PART I

(SHOW YOUR DETAILED SOLUTIONS CLEARLY)

1. Sketch a graph of $f(x)$ with the following properties

$Vertical\ Asymptotes: x = 4, x = -2$

$f(x) \geq 0 \quad on \quad (-\infty,-2) \cup [0,3] \cup (4,\infty), \quad f(x) < 0 \quad elsewhere$

$x-intercepts = 0, 3$

$y-intercept = 0$

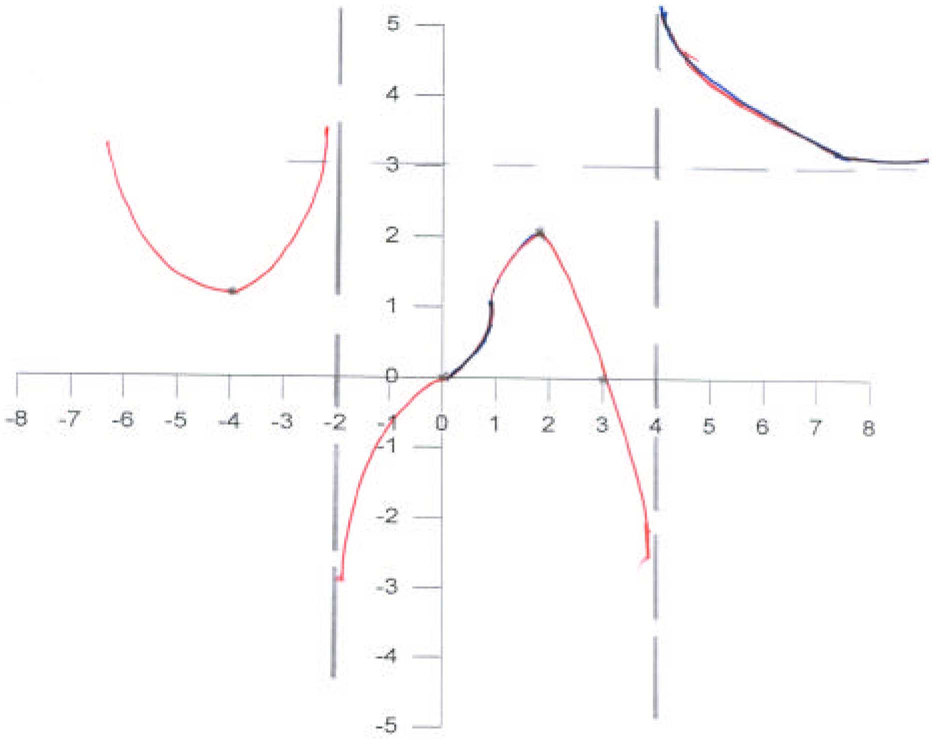
$Relative\ Max.\ at\ (2,2)$

$Relative\ Min.\ at\ (-4,1)$

$Inflection\ Points\ \ at\ (0,0)\ and\ (1,1)$

$Concave\ up\ on\ (-\infty,-2) \cup (0,1) \cup (4,\infty), \quad Concave\ down\ elsewhere$

$\lim_{x\to+\infty} f(x) = 3, \lim_{x\to-\infty} f(x) = +\infty$



2. Consider the function $f(x) = x^{\frac{1}{3}}(x-8)$ with first and second derivatives given by $f'(x) = x^{\frac{1}{3}} + \frac{(x-8)}{3x^{\frac{2}{3}}}$ and $f''(x) = \frac{4x+16}{9x^{\frac{5}{3}}}$. May use $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt[3]{2} \approx 1.3$, $\sqrt[3]{4} \approx 1.6$

a) Find the $x-$ and $y-$ intercepts, the relative extrema, if any exist, and where f is increasing or decreasing

x-int. $(0,0)$, $(8,0)$

$f' = \frac{4(x-2)}{3x^{2/3}}$ c.p. are $x = 2, 0$

f': $-$ (0) $-$ (2) $+$

relative min.

