

3.3 The Product and Quotient Rules

The product

If f and g are both differentiable, then

$$\frac{d}{dx} f(x) g(x) = f(x) \frac{dg(x)}{dx} + \frac{df(x)}{dx} \cdot g(x).$$

The Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x) \cdot \frac{d}{dx} f(x) - f(x) \cdot \frac{d}{dx} g(x)}{\left[g(x)\right]^2}$$

Ex $y = \frac{e^x}{1+x^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1+x^2) \frac{d}{dx} e^x - e^x \frac{d}{dx} (1+x^2)}{(1+x^2)^2} \\ &= \frac{e^x (1+x^2) - 2x e^x}{(1+x^2)^2} \end{aligned}$$

Find the equation of the tangent line to y at $(1, \frac{1}{2}e)$

The slope at 1 is 0 then the equation is

$$y = \frac{1}{2}e.$$

Second derivative and Higher

 $f'(x), f''(x), \dots, f^{(4)}(x), \dots, f^{(n)}(x)$

$\frac{df}{dx} \rightarrow \frac{d^2f}{dx^2} \dots \frac{d^n f}{dx^n}$

Examples

① $y = 3 \quad y' = 0$

② $y = \sqrt{x} \quad y' = \frac{1}{2\sqrt{x}} + 0$

③ $y = ax^3 + bx^2 \quad y' = 3ax^2 + 2bx$

④ $y = x^{-8} \quad y' = -8x^{-9}$

⑤ $y = \left(\frac{1}{x} + \frac{1}{x^2}\right) \cdot x^2 \quad y' = \left[-\frac{1}{x^2} - \frac{2}{x^3}\right] x^2 + \left(\frac{1}{x} + \frac{1}{x^2}\right) 2x$

⑥ $y = (3x^2 + 1)e^x \quad y' = 6xe^x + (3x^2 + 1)e^x$

⑦ find $y'(1)$ if $y = \frac{1}{5x-3}$
 $y' = \frac{-5}{(5x-3)^2} \quad y'(1) = \frac{-5}{(5-3)^2} = -\frac{5}{4}$

⑧ $y = \left(\frac{3x+2}{x}\right)(x^{-5} + 1)$

$y' = \left(\frac{3x+2}{x}\right)(-5x^{-6}) + \frac{3x - (3x+2)1}{x^2} \cdot (x^{-5} + 1)$

⑨ $y = 2(x^3 - 5) + 3x^2(2x+3)$

Find all derivatives

$y' = 2(3x^2) + 6x(2x+3) + 3x^2(2)$

$y'' = 12x + 6(2x+3) + 6x(2) + 12x$

$y''' = 12 + 12 + 12 + 12 = 48$

$y^{(4)} = 0 \quad y^{(n)} = 0 \quad \forall n > 4$

⑩ $f(x) = x^k$ find $f^{(k)}$

$f' = kx^{k-1} \quad f'' = k(k-1)x^{k-2} \dots \quad f^{(k)} = k!$

$f^{(k+n)} = 0 \quad \forall n > 1$

- ⑪ Find the coordinate of all points on the graph of $y = 1 - x^2$ at which the tangent line passes through the point $(2, 0)$.

Ans

The tangent slope is the derivative $y' = -2x$

So any point (x_0, y_0) on the graph of y has the equation of the tangent line $y - y_0 = -2x_0(x - x_0)$ ①

Als (x_0, y_0) on the curve so $y_0 = 1 - x_0^2$ ②

Now $(2, 0)$ on both tangent lines

So from ①

$$-y_0 = -2x_0(2 - x_0)$$

$$\Rightarrow 0 = y_0 - 4x_0 + 2x_0^2 \quad \text{from ②}$$

$$x_0^2 - 4x_0 + 1 = 0$$

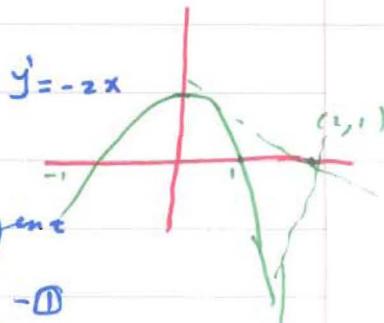
$$\Rightarrow x_0 = 2 \pm \sqrt{3}.$$

$$⑫ f(x) = \begin{cases} x^2 & x \leq 1 \\ \sqrt{x} & x > 1 \end{cases} \quad \text{cont } f'(1) = \lim_{x \rightarrow 1} f(x)$$

$$f'_-(1) = 2x \Big|_{x=1} = 2 \quad \text{while } f'_+(1) = \frac{1}{2\sqrt{x}} \Big|_{x=1} = \frac{1}{2}$$

$$f'(1) \text{ DNE}$$

$$⑬ f(x) = |x - 2| \quad \text{cont but not differentiable at } x = 2.$$

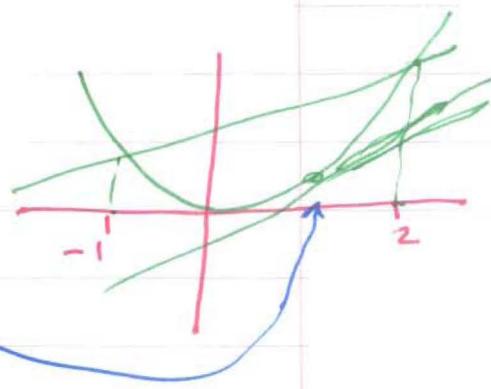


- (14) Find the x -coordinate of the point on the graph of $y = x^2$ when the tangent line is parallel to the secant line that cuts the curve at $x = -1$ & $x = 2$

$$m = \frac{4-1}{2+1} = 1$$

$$y' = 2x = 1 \Rightarrow x = \frac{1}{2}$$

$(\frac{1}{2}, \frac{1}{4})$



- (15) Show that the tangent line to the graph of $y = \frac{1}{x}$ and the coordinate axes form triangle with area equal 2 units.

$$y' = -\frac{1}{x^2}$$

So the slope at x_0 is $-\frac{1}{x_0^2}$
the tangent line equation is

$$y - y_0 = -\frac{1}{x_0^2}(x - x_0)$$

but $y_0 = \frac{1}{x_0}$

then $y = -\frac{x}{x_0^2} + \frac{2}{x_0}$ so the tangent line cross the x -axis at $2x_0$ and the y -axis at $\frac{2}{x_0}$ \Rightarrow Area = $\frac{1}{2} \cdot 2x_0 \cdot \frac{2}{x_0} = 2$.

