King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
MATH 101
Final Exam (013)
Aug. 8, 2002, 12:30 p.m.

Key

100

Each question worth 5 points

Name: **I.D.:** **Time:**3hours. **Section:**01

1. Evaluate $\lim\limits_{x\to 0}\sqrt{\dfrac{\sin 2x - x}{4x}}$ $= \frac{0}{0}$

$$\sqrt{\lim_{x\to 0} \frac{\sin 2x}{2(2x)} - \frac{x}{4x}}$$

$$= \sqrt{\frac{1}{2} - \frac{1}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

another method $\left(\frac{0}{0}\right)$ L. Hopital

$$\lim_{x\to 0} \sqrt{\frac{2\cos 2x - 1}{4}} = \sqrt{\frac{2-1}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

2. Evaluate $\lim\limits_{x\to 0^+}\dfrac{\ln \sin x}{\ln \tan x}$ $\left(\frac{-\infty}{-\infty}\right)$

$$\frac{\frac{\cos x}{\sin x}}{\frac{\sec^2 x}{\tan x}} = \frac{\cos x}{\sin x} \cdot \frac{\tan x}{\sec^2 x} = \frac{\cancel{\cos x}}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{\cancel{\cos x}} \cdot \cos^2 x$$

$$= \cos^2 x = 1$$

3. Evaluate $\lim_{x \to -1} \sqrt[3]{\dfrac{x+1}{x^3+1}}$ $= \lim_{x \to -1} \sqrt[3]{\dfrac{0}{0}}$

$$\sqrt[3]{\lim_{x \to -1} \frac{\cancel{(x+1)}}{\cancel{(x+1)}(x^2-x+1)}} = \sqrt[3]{\lim_{x \to -1} \frac{1}{x^2-x+1}}$$

$$= \sqrt[3]{\frac{1}{1+1+1}} = \sqrt[3]{\frac{1}{3}}$$

L' Hopital

$$\lim_{x \to -1} \sqrt{\frac{1}{3x^2}} = \sqrt[3]{\frac{1}{3}}$$

4. Evaluate $\lim_{x \to 0} (\cos x)^{1/x}$ 1^{∞}

$$y = (\cos x)^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x} \ln(\cos x)$$

$$\ln y = \frac{\ln \cos x}{x} = \frac{0}{0}$$

$$\ln y = \frac{-\frac{\sin x}{\cos x}}{1} = -\frac{\sin x}{\cos x} = 0$$

$$e^0 = 1$$

5. Find y' if $y = \sin^2\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$

$$y' = 2\sin\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\cos\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) \cdot \frac{\frac{1}{2\sqrt{x}}(1-\sqrt{x}) - \left(-\frac{1}{2\sqrt{x}}\right)(1+\sqrt{x})}{(1-\sqrt{x})^2}$$

$$= \sin\left\{2\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\right\}\left[\frac{\frac{1}{2\sqrt{x}} - \frac{\cancel{\sqrt{x}}}{2\cancel{\sqrt{x}}} + \frac{1}{2\sqrt{x}} + \frac{\cancel{\sqrt{x}}}{2\cancel{\sqrt{x}}}}{(1-\sqrt{x})^2}\right]$$

$$= \sin\left\{\frac{2+2\sqrt{x}}{1-\sqrt{x}}\right\}\left(\frac{1}{\sqrt{x}\,(1-\sqrt{x})^2}\right)$$

6. Find y' if $x + y\sqrt{1+2x} = 2x\cos y$

$$x + y\sqrt{1+2x} - 2x\cos y = 0$$

$$1 + \frac{dy}{dx}\left(\sqrt{1+2x}\right) + y\frac{\cancel{2}}{\cancel{2}\sqrt{1+2x}} - \left(2\cos y - 2x\sin y\frac{dy}{dx}\right) = 0$$

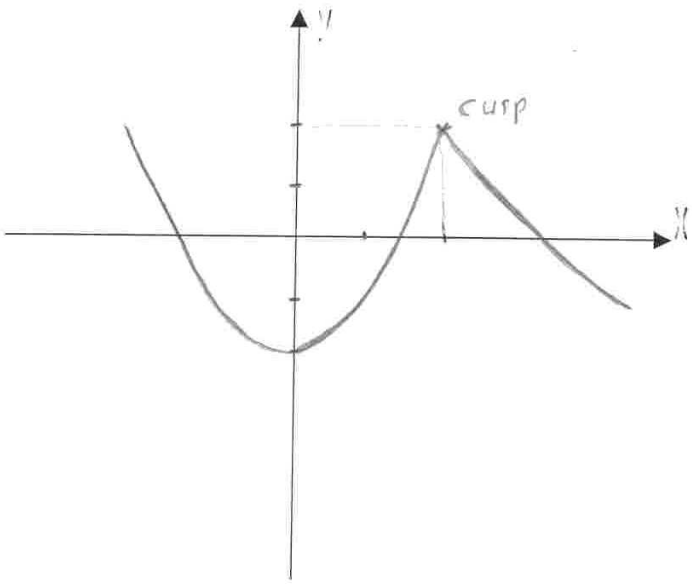
$$\frac{dy}{dx}\left(\sqrt{1+2x}\right) + 2x\sin y\frac{dy}{dx} + 1 + \frac{y}{\sqrt{1+2x}} - 2\cos y = 0$$

$$\frac{dy}{dx}\left(\sqrt{1+2x} + 2x\sin y\right) = 2\cos y - \frac{y}{\sqrt{1+2x}} - 1$$

$$\frac{dy}{dx} = \frac{2\cos y - \frac{y}{\sqrt{1+2x}} - 1}{\sqrt{1+2x} + 2x\sin y}$$