decreasing $(-\infty, 2]$

increasing $[2, \infty)$

b) Find inflection points, if any exist, and where f is concave up or down

$f'' = \frac{4(x+4)}{9x^{5/3}}$ $f'' = 0$ if $x = -4$

f'' undefine if $x = 0$

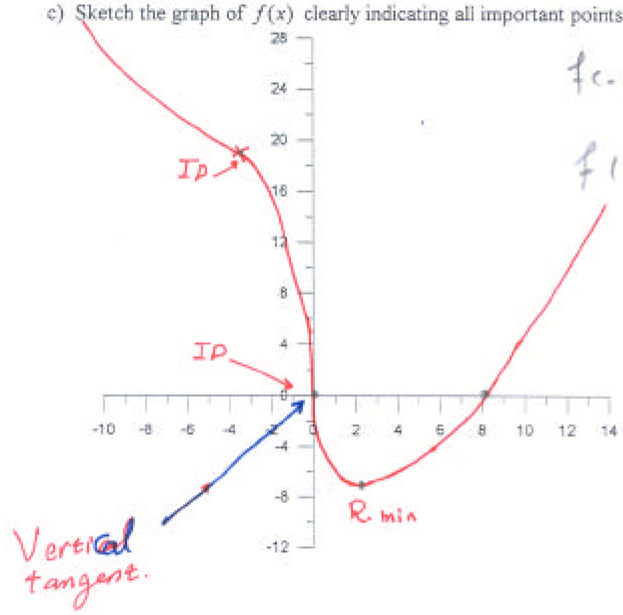
$+$ (-4) $-$ (0) $+$

$\cup$ $\cap$ $\cup$

Concave up $(-\infty, -4) \cup (0, \infty)$

Concave down $(-4, 0)$

c) Sketch the graph of $f(x)$ clearly indicating all important points and concavity



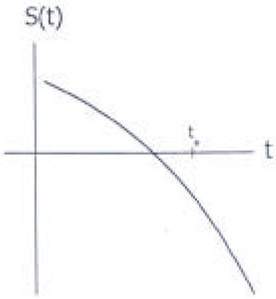
$f(-4) \approx -1.6(-12)$

≈ 19.2

$f(2) \approx 1.3(-6)$

≈ -7.8

3. The figure below represents the position versus time curve of a particle in rectilinear motion. Fill in the spaces appropriately to describe the behavior of the particle at time $t = t_0$.



STATEMNET	CHOICES
The particle is to the left of the origin because Negative direction (decreasing curve)	LEFT/RIGHT
The particle is moving in the Negative direction because Curve below time axis (d~~irected to left~~)	NEGATIVE /POSITIVE
The velocity is Decreasing because Curve concave down	INCREASING/DECREASING
The particle is Speeding up because Curve decreasing -ve velocity Curve concave down -ve acceleration	SPEEDING UP / SLOWING DOWN

3. If $f(x) = 10^{\sqrt{1-x}}$, then $f'(0) =$

a) $5\ln 10$

b) -5

c) $\boxed{-5\ln 10}$

d) 5

e) ∞

$$f' = \frac{-1}{2\sqrt{1-x}}\, 10^{\sqrt{1-x}} \ln 10$$

$$f'(0) = \frac{-1}{2\sqrt{1}}\, 10^{\sqrt{1}} \ln 10$$

$$= -5\ln 10$$

4. A rectangle has two vertices on the x-axis and the other two vertices lie on the graph of $y = 9 - x^2$, $-3 \le x \le 3$. The maximum area of the rectangle is equal to

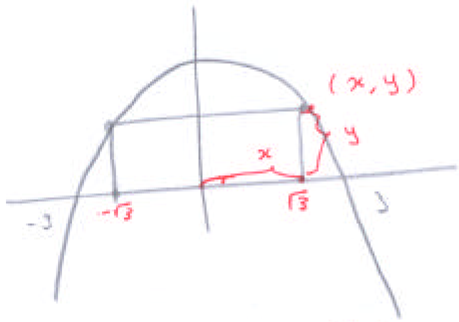
a) $8\sqrt{3}$

b) $\boxed{12\sqrt{3}}$

c) $16\sqrt{3}$

d) $20\sqrt{3}$

e) $24\sqrt{3}$



$\frac{1}{2}$ $A = y \cdot x = (9 - x^2)x$

$= 9x - x^3$

$\frac{1}{2}$ $A'(x) = 9 - 3x^2 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

