A' = -4\sin^2\theta + 4\cos\theta(1+\cos\theta)$

$2\cos^2\theta + \cos\theta - 1 = 0 \qquad w = \cos\theta \quad (2w^2 + w - 1) = 0$

$(2w-1)(w+1) = 0 \quad w = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60° \Rightarrow A = 3\sqrt{3}$

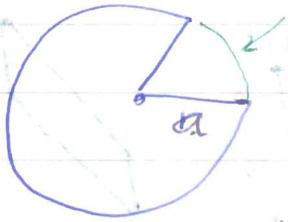
or $w = -1 \Rightarrow \cos\theta = -1 \Rightarrow \theta = 270°$ outside.

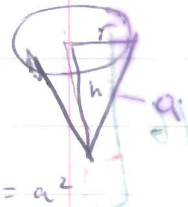
✱ Ex 4. A hotel charges \$80 per day for a room, A special rate is offered for rooms between 30 and 60 (reserved). charge is less by \$1 times the number of rooms over 30. How many rooms must be rented to get max income per day

Ans

$I = [80 - (x - 30)] \times x$

Ex5 A water cup; right circular cone.
by cutting out sector
Find the largest volume





$r^2+h^2=a^2$

$V=\frac{1}{3}\pi r^2 h$

$V=\frac{1}{3}\pi(a^2-h^2)h \qquad 0\le h\le a \qquad V=\frac{1}{3}\pi(ha^2-h^3)$

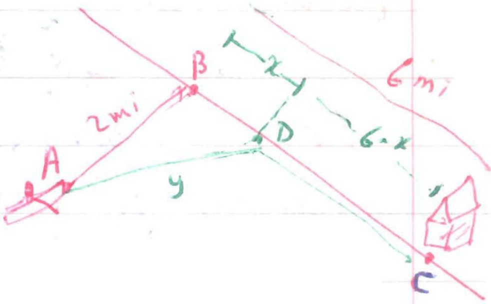
$V'=\frac{1}{3}\pi(a^2-3h^2) \qquad h=\pm\sqrt{\frac{a^2}{3}}=\pm\frac{a}{\sqrt{3}}$ — out

$h=a/\sqrt{3} \qquad r=a\sqrt{2/3}$

$P_1=2\pi a \qquad P_2=2\pi r=2\pi a\sqrt{2/3}$

cutting. $2\pi a-2\pi a\sqrt{2/3}=2\pi a\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}\right)$

Ex. a person in a boat 2 mi from the nearest point on a straight shoreline wishes to reach a house 6 mi farther down the shore. If the person can row at a rate of 3 mi/hr & walk at a rate of 5 mi/hr. Find the least amount of time required to reach the house.



Ans. $y=\sqrt{x^2+4} \qquad 0\le x\le 6$

$\text{time}=\frac{\text{distance}}{\text{rate (speed)}} \qquad t_1=\frac{\sqrt{x^2+4}}{3}$ time from A to D

$x=0$ row directly to B then walk 6 mi
$x=6$ then row directly to home

$t_2=\frac{6-x}{5}$ time from D to C

Total time $T=t_1+t_2=\frac{\sqrt{x^2+4}}{3}+\frac{6-x}{5}=\frac{1}{3}(x^2+4)^{1/2}+\frac{6}{5}-\frac{x}{5}$

$x=0 \quad T=\frac{28}{15}$
$x=6 \quad T\approx 2.11$

$T'=\frac{x}{3(x^2+4)^{1/2}}-\frac{1}{5} \qquad x=\pm\frac{3}{2}$ is critical point

$T(\frac{3}{2})$ only $x=\frac{3}{2}$, $T=\frac{26}{15}$ 1 hr 44 min. Abs min
$x=0 \qquad \frac{28}{15}$ 1 hr 52 min walk = 4.5
$x=6 \qquad \approx 2.11$ 2 hr 7 min row $=y=\sqrt{(\frac{3}{2})^2+4}=5$