

King Fahd University of Petroleum and Minerals  
Department of Mathematics and Statistics  
Math 101 – Exam 1 - Term 152  
**Time allowed 2 hours**

Full name: ..... "Key Solutions"

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Good Luck!

1- Show that  $\lim_{x \rightarrow 0} f(x)$  does not exist for

$$f(x) = \begin{cases} \frac{1-\cos x}{\sin x}, & \text{for } -2 < x < 0 \\ \frac{(2+x)^2 - 4}{x}, & \text{for } x > 0 \end{cases}$$

$$\begin{aligned} * \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \left( \frac{1-\cos x}{\sin x} \right) = \lim_{x \rightarrow 0^-} \left( \frac{1-\cos x}{\sin x} \cdot \frac{1+\cos x}{1+\cos x} \right) \\ &= \lim_{x \rightarrow 0^-} \left( \frac{1-\cos^2 x}{\sin x (1+\cos x)} \right) = \lim_{x \rightarrow 0^-} \frac{\sin^2 x}{\sin x (1+\cos x)} \\ &= \lim_{x \rightarrow 0^-} \frac{\sin x}{1+\cos x} = \frac{0}{2} = \boxed{0}. \end{aligned}$$

(4 points)

2 pts

$$\begin{aligned} * \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \left( \frac{(2+x)^2 - 4}{x} \right) = \lim_{x \rightarrow 0^+} \left( \frac{4+4x+x^2-4}{x} \right) = \lim_{x \rightarrow 0^+} \frac{x(4+x)}{x} \\ &= \lim_{x \rightarrow 0^+} (4+x) = \boxed{4}. \end{aligned}$$

2 pts

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x) \Rightarrow \lim_{x \rightarrow 0} f(x) \text{ DNE}.$$

2- Find the horizontal asymptote of

$$f(x) = \sqrt{4x^2 + 2x} - \sqrt{4x^2 + 5x}.$$

(4 points)

Note:  $\sqrt{4x^2 + 2x} - \sqrt{4x^2 + 5x}$

$$\begin{aligned} &= \sqrt{4x^2 + 2x} - \sqrt{4x^2 + 5x} \cdot \frac{\sqrt{4x^2 + 2x} + \sqrt{4x^2 + 5x}}{\sqrt{4x^2 + 2x} + \sqrt{4x^2 + 5x}} = \frac{4x^2 + 2x - (4x^2 + 5x)}{\sqrt{4x^2 + 2x} + \sqrt{4x^2 + 5x}} \\ &= \frac{-3x}{|x| \left( \sqrt{4 + \frac{2}{x}} + \sqrt{4 + \frac{5}{x}} \right)}. \end{aligned}$$

1 pt

$$* \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{-3x}{|x| \left( \sqrt{4 + \frac{2}{x}} + \sqrt{4 + \frac{5}{x}} \right)} = \frac{-3}{4}.$$

1 pt

$$* \lim_{x \rightarrow -\infty} f(x) = \frac{-3}{|x| \left( \sqrt{4 + \frac{2}{x}} + \sqrt{4 + \frac{5}{x}} \right)} = \frac{3}{4}.$$

1 pt

$\therefore y = -\frac{3}{4}$  &  $y = \frac{3}{4}$  are Horizontal Asymptotes

3- Using the precise definition of limit, show that

$$\lim_{x \rightarrow 3} \left(1 + \frac{1}{3}x\right) = 2.$$

(4 points)

\* Let  $\epsilon > 0$ , we want to find a number  $\delta > 0$  such that  
if  $0 < |x - 3| < \delta$  then  $\left|\left(1 + \frac{1}{3}x\right) - 2\right| < \epsilon$ .

\* But  $\left|1 + \frac{1}{3}x - 2\right| < \epsilon \Leftrightarrow \left|\frac{1}{3}x - 1\right| < \epsilon$   
 $\Leftrightarrow \frac{1}{3}|x - 3| < \epsilon$   
 $\Leftrightarrow |x - 3| < 3\epsilon$ .

\* So, if we choose  $\delta = 3\epsilon$ , then

$$0 < |x - 3| < \delta \Rightarrow |x - 3| < 3\epsilon \Rightarrow \frac{1}{3}|x - 3| < \epsilon \Rightarrow \left|\left(1 + \frac{1}{3}x\right) - 2\right| < \epsilon.$$

\* Thus  $\lim_{x \rightarrow 3} \left(1 + \frac{1}{3}x\right) = 2$  by the definition of limit.

4- Suppose that

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & \text{if } x < 2 \\ ax^2 - bx + 3, & \text{if } 2 \leq x < 3 \\ 2x - a + b, & \text{if } x \geq 3 \end{cases}$$

Find the values of  $a$  &  $b$  that make  $f$  continuous everywhere.

(6 points)

\*  $f(x)$  is cont. on  $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$ .

So, we need to make it cont. at  $x=2$  and  $x=3$ .