$A = 2(\sqrt{3})(9-3)$

$= 12\sqrt{3}$

1. If $y = x^{x-1}$, then $y' =$

a) $(x-1)x^{x-2}$

$\boxed{\text{b)}}$ $(x-1)x^{x-2} + x^{x-1}\ln x$

c) $(\ln x + 1)x^{x-1}$

d) $(\frac{x-1}{x} - \ln x)x^{x-1}$

e) $(x-1)x^{x-2}\ln x$

$\ln y = x-1 \ln x$

$\frac{y'}{y} = \ln x + \frac{x-1}{x}$

$y' = (\ln x) x^{x-1} + x^{-2}(x-1)$

2. If $f(x) = \begin{cases} x-2 & if \quad x \le 0 \\ -2 & if \quad 0 < x < 1 \\ 2x^2 - 4x & if \quad x \ge 1 \end{cases}$, which one of the following is **FALSE**?

a) $f'(3) = 8$ ✓

b) $f'(1) = 0$ ✓

c) $f'(\frac{1}{2}) = 0$ ✓

$\boxed{\text{d) } f'(0) = 1}$

e) $f'(-2) = 1$ ✓

$f' = \begin{cases} 1 & x \le 0 & ① \\ 0 & 0 < x \le 1 & ② \\ 4x - 4 & x > 1 & ③ \end{cases}$

$f'(3) = 4(3) - 4 = 8$ from ③

$f'(1) = \begin{cases} f'_-(1) = 0 & \text{from ②} \\ f'_+(1) = 4(1) - 4 = 0 & \text{from ③} \end{cases}$

$f'(\frac{1}{2}) = 0$ from ②

$f'(0) = \begin{cases} f'_-(0) = 1 & \text{from ①} \\ f'_+(0) = 0 & \text{from ②} \end{cases}$ $f'(0)$ DNE

$f'(-2) = 1$ from ①

7. If $y=\sqrt{u-1}$, *and* $u=\sqrt{1-x}$, then $\frac{dy}{dx}$ when $x=-3$ is

a) $\frac{1}{4}$

b) $\boxed{-\frac{1}{8}}$

c) 4

d) $-\frac{1}{4}$

e) $-\frac{1}{2}$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{2\sqrt{u-1}}\cdot\frac{-1}{2\sqrt{1-x}}$$

$$=\frac{-1}{4\sqrt{\sqrt{1-x}-1}\ \sqrt{1-x}}$$

$$\left.\frac{dy}{dx}\right|_{x=-3}=\frac{-1}{4\sqrt{\sqrt{4}-1}\ \sqrt{4}}$$

$$=\frac{-1}{4\sqrt{1}\ \ 2}$$

$$=\frac{-1}{8}$$

8. The function $y=\frac{\ln x}{x}$ has

a) $\boxed{\text{a relative maximum at } x=e \text{ and a point of inflection at } x=e^{\frac{3}{2}}}$ ✓

b) a relative minimum at $x=0$ and a point of inflection at $x=e$

c) a relative maximum at $x=1$ and a point of inflection at $x=e$

d) a relative minimum at $x=1$ and a point of inflection at $x=0$

e) a relative minimum at $x=e^{-1}$ and a point of inflection at $x=1$

Domain $(0,\infty)$

$$y'=\frac{\frac{1}{x}\cdot x-\ln x}{x^2}=\frac{1-\ln x}{x^2}=0 \qquad \ln x=1\Rightarrow x=e$$

$$y''=\frac{-\frac{1}{x}\cdot x^2-2x(1-\ln x)}{x^4}=\frac{-1-2+2\ln x}{x^3}$$

$$=\frac{-3+2\ln x}{x^3}=0\Rightarrow \ln x=\frac{3}{2}\Rightarrow x=e^{\frac{3}{2}}$$

0 + e −

Rel. max

− $e^{3/2}$ +

IP

5. If $f(x)=\dfrac{\sin 2x+\cos 3x}{1+\tan 4x}$, then $f'(0)=$

a) 1

b) 2

c) $\boxed{-2}$

d) 4

e) -4

$(4\sec^2 4x)$

$$f'=\frac{(2\cos 2x-3\sin 3x)(1+\tan 4x)-[\sin 2x+\cos 3x]}{(1+\tan 4x)^2}$$

$$f'(0)=\frac{(2-0)(1+0)-(0+1)(4)}{1}$$

$$=\frac{2-4}{1}=\underline{\underline{-2}}$$

6. If $f(x)=2x^2-3x+1$, then the number which satisfies the conclusion of the Mean Value Theorem for $f(x)$ on the closed interval $[0,4]$ is

