# Math260 Revision Notes

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The following notes are to help you revise for Math260. A few important points need to be made:

- These notes do *not* contain everything you need to know for the exam; they only contain some of the most important material.
- The best way to study for this exam is to *practice*. You should spend time solving the questions at the end of each section and solving old exam questions.
- These notes will be a convenient resource while you are doing this practice.
- *Note:* These notes may contain small errors. If you spot any please send me an email and I will update the file. Thank you.

## 1 First-order differential equations

### 1.1 Separable equations

The first-order differential equation

$$\frac{dy}{dx} = H(x, y)$$

is called *separable* provided that $H(x, y)$ can be written as the product of a function of $x$ and a function of $y$:

$$\frac{dy}{dx} = g(x)h(y) = \frac{g(x)}{f(y)} \text{ (where } h(y) = 1/f(y)).$$

We can solve this by integrating:

$$\int f(y)dy = \int g(x)dx + C$$

and then (possibly) solving for $y$.

### 1.2 Linear equations

A *linear* first-order differential equation is of the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

To solve this, first calculate the integrating factor

$$\rho(x) = e^{\int P(x)dx}$$

then multiply both sides of the equation to get

$$D_x[\rho(x)y(x)] = \rho(x)Q(x).$$

We then integrate to get

$$\rho(x)y(x) = \int \rho(x)Q(x)dx + C$$

and then solve for $y$.

### 1.3 Homogeneous equations

A *homogeneous* first-order differential equation is of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

To solve this we make the substitutions

$$v = \frac{y}{x}, y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$$

which will transform our original equation into the separable equation

$$x\frac{dv}{dx} = F(v) - v$$

which we then solve using the method above.

### 1.4 Bernoulli equations

A *Bernoulli* first-order differential equation is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

To solve this we make the substitution $y = y^{1-n}$ which transforms our original equation into a linear equation:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

We then solve this using the method above.

### 1.5 Exact equations

A differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

ie.

$$M(x,y)dx + N(x,y)dy = 0$$

is *exact* if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

To solve such an equation, first integrate $M(x,y)$ with respect to $x$ to get

$$F(x,y) = \int M(x,y)dx + g(y).$$

Then determine $g(y)$ be imposing the condition that

$$\frac{\partial F}{\partial y} = N(x,y).$$

This will yield a general solution of the form $F(x,y) = C$.

## 2 Matrices

We will not summarise this material here (since you know this already from the preparatory year). You should know

- How to solve a linear system using the method of elimination;

- How to reduce a matrix to echelon form using Gaussian elimination;
- How to reduce a matrix to reduced echelon form using Gauss-Jordan elimination;
- How to use the above to solve systems of equations;
- How to add and multiply matrices together;
- How to compute the inverse of a matrix;
- How to work with matrix equations;
- How to compute the determinant of a matrix;
- That a matrix has an inverse if and only if it has nonzero determinant;
- How to use Cramer's rule;
- How to compute the inverse of a matrix using the formula

$$\mathbf{A}^{-1} = \frac{[A_{ij}]^T}{|\mathbf{A}|}.$$

If you cannot do any of this, then you should revise the material in Chapter 3.

# 3 Vector Spaces

To begin, you should know:

- What a vector is;
- How to add vectors together and multiply vectors by scalars.

The *n-dimensional space* $\mathbb{R}^n$ is the set of all $n$-tuples (vectors) $(x_1, x_2, \ldots, x_n)$ of real numbers. We mainly deal with $\mathbb{R}^2$ and $\mathbb{R}^3$ (but sometimes with $\mathbb{R}^n$ for higher $n$).

## 3.1 Subspaces

Let $W$ be a nonempty subset of the vector space $V$. Then $W$ is a *subspace* of $V$ provided that $W$ is itself a vector space with the same operations of addition and multiplication by scalars as in $V$. To check that a subset $W$ is a subspace, you need to check that the following hold for $W$:

1. If $\vec{u}$ and $\vec{v}$ are vectors in $W$, then $\vec{u} + \vec{v}$ is also in $W$;
2. If $\vec{u}$ is in $W$ and $c$ is a scalar, then the vector $c\vec{u}$ is also in $W$.