\* at  $x=2$ ,  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$

$$\lim_{x \rightarrow 2^-} \left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x \rightarrow 2^-} \frac{(x-2)(x+2)}{x-2} = 4$$

$$\lim_{x \rightarrow 2^+} (ax^2 - bx + 3) = 4a - 2b + 3$$

$$4a - 2b + 3 = 4$$

$$\text{or } 4a - 2b = 1 \quad (1)$$

1pt

\* at  $x=3$ ,  $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$

$$10a - 4b = 3 \quad (2)$$

Solving (1) & (2)

$$a = \frac{1}{2}$$

$$b = \frac{1}{2}$$

1pt

5- Show that

$$\lim_{x \rightarrow 0^-} x^3 \cos\left(\frac{2}{x}\right) = 0.$$

(4 points)

1 pt

We know  $-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$

1 pt

Multiply by  $x^3$  ( $\text{as } x \rightarrow 0^-, x^3 < 0$ ), we get

$$-x^3 \geq x^3 \cos\left(\frac{2}{x}\right) \geq x^3, \text{ and}$$

1 pt

Since  $\lim_{x \rightarrow 0^-} x^3 = 0 = \lim_{x \rightarrow 0^-} -x^3$ ,

1 pt

by the squeeze theorem we have  $\lim_{x \rightarrow 0^-} x^3 \cos\left(\frac{2}{x}\right) = 0$

6- Use limits to find the vertical asymptotes of the function

$$f(x) = \frac{(x+1)(x^2+x-6)}{(x^2-x)(x^2-4)}.$$

1 pt

Note:  $f(x) = \frac{(x+1)(x-2)(x+3)}{x(x-1)(x+2)(x+2)} = \frac{(x+1)(x+3)}{x(x-1)(x+2)}$  for  $x \neq -2$ .

(4 points)

1 pt

\*  $\lim_{x \rightarrow 0^+} \frac{(x+1)(x+3)}{x(x-1)(x+2)} = -\infty$ ,  $\therefore \boxed{x=0}$  is V.A.

1 pt

\*  $\lim_{x \rightarrow 1^+} \frac{(x+1)(x+3)}{x(x-1)(x+2)} = +\infty$ ,  $\therefore \boxed{x=1}$  is V.A.

1 pt

\*  $\lim_{x \rightarrow -2^+} \frac{(x+1)(x+3)}{x(x-1)(x+2)} = -\infty$ ,  $\therefore \boxed{x=-2}$  is V.A.

7- Find the limit of

$$\lim_{x \rightarrow \frac{1}{2}^-} (x - \lfloor 2x \rfloor) , \text{ where } \lfloor x \rfloor = \text{the largest integer that is less than or equal to } x.$$

1 pt

as  $x \rightarrow \frac{1}{2}^-$ ,  $2x \rightarrow 1^-$ , and then  
 $\lfloor 2x \rfloor \rightarrow 0$ .

(2 points)

1 pt

$$\text{so } \lim_{x \rightarrow \frac{1}{2}^-} (x - \lfloor 2x \rfloor) = \frac{1}{2} - 0 = \frac{1}{2}.$$

8- Use Intermediate Value Theorem to show that  $\sin x = x - 1$  has a solution.

1 pt

(4 points)

$$f(x) = \sin x - x + 1 \text{ which is cont. on } \mathbb{R}$$

Now,

1 pt

$$f(0) = \sin 0 - 0 + 1 = 1 > 0, \text{ and}$$

1 pt

$$f(\pi) = \sin \pi - \pi + 1 = -\pi + 1 < 0.$$

1 pt

So, by I.V.T there is at least a  $c \in (0, \pi)$   
such that  $f(c) = 0$ . Therefore, eq.  $\sin x = x - 1$  has a solution.

9- Let  $f(x) = \sqrt{1 + \sqrt{x}}$ .

(4 points)

a) Use the definition of derivative to find  $f'(x)$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \left( \frac{f(x+h) - f(x)}{h} \right) = \lim_{h \rightarrow 0} \left( \frac{\sqrt{1+\sqrt{x+h}} - \sqrt{1+\sqrt{x}}}{h} \right) \cdot \frac{\sqrt{1+\sqrt{x+h}} + \sqrt{1+\sqrt{x}}}{\sqrt{1+\sqrt{x+h}} + \sqrt{1+\sqrt{x}}} \quad \text{multiply with conj.} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{1+\sqrt{x+h}} - \sqrt{1+\sqrt{x}}}{h(\sqrt{1+\sqrt{x+h}} + \sqrt{1+\sqrt{x}})} = \lim_{h \rightarrow 0} \left( \frac{\sqrt{x+h} - \sqrt{x}}{h(\sqrt{1+\sqrt{x+h}} + \sqrt{1+\sqrt{x}})} \right) \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \quad \text{multiply with conj.} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h(\sqrt{1+\sqrt{x+h}} + \sqrt{1+\sqrt{x}})(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{(\sqrt{1+\sqrt{x+h}} + \sqrt{1+\sqrt{x}})(\sqrt{x+h} + \sqrt{x})} \quad 1 \\ &= \frac{1}{4(\sqrt{1+\sqrt{x}})(\sqrt{x})} \quad 1 \text{ pt.} \end{aligned}$$

b) If  $f'(1) = \frac{1}{4\sqrt{2}}$ , find an equation for the tangent line to the curve  $y = f(x)$  when  $x = 1$ .