a) $\frac{1}{2}$

b) 1

c) $\frac{3}{2}$

d) $\boxed{2}$

e) $\frac{5}{2}$

$f'(x)=4x-3$

$$\frac{f(4)-f(0)}{4-0}=f'(c)$$

$$\Rightarrow \frac{32-12+1-1}{4}=4c-3$$

$$\Rightarrow 5=4c-3$$

$$\Rightarrow \frac{8}{4}=c \Rightarrow \underline{\underline{c=2}}$$

11. A local linear approximation of $(1+2x)^{-5}$ at $x_o = 0$ is

a) $1+5x$

b) $2+5x$

c) $1-10x$

d) $1+10x$

e) $2-5x$

$f(x_0) + f'(x_0)(x - x_0)$

$= (1+0)^{-5} + -5(1+2(0))^{-4}(2)(x-0)$

$\Rightarrow 1 + (-10)(1)x$

$\Rightarrow 1 - 10x$

12. The set of all critical numbers of $f(x) = x + \frac{1}{x-1}$ is

a) $\{0,2\}$

b) $\{1\}$

c) empty

d) $\{0,1,2\}$

e) $\{0,1\}$

$f' = 1 + \frac{-1}{(x-1)^2} = \frac{(x-1)^2 - 1}{(x-1)^2} = 0$

$x^2 - 2x + 1 - 1 = 0$

$x^2 - 2x = 0$

$x = 0$

or $x = 2$

$x = 1$ is not belong to ~~to~~ the Domain

9. If Newton's method is used to find a root of the equation $x - 2\cos x = 0$, and the first approximation is $x_1 = \frac{\pi}{2}$, then the next approximation x_2 is equal to

a) 0

b) π

c) $\frac{\pi}{4}$

d) $\frac{2\pi}{3}$

e) $\boxed{\frac{\pi}{3}}$

$$X_2 = X_1 - \frac{X_1 - 2\cos X_1}{1 + 2\sin X_1}$$

$$= \frac{\pi}{2} - \frac{\frac{\pi}{2} - 0}{1 + 2(1)}$$

$$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{3\pi}{6} - \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

10. $\lim_{x \to +\infty} \left(e^{2x} - 1\right)^{\frac{1}{x}} = \infty^0$

$$\lim_{x \to \infty} \frac{1}{x} \ln\left(e^{2x} - 1\right) \quad \frac{\infty}{\infty}$$

a) ∞

b) 1

c) $\frac{1}{2}$

d) e

e) $\boxed{e^2}$

(L'Hopital) $= \lim_{x \to \infty} \frac{2e^{2x}}{e^{2x} - 1} \quad \left(\frac{\infty}{\infty}\right)$

(L'Hopital) $= \lim_{x \to \infty} \frac{4\cancel{e^{2x}}}{2\cancel{e^{2x}}}$

$$= 2$$

then e^2 (because of the ln)

15. If $f(x)=\left(2x^2+1\right)^2\left(2-x^2\right)^3$, then $f'(1)=$

a) 30

b) -30

c) 6

d) 78

e) -78

$f' = 2(2x^2+1)(4x)(2-x^2)^3 + (2x^2+1)^2 3(2-x^2)^2(-2x)$

$f'(1) = 2(3)(4)(1) + 9(3)(1)(-2)$

$= 24 + (-54) = -30$

16. The slope of the tangent line to the graph of $x^2+xy+y^5=3$ at the point $(1,1)$ is equal to

a) -8

b) 8

c) $-\frac{1}{2}$

d) $\frac{1}{2}$

e) -2

$2x + y + xy' + 5y^4y' = 0$

$y'\Big|_{(1,1)} = \dfrac{-2x-y}{x+5y^4}$

$y'(\text{slope}) = \dfrac{-2-1}{1+5} = \dfrac{-3}{6}$

$= -1/2$

17. If $f(x)$, $f'(x)$, and $f''(x)$ are continuous, $f(1) = f'(1) = 1$, and $\lim_{x\to 1} \frac{f(x)-x}{\sin^2(\pi x)} = \frac{1}{2}$, then

