3.6 Derivative of logarithmic function.

$$\frac{d}{dx} \log_b(x) = \frac{1}{x \ln b}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

How
$y = \log_b(x)$
$\Rightarrow b^y = x$
$b^y \ln b \frac{dy}{dx} = 1$
$\Rightarrow y' = \frac{1}{x \ln b}.$

Ex. $y = \ln(\sin x)$ $\quad y' = \frac{\cos x}{\sin x} = \cot x.$

Ex. $y = \ln \frac{x+2}{\sqrt{x}} \longrightarrow y' = \frac{1}{\frac{x+2}{\sqrt{x}}} \cdot \frac{d}{dx} \frac{x+2}{\sqrt{x}}$

$$= \frac{\sqrt{x}}{x+2} \cdot \frac{\sqrt{x} - (x+2)\frac{1}{2\sqrt{x}}}{x} = \frac{\sqrt{x}}{x+2} \cdot \frac{2x - x - 2}{2\sqrt{x}\, x} = \frac{x-2}{2x(x+2)} = A$$

or $y = \ln x+2 - \ln\sqrt{x}$ $\quad y' = \frac{1}{x+2} - \frac{1}{2x} = A.$

Ex. $y = \ln|x|$

$$y = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

$y' = \frac{1}{x}$ $\qquad y' = \frac{-1}{-x} = \frac{1}{x}$ for both cases.

Ex. $y = \ln|\tan x|$

$$y' = \frac{\sec^2 x}{\tan x}$$

Ex $y = \ln \ln x$ $\qquad y' = \frac{1}{x \ln x}$

Ex $y = [\ln x]^2$ $\qquad\qquad y = \ln x^2 = 2\ln x$

$y' = 2(\ln x)\frac{1}{x}$ $\qquad\qquad y' = \frac{2}{x}$

Logarithmic differentiation

. take ln for both side
diff the solve

Ex $y = x^{x}$

$\ln y = \ln x^{x} = x \ln x$

$y'/y = \ln x + 1$

Ex $y = \sqrt[5]{\frac{x-1}{x+1}} \Rightarrow \ln y = \frac{1}{5} \ln(x-1) - \ln(x+1)$

$y' / y = \frac{1}{5}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$

Ex. $y = [\sin x]^{\ln x}$

$\ln y = \ln x \ln \sin x$

$\frac{y'}{y} = \frac{\ln \sin x}{x} + \frac{\ln x \cdot \cos x}{\sin x}$

Ex $y = \pi^{\sin x}$ $y' = \pi^{\sin x} \ln \pi \cos x$

Ex. $y = [\ln(x \tan y)]^{x}$

$\frac{y'}{y} = \frac{x(\tan y + x y' \sec^2 y)}{x \tan y} + \ln(x \tan y)$

$\ln y = x \ln(x \tan y)$

Ex. find y' if $x^y = y^x$

$\Rightarrow y \ln x = x \ln y \Rightarrow y' \ln x + \frac{y}{x} = \ln y + \frac{x}{y} y'$

$$y' = \frac{\frac{y}{x} - \ln y}{\frac{x}{y} - \ln x}$$

The natural number e

$f(x) = \ln x$ then $f'(x) = \frac{1}{x}$ $f'(1) = 1$

but $f'(1) = \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \ln(1+x)^{1/x}$

So $1 = \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}$

but

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}} = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$$

$$\boxed{e = \lim_{x \to 0} (1+x)^{\frac{1}{x}} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$ or $e^{\frac{3x}{2}} = \lim_{n \to \infty} \left(1 + \frac{3x}{2n}\right)^n$

$$\frac{d}{dx} \lim_{n \to \infty} \left(1 + \frac{3x}{2n}\right)^n = \frac{d}{dx} e^{\frac{3x}{2}} = \frac{3}{2} e^{\frac{3x}{2}}$$