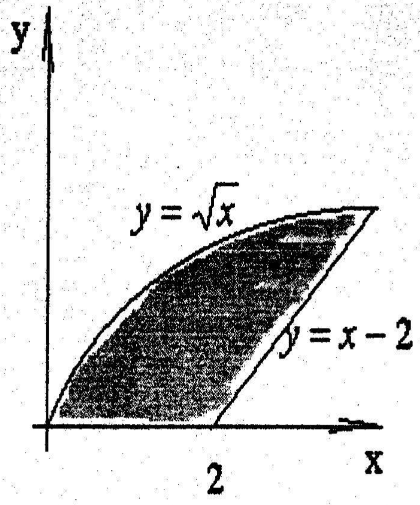
King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 102
Second Major (012) 22 April 2002 Time:7-9 pm.

100

Name :--ID # :--------------------

Q1 **(15 points)** I. Find the area of the shaded region.



Setup (but do not evaluate) an integral that gives the volume of the solid generated by rotating the shaded region about

II. x-axis using cylindrical shells.

III. y-axis using washers.

I. $\int_0^2 y+2-y^2\,dy = \frac{y^2}{2}+2y-\frac{y^3}{3}\Big|_0^2 = \frac{10}{3}$

II. $2\pi\int_0^2 y\,[(y+2)-y^2]\,dy$

III. $\pi\int_0^2 (y+2)^2 - y^4\,dy.$

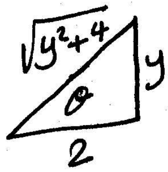
Q2 **(12 points)** The graph of the equation $y=2\sqrt{x+1}$ from A(0,2) to B(3,4).

I. Find the arc length.

II. If revolved about the x-axis then find area of the resulting surface.

I. Solve for x; $x=\frac{y^2}{4}-1$ $\quad \frac{dx}{dy}=\frac{y}{2}$ $\quad$ the arc length $\int_2^4\sqrt{1+\frac{y^2}{4}}\,dy$

let $y=2\tan\theta$ $\quad I=\int\sqrt{4+4\tan^2\theta}\,\sec^2\theta\,d\theta=2\int\sec^3\theta\,d\theta$



$$=2\left(\frac{\sec\theta\tan\theta}{2}-\frac{1}{2}\ln|\sec\theta+\tan\theta|\right)$$

$$\Rightarrow I=2\left[\frac{\sqrt{y^2+4}}{2}\cdot\frac{y}{2}-\frac{1}{2}\ln\left|\frac{\sqrt{y^2+4}}{2}+\frac{y}{2}\right|\right]_2^4$$

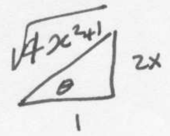
$$=2\sqrt{5}-\sqrt{2}-\ln\frac{1+\sqrt{2}}{2+\sqrt{5}} \quad ■$$

II. $2\pi\int_0^3 2\sqrt{x+1}\sqrt{1+\frac{1}{x+1}}\,dx \Rightarrow 4\pi\int_0^3\sqrt{x+2}\,dx=\frac{8}{3}\pi(x+2)^{\frac{3}{2}}\Big|_0^3$

$$=\frac{8\pi}{3}\left[5\sqrt{5}-2\sqrt{2}\right].$$

Q3 **(10 points)** Find the area of the surface generated when the curve $y=x^2$ from $x=0$ to $x=1$ is revolved about the x-axis.

$2\pi\int_0^1 x^2\sqrt{1+4x^2}\,dx \qquad 2x=\tan\theta \quad 2\,dx=\sec^2\theta\,d\theta$



$$\Rightarrow \frac{2\pi}{4}\int\tan^2\theta\sec^3\theta\,d\theta$$

$$=2\pi\int\sec^5\theta-\sec^3\theta\,d\theta$$

$$=2\pi\left\{\left[\frac{\sec^3\theta\tan\theta}{4}+\frac{3}{4}\left[\frac{\sec\theta\tan\theta}{2}+\frac{1}{2}\ln|\sec\theta+\tan\theta|\right]\right]\right\}-\left[\frac{\sec\theta\tan\theta}{2}+\frac{1}{2}\ln|\sec\theta+\tan\theta|\right]\Big\}$$

$$=2\pi\left(\left\{\frac{(\sqrt{4x^2+1})^3\,2x}{4}+\frac{3}{4}\left[\frac{\sqrt{4x^2+1}\,2x}{2}+\frac{1}{2}\ln\left|\sqrt{4x^2+1}+2x\right|\right]\right\}-\left[\frac{\sqrt{4x^2+1}\,2x}{2}+\ln\left|\sqrt{4x^2+1}+2x\right|\right]\right)\Bigg|_0^1$$

$$=2\pi\left(\left\{\frac{10\sqrt{5}}{4}+\frac{3}{4}\left[\frac{2\sqrt{5}}{2}+\frac{1}{2}\ln|\sqrt{5}+2|\right]\right\}-\left[\frac{2\sqrt{5}}{2}+\ln|\sqrt{5}+2|\right]\right)-0$$

$$=\pi\left(\frac{9\sqrt{5}}{2}-\frac{5}{4}\ln(\sqrt{5}+2)\right)$$

Q4. **(7 points each; total 14 points)** Determine if the integral converges or diverges