$f''(1) =$

a) π

b) $-\pi$

c) $-\pi^2$

d) π^2

e) 1

$$\lim_{x\to 1} \frac{f(x)-x}{\sin^2(\pi x)} \quad \left(\frac{1-1=0}{0}\right)$$

Using L'Hopital $\lim_{x\to 1} \frac{f'(x)-1}{2\pi \sin \pi x \cos \pi x}$ $\left(\frac{1-1=0}{0}\right)$

Using L'Hopital again

$$\lim_{x\to 1} \frac{f''(x)}{2\pi^2(\cos^2 \pi x - \sin^2 \pi x)} = \boxed{\frac{f''(1)}{2\pi^2(1-0)} = \frac{1}{2}}$$

$$\Rightarrow f''(1) = \pi^2$$

18. If $y = \ln(\tan^{-1} x^2) + \tan^{-1}(\ln x^2)$, then $y'(1) =$

a) $\frac{4}{\pi}+2$

b) $\frac{\pi}{4}$

c) $\frac{8}{\pi}-2$

d) $-\frac{\pi}{4}+4$

e) $\pi+2$

$$y' = \frac{\frac{2x}{1+x^4}}{\tan^{-1} x^2} + \frac{1}{1+(\ln x^2)^2} \cdot \frac{2x}{x^2}$$

$$y'(1) = \frac{\frac{2}{2}}{\tan^{-1} 1} + \frac{1}{1+\ln 1} \cdot 2$$

$$= \frac{1}{\frac{\pi}{4}} + 2$$

13. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=$ $(\infty-\infty)$

a) 0

$\boxed{\text{b) } \frac{1}{2}}$

c) $-\infty$

d) 1

e) ∞

$$\lim_{x\to 1}\frac{(x-1)-\ln x}{(\ln x)(x-1)}=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}\ \left(\frac{0}{0}\right)$$

$$\text{L'Hopital} = \lim_{x\to 1}\frac{\cancel{x}-\frac{1}{\cancel{x}}}{\frac{x\ln x+(x-1)}{\cancel{x}}}\ \left(\frac{0}{0}\right)$$

$$\text{L'Hopital} = \lim_{x\to 1}\frac{1}{\ln x+\frac{x}{x}+1}=\frac{1}{0+1+1}=\frac{1}{2}.$$

14. $\lim_{x\to 0^-}\frac{\sin x+\tan x}{x^2}=$ $\left(\frac{0}{0}\right)$ L'Hopital

a) $-\frac{1}{2}$

b) -1

c) 1

d) ∞

e) $-\infty$

$$\lim_{x\to 0^-}\frac{\cos x+\sec^2 x}{2x}=-\infty$$

$\frac{1+1}{\text{v. small no.}}$

19. The *sum* of the absolute maximum and minimum values of the function

$f(x)=\begin{cases} 4x-2 & if \quad x<1 \\ x^2-5x+6 & if \quad x\geq 1 \end{cases}$ on the interval $\left[\frac{1}{2},\frac{7}{2}\right]$ is

a) $\boxed{\frac{7}{4}}$

end point $\frac{1}{2}$, 1, $\frac{7}{2}$

not diff.

b) $\frac{3}{4}$

$f' = 4 \quad x<1$

c) 2

d) 0

$f' = 2x-5 \Rightarrow$ c.p. if $x=\frac{5}{2}$

e) $\frac{9}{4}$

$f(\frac{1}{2}) = 0$

$f(\frac{7}{2}) = 0.75$

$f(1) = 2$ ← Abs max

$f(\frac{5}{2}) = -0.25$ ← Abs min

$7/4$ — $2-0.25$

20. The set of values of x for which $f(x)=\begin{cases} x+1 & if \quad x<0 \\ 1-x^2 & if \quad 0\leq x\leq 1 \\ \sqrt{x} & if \quad 1<x<2 \\ \sqrt{2}+x & if \quad x\geq 2 \end{cases}$ is **discontinuous**, is

given by

a) $\{0,1,2\}$

discontinuous at 2

" at 1

b) $\{0,1\}$

Not discontinuous at 0

c) $\{0,2\}$

d) $\boxed{\{1,2\}}$

Since $\lim_{x\to 0^+} f(x) = 1-0 = 1$

e) $\{1\}$

$\lim_{x\to 0^-} f(x) = 0+1 = 1$