

(1B)

## 2. 3 The precise definition of a limit (up Ex 4)

Ex.  $f(x) = 2x - 1 \quad \lim_{x \rightarrow 3} 2x - 1 = 5$

$y$  close to 5 when  $x$  close to 3

How  $x$  close to 3 so that  $y$  differs from 5. Let say by 1

Given  $|y - 5| < 1$  find  $|x - 3| < ?$

$$|2x - 1 - 5| = |2x - 6| = 2|x - 3|$$

$$\text{So if } |y - 5| < 1$$

$$\text{then } 2|x - 3| < 1$$

$$\Rightarrow |x - 3| < \frac{1}{2} = \delta \text{ or any larger number}$$

also we can write

$$-\frac{1}{2} < x - 3 < \frac{1}{2}$$

$$2.5 < x < 3.5$$

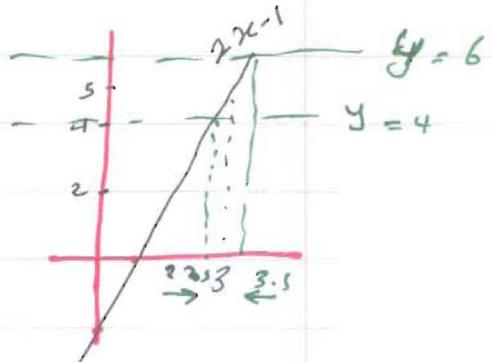
### Definition

Let  $f(x)$  be defined on an open interval about  $x_0$ , except possibly at  $x_0$  itself. We say that the limit of  $f(x)$  as  $x$  approaches  $x_0$  is the number  $L$  and write

$$\lim_{x \rightarrow x_0} f(x) = L$$

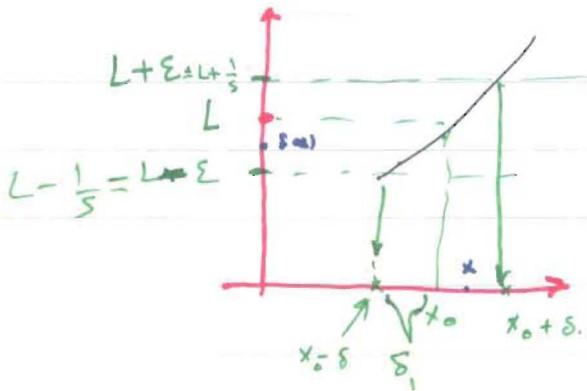
If  $\forall \varepsilon > 0, \exists \delta > 0$  such that  $\forall x |x - x_0| < \delta \Rightarrow$

$$|f(x) - L| < \varepsilon$$



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$$\varepsilon = \frac{1}{5}$$



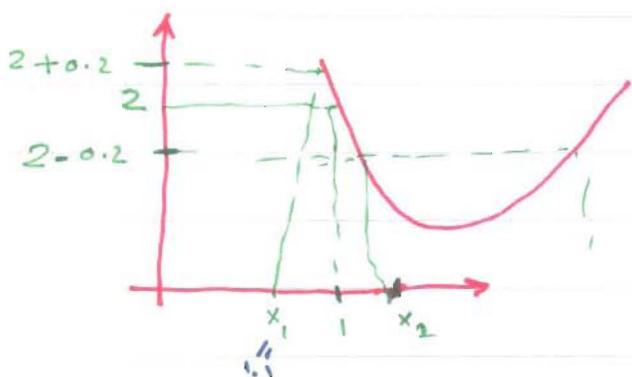
Ex.  $\lim_{x \rightarrow 2} 3x - 5 = 1$  prove using  $\varepsilon - \delta$

$$|3x - 5 - 1| < \varepsilon \Rightarrow |3x - 6| < \varepsilon \Rightarrow 3|x - 2| < \varepsilon \Rightarrow |x - 2| < \frac{\varepsilon}{3}$$

you can ~~also~~ take  $\delta \leq \frac{\varepsilon}{3}$  (any number)

Ex.  $f(x) = x^2 - 5x + 6 \quad \lim_{x \rightarrow 1} f(x)$

find  $\delta$  if  $\varepsilon = 0.2$  geometrically



$$|x^2 - 5x + 6 - 2| < 0.2$$

$$1.8 < f(x) < 2.2$$

$$\text{find } x_2 \text{ when } y = 1.8 = f(x)$$

$$\text{find } x_1 \text{ s.t. } y = 2.2 = f(x)$$

For  $x_2$ ,  $y = 1.8$  or  $x^2 - 5x + 6 = 1.8$

$$x^2 - 5x + 4.2 = 0$$

$$x_2 = \frac{5 \pm \sqrt{18.2}}{2} \approx \frac{5 \pm 4.28}{2} \approx 1.1 \text{ or } 3.9$$