7. Sketch the graph of f that satisfies the condition

$$\begin{array}{ll} f(0) = -2 & f(2) = 2 \\ f'(0) = 0 & f'(2) \text{ undefined} \\ f'(x) > 0 \text{ if} & 0 < x < 2 \\ f'(x) < 0 \text{ if} & x < 0 \text{ or } x > 2 \\ f''(x) > 0 \text{ for} & x < 2 \text{ or } x > 2 \end{array}$$



8. Find y' if $y = (\ln x)^{\tan x}$

$$\ln y = \tan x \left[\ln(\ln x) \right]$$

$$\frac{y'}{y} = \tan x \left[\frac{1}{x \ln x} \right] + \sec^2 x \left(\ln(\ln x) \right)$$

$$y' = (\ln x)^{\tan x} \left[\frac{\tan x}{x \ln x} + \sec^2 x \left(\ln(\ln x) \right) \right]$$

9. Find the absolute extrema for the function $f(x) = \frac{x^2}{x+1}$ in $(-1,5)$

$x = 1$ is V.A. $\quad \lim_{x \to -1^+} f(x) = \infty$

$$f' = \frac{2x^2 + 2x - x^2}{(x+1)^2}$$

$x(x+2)$

$x = 0$ c.p.

$x = -2$ out of range

$x - 1$ ← oblique As.

$$\begin{array}{r} x-1 \\ x+1 \overline{) x^2} \\ x^2 + x \\ \hline -x \end{array}$$

No Abs max

Abs min at $x = 0$

10. Use the Mean Value Theorem to prove that $|\sin x - \sin y| \le |x - y| \quad \forall x, y \in \Re$

let $f(x) = \sin x$ then, $|f'(x)| \leqslant 1, \ \forall x \in \mathbb{R}$

From MVT, $\forall x, y \in \mathbb{R}$

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \leqslant 1$$

$$\frac{|f(x) - f(y)|}{|x - y|} \leqslant 1 \Rightarrow$$

$$|f(x) - f(y)| \leqslant |x - y| \Rightarrow$$

$$|\sin x - \sin y| \leqslant |x - y|$$

11. A particle moves in straight line so that its distance s from the starting point is $s(t) = \frac{1}{4}t^4 - 4t^3 + 16t^2$. Then find the time when both distance and velocity are 0.

when distance $= 0 = S(t) = t^4 - 16t^3 + 64t^2$

$= t^2(t-8)^2$; $t = 8$

when velocity $= 0 = v(t) = 4t^3 - 48t^2 + 128t = 0$

$= t^3 - 12t^2 + 32t$

$= t(t-8)(t-4)$

$t = 8$

At time $t = 8$ both $v(t)$ & $S(t) = 0$

12. If $y = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.1, find Δy and dy

$\Delta y = f(x + \Delta x) - f(x)$, $x = 2$, $\Delta x = 0.1$

$= (x+\Delta x)^3 + (x+\Delta x)^2 - 2(x+\Delta x) + \cancel{1} - x^3 - x^2 + 2x - \cancel{1}$

$= (2.1)^3 + (2.1)^2 - 2(2.1) - (2)^3 - (2)^2 + 2(2)$

$= 9.26 + 4.41 - 4.2 - 8 - \cancel{4} + \cancel{4}$

$= 1.47$

$\frac{dy}{dx} = 3x^2 + 2x - 2$

$dy = (3x^2 + 2x - 2)\,dx$

$= [3(4) + 2(2) - 2](0.1) = 1.4$

$[12 + 4 - 2](0.1)$

13. Find c promised by Mean Value Theorem for $f(x) = 1 + \frac{7}{2}x^{3/4}$ in [0,16].

$$f'(x) = \frac{21}{8} x^{-\frac{1}{4}}$$

$$\frac{21}{8\sqrt[4]{c}} = \frac{29-1}{16} \Rightarrow \frac{21}{8\sqrt[4]{c}} = \frac{28}{16} = \frac{7}{4}$$

$$\Rightarrow \frac{8\sqrt[4]{c}}{21} = \frac{4}{7} \qquad \Rightarrow \sqrt[4]{c} = \frac{3}{2}$$

$$\Rightarrow \qquad c = \left(\frac{3}{2}\right)^4 = \frac{81}{16}$$

14. Given $T(l) = 2\pi\sqrt{l/g}$ where g is a constant, using differentials to approximate the change in l, if T is increased by 1% then find the increases in l.