(2 points)

When  $x_1 = 1$ ,  $y_1 = f(1) = \sqrt{1+\sqrt{1}} = \sqrt{2}$

1 pt

then an eq. for tangent line:

$$y - y_1 = m(x - x_1)$$

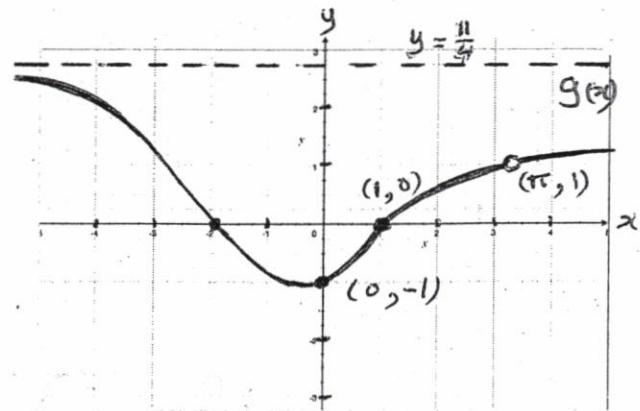
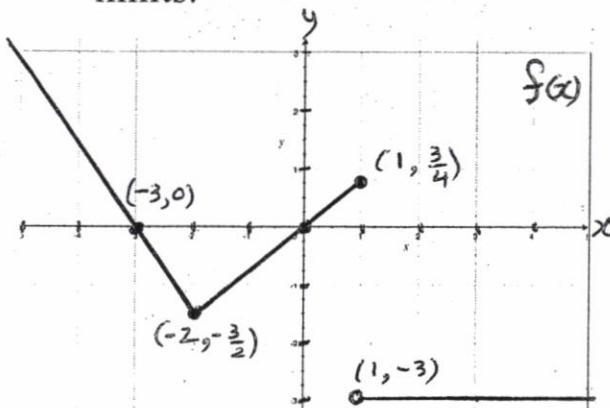
$$y - \sqrt{2} = \frac{1}{4\sqrt{2}}(x - 1)$$

1 pt

OR

$$y = \frac{1}{4\sqrt{2}}x + \frac{7\sqrt{2}}{8}$$

10- Referring to the graph given below, find the value of the following limits:



a.  $\lim_{x \rightarrow 2} f(2x)$

(2 points each)

1 pt

$$= f\left(\lim_{x \rightarrow 2} (2x)\right) = f\left(2 \lim_{x \rightarrow 2} x\right) = f(4) = -3.$$

(since  $f$  is cont. at  $x = -4$ )

1 pt

b.  $\lim_{x \rightarrow 0} [f(x) - g(x)]$

1 pt

$$= \lim_{x \rightarrow 0} f(x) - \lim_{x \rightarrow 0} g(x) = 0 - (-1) = 1.$$

(both  $f$  &  $g$  are cont. at  $x = 0$ )

1 pt

c.  $\lim_{x \rightarrow 1^+} \left[ \frac{g(x)}{f(x)} \right]$

1 pt

$$= \frac{\lim_{x \rightarrow 1^+} (g(x))}{\lim_{x \rightarrow 1^+} (f(x))} = \frac{0}{-3} = 0.$$

1 pt

(both limits exist as  $x \rightarrow 1^+$ )

d.  $\lim_{x \rightarrow \pi} [x^2 f(x) g(x)]$

1 pt

$$= \pi^2 \left( \lim_{x \rightarrow \pi} f(x) \right) \cdot \left( \lim_{x \rightarrow \pi} g(x) \right) = \pi^2 (-3) \cdot (1) = -3\pi^2.$$

1 pt

(both limits exist as  $x \rightarrow \pi$ )

e.  $\lim_{x \rightarrow 1^-} \left[ g(f(x) + \frac{1}{4}) \right]$

1 pt

$$= g\left(\lim_{x \rightarrow 1^-} \left(f(x) + \frac{1}{4}\right)\right) = g\left(\frac{3}{4} + \frac{1}{4}\right) = g(1) = 0$$

(both limits exist)

1 pt

f.  $\lim_{x \rightarrow 0} \left[ \frac{f(x-1) - f(-1)}{x} \right]$

1 pt

$$= \lim_{h \rightarrow 0} \left( \frac{f(-1+h) - f(-1)}{h} \right) = f'(-1) = \frac{\frac{3}{4} - \left(-\frac{3}{2}\right)}{1 - (-2)} = \frac{\frac{3}{4} + \frac{6}{4}}{3} = \frac{3}{4}.$$

1 pt