

3.5 Implicit Differentiation

- $y = f(x)$ defined explicitly
 y appears alone
- $2y + 3x = 1$ implicit equation
 $y = \frac{1-3x}{2}$ easy to make it explicit

Some time it is difficult or impossible

Ex.

$$x^2 + y^2 = 1 \rightarrow \begin{cases} y = \sqrt{1-x^2} \\ y = -\sqrt{1-x^2} \end{cases}$$

or

$$y^2 x + \sin^{xy} + cxy = 1 + x$$

In general for derivative no need for solving the equation.

Ex. find $\frac{dy}{dx}$ $y^2 - x = -2$

$$y^2 = x - 2 \quad y_1 = \sqrt{x-2} \quad \text{or} \quad y_2 = -\sqrt{x-2}$$

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x-2}}$$

$$\frac{dy_2}{dx} = \frac{-1}{2\sqrt{x-2}}$$

Now $y(x)$ is function of x so derivative using chain rule

$$\frac{d}{dx} y^2 = 2yy' \quad \text{so for } 2yy' = 1 \Rightarrow y' = \frac{1}{2y} \text{ as above}$$

So you differentiate both sides of the equation then collect y' in one side (solve for y')

Ex. Find y' if $x^3 + y^3 = 6xy$ ①

$$3x^2 + 3y^2 y' = 6(y + xy')$$

$$(3y^2 - 6xy) y' = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

Find ^{one} ~~the~~ equation of the tangent lines at $x=3$

$x=3$ $27 + y^3 = 18y$ or

$$y^3 - 18y + 27 = 0 \quad \underline{y=3}$$

$$(y-3)(y^2 + 3y - 9) = 0$$

$$y' \Big|_{(3,3)} = \frac{6(3) - 3(3)^2}{3(3)^2 - 6(3)} = \frac{18 - 27}{27 - 18} = \frac{-9}{9} = -1 \quad (3,3)$$

$$y - 3 = -1(x - 3) \quad y = -x + 6$$

Find when the tangent line horizontal

$$y' = 0 = \frac{2y - x^2}{y^2 - 2x} \Rightarrow 2y = x^2 \quad y = \frac{1}{2}x^2$$

Substituting in ① $x^3 + \frac{1}{8}x^6 = 6x(\frac{1}{2}x^2) \Rightarrow x^3 = 16$

$$x = \sqrt[3]{16} \quad y = \sqrt[3]{4}$$

Tangent Horizontal $(\sqrt[3]{16}, \sqrt[3]{4})$

Ex. $\sin(x^2y^2) = x$

$$y' = \frac{-2x^2y^2}{\cos x^2y^2} \cdot 2x^2y$$

Ex. Find Slope $y^4 + 3y - 4x^3 - 5x - 1 = 0$
at the point $(1, -2)$

$$y' = \frac{12x^2 + 5}{4y^3 + 3} \Big|_{(1, -2)} = -\frac{17}{29}$$

Ex. If $x = \frac{\cot y}{1 + \csc y}$

Show that $y'' = \cos y (\sin y + 1)$

always change to \sin & \cos

So $x = \frac{\cos y}{\sin y + 1} \Rightarrow y' = -(\sin y + 1)$

$$y'' = \cos y (\sin y + 1)$$

Ex $y + \sin y = x$ y'' ?

Ex $y^2 - x + 1 = 0$ slope at $x = 10$

$$x = 10 \Rightarrow y^2 - 10 + 1 = 0$$

$$y^2 = 9 \quad y = 3, -3$$

$$2yy' - 1 = 0 \quad y' = \frac{1}{2y} \Big|_3 = \frac{1}{6}$$

$$y' \Big|_{-3} = -\frac{1}{6}$$

normal line is the line perpendicular to the tangent line at x_0

Ex. $x^{2/3} + y^{2/3} = 4$ find the normal line at $x = -1$

first find y coordinate

$$\sqrt[3]{(-1)^2} + y^{2/3} = 4 \Rightarrow y = \pm 3\sqrt{3}$$

find y'

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} y' = 0$$

$$y' = -\sqrt[3]{y/x}$$

the slope $(-1, 3\sqrt{3})$

$$y'| = \sqrt{3}$$

$(-1, -3\sqrt{3})$

$$y'| = -\sqrt{3}$$

normal line slope $-\frac{1}{\sqrt{3}}$

$$\frac{1}{\sqrt{3}}$$

$m_s \cdot m_n = -1$

$m_s \cdot m_n = -1$

normal lines

$$y - 3\sqrt{3} = -\frac{1}{\sqrt{3}}(x + 1)$$

$$y + 3\sqrt{3} = \frac{1}{\sqrt{3}}(x + 1)$$

Ex $a^2 w^2 + b^2 \lambda^2 = 1$ find $\frac{dw}{d\lambda}$

$$a^2 2w \frac{dw}{d\lambda} + b^2 2\lambda = 0, \quad \frac{dw}{d\lambda} = -\frac{2b^2 \lambda}{2a^2 w}$$

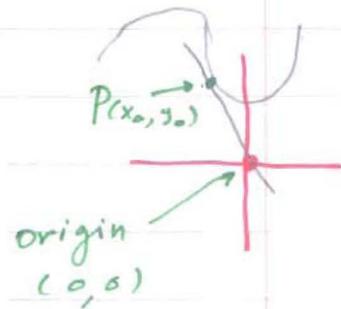
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Ex Find equation of two lines through the origin tangent to $x^2 - 4x + y^2 + 3 = 0$

$P(x_0, y_0)$ is the tangent point(s)

$$y' = \frac{2-x}{y} \quad \text{Slope}$$

origin $(0,0)$



we can find the slope in two ways

$$\begin{aligned} \text{from } y' \text{ at } (x_0, y_0) &\Rightarrow m = \frac{2-x_0}{y_0} \\ \text{from } (0,0) \text{ \& } (x_0, y_0) &\Rightarrow m = \frac{y_0-0}{x_0-0} \end{aligned}$$

$$\Rightarrow \frac{2-x_0}{y_0} = \frac{y_0}{x_0} \Rightarrow y_0^2 = x_0(2-x_0) \quad \text{--- (1)}$$

But (x_0, y_0) on the curve so

$$x_0^2 - 4x_0 + y_0^2 + 3 = 0 \quad \text{--- (2)}$$

$$\Rightarrow x_0^2 - 4x_0 + 2x_0 - x_0^2 + 3 = 0$$

$$x_0 = \frac{3}{2} \quad y_0^2 = \frac{3}{4} \Rightarrow y_0 = \pm \sqrt{\frac{3}{4}}$$

$$\begin{aligned} \text{slope}_1 = m_1 &= -\frac{\sqrt{3}}{3} \\ m_2 &= \frac{\sqrt{3}}{3} \end{aligned}$$

$$\text{e.g., } \Rightarrow y - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{3} \left(x - \frac{3}{2}\right)$$

$$\text{e.g., } \Rightarrow y + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3} \left(x - \frac{3}{2}\right)$$