~~or~~ we take  $x_2 = 1.1$  so  $\delta_1 = 1.1 - 1 = 0.1$

For  $x_1$ ,  $y = 2.2$  or  $x^2 - 5x + 6 = 2.2$

$$x^2 - 5x + 3.8 = 0 \quad x_1 = \frac{5 \pm \sqrt{3.18}}{2} \approx 0.91 \text{ or } 4.09$$

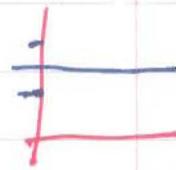
$$\delta_2 = 0.089$$

$$\min(0.1, 0.089) = 0.089.$$

**Ex.**  $\lim_{x \rightarrow x_0} k = k$

$$|k - k| < \epsilon$$

$\delta$  any positive number



**Ex.**  $\lim_{x \rightarrow x_0} x = x_0$   $\epsilon = \delta$ .

**Ex.** Find  $\delta$  when  $f(x) = \frac{1}{x}$   $\epsilon = 0.2$   $L = 0.5$   $a = 2$

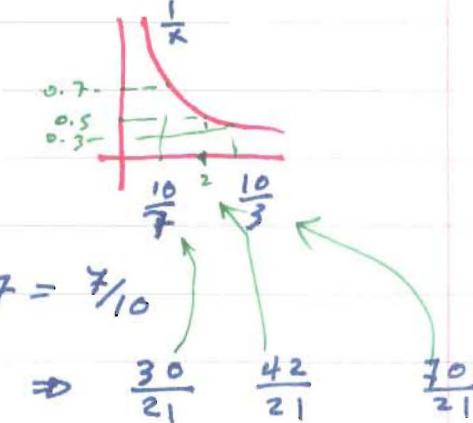
$$\left| \frac{1}{x} - 0.5 \right| < 0.2$$

①  $y = 0.7$

$$\frac{1}{x} = 0.7 \Rightarrow x = \frac{10}{7}$$

$$\frac{3}{10} = 0.3 < \frac{1}{x} < 0.7 = \frac{7}{10}$$

$$\Rightarrow \frac{10}{3} > x > \frac{10}{7} \Rightarrow \frac{30}{21}, \frac{42}{21}, \frac{70}{21}$$



$$\text{So } \delta_1 = 2 - \frac{10}{7} = \frac{3}{7}, \quad \delta_2 = \frac{10}{3} - 2 = \frac{4}{3}$$

$$= \frac{12}{21} \quad = \frac{28}{21} \quad \delta = \min = \frac{12}{21}.$$

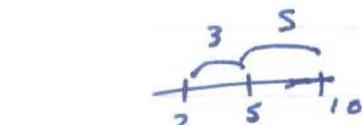
**Ex.**

$$\lim_{x \rightarrow s} \sqrt{x-1} = 2 \quad \text{if } \epsilon = 1 \quad \text{find } \delta$$

$$3 < \sqrt{x-1} < 1$$

$$9 > x-1 > 1$$

$$10 > x > 2$$



$$\min(3, \delta) = 3.$$

or  $-3 < x - s < \delta$

$$\underline{\underline{5, |-3|}}$$

Ex.

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Use the graph of  $f(x) = \sqrt{x}$  to find a number  $\delta > 0$  such that if  $|x - 9| < \delta$  then  $|\sqrt{x} - 3| < 0.2$

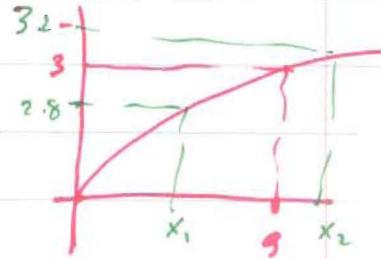
So  $\varepsilon = 0.2$

Find  $x_1$  such that  $f(x_1) = 2.8$

Find  $x_2$  such that  $f(x_2) = 3.2$

$$\sqrt{x_1} = \frac{2.8}{10} \Rightarrow x_1 = 7.84$$

$$\sqrt{x_2} = \frac{3.2}{10} \Rightarrow x_2 = 10.24$$



$$\delta = \min \{ 9 - 7.84, 10.24 - 9 \} \\ = 1.16$$

Ex. Use the  $\varepsilon, \delta$  definition of limit to prove that

$$\lim_{x \rightarrow 6} \left( \frac{x}{4} + 3 \right) = \frac{9}{2}$$

$$\left| \frac{x}{4} + 3 - \frac{9}{2} \right| \leq \varepsilon$$

$$\Leftrightarrow \left| \frac{x}{4} - \frac{3}{2} \right| \leq \varepsilon \text{ or } |x - 6| \leq 4\varepsilon$$

thus we can choose  $0 < \delta \leq 4\varepsilon$

$$\underline{\delta = 4\varepsilon}$$