If $\mathbf{A}$ is a constant $m \times n$ matrix, then the solution set of the homogeneous linear system

$$\mathbf{A}\vec{x} = \vec{0}$$

is a subspace of $\mathbb{R}^n$. We call this the *solution space.*

## 3.2 Bases

Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ be vectors in the vector space $V$. Then the set $W$ of all linear combinations of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ is a subspace of $V$. We say that $W$ is *spanned* by the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$.

The vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ in a vector space $V$ are said to be *linearly independent* provided that the equation

$$c_1\vec{v_1} + c_2\vec{v_2} + \ldots c_k\vec{v_k} = \vec{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$.

The $n$ vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ in $\mathbb{R}^n$ are linearly independent if and only if the $m \times n$ matrix

$$\mathbf{A} = [\vec{v_1}\vec{v_2}\cdots\vec{v_n}]$$

has nonzero determinant.

A finite set $S$ of vectors in a vector space $V$ is called a *basis* for $V$ provided that

1. the vectors in $S$ are linearly independent, and
2. the vectors in $S$ span $V$.

Any two bases for a vector space consist of the same number of vectors. We call this number the *dimension* of the vector space. If a vector space $V$ has dimension $n$, then any set of $k > n$ vectors is linearly dependent.

If $V$ is an $n$-dimensional vector space and $S$ is a subset of $V$, then if $S$ is linearly independent, then $S$ is contained in a basis for $V$. In particular, if $S$ in linearly independent and consists of $n$ vectors, then $S$ is a basis for $V$.

### 3.3 Bases for solution spaces

If you have to homogeneous linear system

$$\mathbf{A}\vec{x} = \vec{0}$$

to find a basis for the solution space $W$:

1. Reduce $\mathbf{A}$ to echelon form;
2. Identify the $r$ leading variables and the $k = n - r$ free variables. If $k = 0$, then $W = \{\vec{0}\}$.
3. Set the free variables to parameters $t_1, t_2, \cdots, t_k$ and the solve by back substitution for the leading variables.
4. Let $\vec{v_j}$ be the solution vector obtained by setting $t_j$ equal to 1 and the other parameters to zero.
5. Then $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a basis for $W$.

## 4 Higher-order equations

### 4.1 Basics

The general $n$th order linear differential equation has the form

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = F(x).$$

We usually assume that $P_0(x) \neq 0$ so that we can divide and get an equation of the form

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x).$$

The associated homogeneous linear equation is then

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

We will assume that the functions $p_i$ and $f$ are continuous on some open interval $I$.

If $y_1, y_2, \ldots, y_n$ are $n$ solutions of the homogeneous linear equation on the interval $I$ and if $c_1, c_2, \ldots, c_n$ are constants, then

$$y = c_1y_1 + c_2y_2 + \cdots c_ny_n$$

is also a solution.

The $n$ functions $f_1, f_2, \ldots, f_n$ are said to be *linearly dependent* on the interval $I$ provided that there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$$

on $I$. If the functions $f_1, f_2, \ldots, f_n$ are not linearly dependent, they are *linearly independent.*

If we have $n$ solutions $y_1, y_2, \ldots, y_n$ of a homogeneous $n$th order linear equation, then we can check to see if they are linearly independent using the *Wronskian*:

$$W = W(x) = W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

To show that the solutions $y_1, y_2, \ldots, y_n$ are linearly independent (on the interval $I$) we must show that their Wronskian is nonzero at some point in $I$.

If $y_1, y_2, \ldots y_n$ are $n$ linearly independent solutions of the homogeneous equation then any other solution $Y$ can be written as a linear combination of these:

$$Y(x) = x_1 y_1(x) + c_2 y_2(x + \cdots c_n y_n(x).$$

Such a linear combination is called a *general solution*.