$$\Delta T \simeq dT = \frac{2\pi}{\sqrt{g}} \cdot \frac{1}{2\sqrt{l}}\, dl$$

$$\frac{\Delta T}{T} = \frac{2\pi\, dl}{2\sqrt{gl}} \Big/ 2\pi\sqrt{\frac{l}{g}}$$

$$= \frac{\sqrt{g}}{2\sqrt{g}\sqrt{l}\sqrt{l}}\, dl = \frac{1}{2}\frac{dl}{l}$$

Now $\frac{\Delta T}{T} \times 100 = 1 \Rightarrow \frac{\Delta T}{T} = 0.01$

but $\frac{\Delta T}{T} = \frac{1}{2}\frac{dl}{l} = 0.01$

$\Rightarrow \frac{dl}{l} = 0.02$

$\Rightarrow$ increases in l is 2%

15. When approximating $\sqrt[5]{1.1}$ with $x_0 = 1$ using Local linear approximation Find $\sqrt[5]{1.1} \approx$

let $f(x) = \sqrt[5]{x}$, $x_0 = 1$, $f'(x) = \frac{1}{5} x^{-\frac{4}{5}}$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$\approx \sqrt[5]{1} + \frac{1}{5\sqrt[5]{1^4}} (0.1)$$

$$\approx 1 + (0.2)(0.1)$$

$$\approx 1 + 0.02 \approx 1.02$$

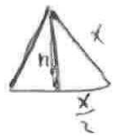
16. When approximating $\sqrt[5]{1.1}$ with $x_0 = 1$ using Newton method (one iteration) Find $\sqrt[5]{1.1} \approx$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad f(x) = x^5 - 1.1$$

$$= x_n - \frac{x_n^5 - 1.1}{5x_n^4}$$

$$x_1 = 1 - \frac{1 - 1.1}{5} = 1 + \frac{0.1}{5} = 1 + \frac{1}{50} = \underline{1.02}$$

17. The area of an equilateral triangle is increasing at the rate of 25 m^2 / *hour*. Find the rate of the sides when the side is *10m*.



$$A = \frac{1}{2} x h$$

$$x^2 = h^2 + \frac{x^2}{4} \Rightarrow h^2 = \frac{4x^2}{4} - \frac{x^2}{4} = \frac{3x^2}{4}$$

$$h^2 = \frac{3x^2}{4} \Rightarrow \boxed{h = \frac{\sqrt{3}}{2} x}$$

$$A = \frac{1}{2} x^2 \left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4} x^2$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} (x) \frac{dx}{dt}$$

$$5 = \frac{dA}{dt} = \frac{\sqrt{3}}{2} (10) \frac{dx}{dt}$$

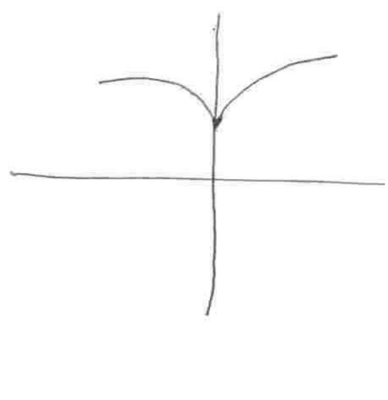
$$\frac{dx}{dt} = \frac{5}{\sqrt{3}} \text{ m / hour.}$$

18. Sketch the Graph of the function $f(x) = 1 + \frac{7}{2} x^{2/3}$ .Is there [redacted] a cusp? f(x) always positive

$$f' = \frac{7}{3} x^{-1/3} = \frac{7}{3\sqrt[3]{x}} \qquad x = 0 \quad \text{c.p} \qquad - \quad + $$

$$f'' = -\frac{7}{9} x^{-\frac{4}{3}} = -\frac{7}{9\sqrt[3]{x^4}} \qquad - \quad 0 \quad -$$

$$\left.\begin{array}{l} \lim_{x \to 0^-} f'(x) = -\infty \\ \lim_{x \to 0^+} f'(x) = +\infty \end{array}\right\} \text{cusp.}$$



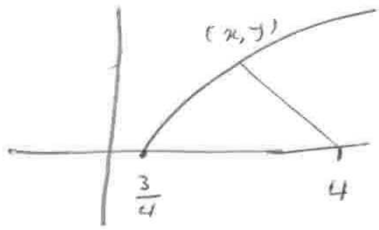
19. Find the shortest distance between the point (4,0) and the curve $y = \sqrt{4x-3}$.

The distance

$$d(x) = \sqrt{(x-4)^2 + (y-0)^2}$$

$$= \sqrt{x^2 - 8x + 16 + 4x - 3}$$

$$= \sqrt{x^2 - 4x + 13}$$



$$\frac{d\,d(x)}{dx} = \frac{2x-4}{2\sqrt{x^2-4x+13}} = 0$$

$$\Rightarrow x = 2; \quad y = \sqrt{5}$$

20. Find the values of h, k in the function

$$f(x) = \begin{cases} k\sin x & x \ge 0 \\ x+h & x < 0 \end{cases}$$

to be differentiable at $x=0$.

First f(x) must be cont. $f(0) = 0 = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = h$

$\boxed{h=0}$

$f'_+ = K\cos x = K\cos 0 = K$

$f'_- = 1$

$\boxed{K=1}$