I. $\frac{1}{2}\int_0^2 \frac{2(x-2)}{x^2-4x+3}\,dx$ $\quad u = x^2-4x+3 \quad du = 2x-4$

$\int \frac{1}{u}\,du = \ln|u|$

at 1 the integration is improper.

$$\lim_{l\to 1^-} \frac{1}{2}\int_0^l \frac{2x-4}{x^2-4x+3}\,dx + \lim_{k\to 1^+} \frac{1}{2}\int_k^2 \frac{2x-4}{x^2-4x+3}\,dx$$

$$= \lim_{l\to 1^-} \frac{1}{2} \ln|x^2-4x+3|\Big|_0^l + \lim_{k\to 1^+} \frac{1}{2} [\ln|x^2-4x+3|\Big|_k^2$$

$4-8+3$

$$= \frac{1}{2}\lim_{l\to 1^-} \ln|l^2-4l+3| - \ln|3| + \frac{1}{2}\lim_{k\to 1^+} |k^2-4k+3| - \ln|-1|$$

$$= \frac{1}{2}(-\infty - \ln|3|) + \frac{1}{2}(-\infty - \ln 1) \text{ diverges.}$$

II. $\int_1^\infty \frac{dx}{x\ln\sqrt{x}} = \int_1^\infty \frac{dx}{\frac{1}{2}x\ln x} = 2\int_1^\infty \frac{dx}{x\ln x}$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$$\lim_{l\to\infty} (\ln\ln x)\Big|_1^l \cancel{= \lim_{l\to\infty} \ln\ln l - \ln\ln 1}$$

$$\lim_{l\to\infty} (\ln\ln x)\Big|_e^l + \lim_{k\to 1^+} (\ln\ln x)\Big|_1^e$$

$$\infty \quad -\infty$$

diverges.

Q5. **(7 points each; total 49 points)** Evaluate

I. $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

$$\int_{-1}^{1} \frac{dx}{(x+1)^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\Big|_{-1}^{1} = \frac{\pi}{8}$$

II. $\int \frac{\sqrt{2}}{\sin x - \sqrt{3}\cos x}\,dx$

$$\sqrt{2}\int \frac{\frac{2}{\cancel{u^2+1}}}{\frac{2u}{\cancel{u^2+1}} - \sqrt{3}\,\frac{1-u^2}{\cancel{u^2+1}}}\,du = 2\sqrt{2}\int \frac{du}{2u-\sqrt{3}+\sqrt{3}u^2}$$

$$= 2\sqrt{2}\int \frac{du}{(\sqrt{3}u-1)(u+\sqrt{3})}$$

$$=$$

$$\frac{A}{\sqrt{3}u-1} + \frac{B}{u+\sqrt{3}}$$

III. $\int \sin^4 x \cos^5 x dx$

$$\int \sin^4 x (1-\sin^2 x)^2 \sin x dx \qquad u = \sin x$$
$$du = \cos x dx$$

IV. $\int x2^x dx$

$$\frac{x2^x}{\ln 2} - \int \frac{2^x}{\ln 2} dx = \frac{x2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + C$$

V. $\int \frac{\tan^{-1} x}{x^2} dx \qquad u = \tan^{-1} x \qquad dx = x^{-2}$

$$\frac{-\tan^{-1} x}{x} + \int \frac{1}{x(x^2+1)} dx \qquad \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

VI. $\int \frac{x\,dx}{\sqrt{3-2x-x^2}}$ use trigonometric substitution

$$\int \frac{x\,dx}{\sqrt{4-(x+1)^2}}$$

$x+1 = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$$= \int \frac{2\sin\theta - 1}{\cancel{\cos\theta}}\,\cancel{2\cos\theta}\,d\theta$$

$$= -2\cos\theta - \theta + C$$

$$= -2\,\frac{\sqrt{4-(x+1)^2}}{2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C.$$

VII. $\int_1^{64} \frac{dx}{\sqrt{x}(\sqrt{x}+\sqrt[3]{x})}$

$u = x^{\frac{1}{6}}$ $\quad u^6 = x$

~~$6u^5\,du\,dx$~~

$6u^5\,du = dx$

$$\int_1^2 \frac{6u^5\,du}{u^3(u^3+u^2)} = \int_1^2 \frac{6\,du}{1+u}$$

$$= 6\ln|1+u|\Big|_1^2 = 6\ln\frac{3}{2}.$$