## 4.2 Constant coefficients

We now consider the homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

where the coefficients $a_0, \ldots, a_n$ are constant.

The *characteristic equation* is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0.$$

### 4.2.1 Distinct real roots

If the roots $r_1, r_2, \ldots, r_n$ of the characteristic equation are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

is a general solution of our equation.

### 4.2.2 Repeated real roots

If the characteristic equation has a repeated root $r$ of multiplicity $k$, then the part of a general solution of the differential equation corresponding to $r$ is of the form

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{rx}.$$

### 4.2.3 Complex roots

We need *Euler's formula*:

$$e^{i\theta} = \cos\theta + i \sin\theta$$

which tells us that if $z = x + iy$, then

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

If the characteristic equation has a complex conjugate pair of roots $a \pm bi$, then the corresponding part of a general solution to the homogeneous differential equation has the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx).$$

### 4.2.4 Repeated complex roots

If the complex conjugate pair of roots $a \pm bi$ has multiplicity $k$, then the corresponding part of the general solution has the form:

$$(A_1 + A_2 x + \cdots A_k x^{k-1}) e^{(a+bi)x} + (B_1 + B_2 x + \cdots + B_k x^{k-1}) e^(a - bi)x.$$

## 4.3 Nonhomogeneous equations

We are still considering a differential equation with constant coefficients (and we will still need the associated homogeneous equation).

A general solution for such an equation will have the form

$$y = y_c + y_p$$

where the *complementary function* $y_c$ is a general solution of the associated homogeneous equation and $y_p$ is a particular solution of our nonhomogeneous equation.

### 4.3.1 The method of undetermined coefficients

**Rule 1:** If the term $f(x)$ in our nonhomogeneous equation is a linear combination of (finite) products of the following types

1. A polynomial in $x$;
2. An exponential function $e^{rx}$;
3. $\cos kx$ or $\sin kx$

and if no term appearing in $f(x)$ or any of its derivatives satisfies the associated homogeneous equation, then we take as a *trial solution* for $y_p$ a linear combination of all linearly independent such terms and their derivatives. We then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation.

There may be *duplication* (ie. Rule 1 does not apply):

**Rule 2:** If the function $f(x)$ is of the form

$$P_m(x)e^{rx}\cos kx \text{ or } P_m(x)e^{rx}\sin kx$$

take as the trial solution

$$y_p(x) = x^s[(A_0 + A_1x + A_2x^2 + \cdots A_mx^m)e^{rx}\cos kx + (B_0 + B_1x + B_2x^2 + \cdots + B_mx^m)e^{rx}\sin kx]$$

where $s$ is the smallest nonnegative integer such that no term in $y_p$ duplicates a term in the complementary function $y_c$. We then determine the coefficients as before.

The best way to understand the method of undetermined coefficients is to look at examples (pp.338–347 in the book).

### 4.3.2 Variation of parameters

If the nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = f(x)$$

has complementary function

$$y_c(x) = c_1y_1(x) + c_2y_2(x)$$

then a particular solution is given by

$$y_p(x) = \left(-\int \frac{y_2(x)f(x)}{W(x)}dx\right)y_1(x) + \left(\int \frac{y_1(x)f(x)}{W(x)}\right)y_2(x)$$

where $W = W(y_1, y_2)$ is the Wronskian of $y_1$ and $y_2$.

## 4.4 Eigenvalues and eigenvectors

A number $\lambda$ is an *eigenvalue* of the $n \times n$ matrix $\mathbf{A}$ if there exists a *nonzero* vector $\vec{v}$ such that

$$\mathbf{A}\vec{v} = \lambda\vec{v}.$$

We call $\vec{v}$ an *eigenvector* of the matrix $\mathbf{A}$ associated to the eigenvalue $\lambda$.

A number $\lambda$ is an eigenvalue of the $n \times n$ matrix $\mathbf{A}$ is and only if it satisfies the *characteristic equation*

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

To find eigenvalues and eigenvectors:

1. First solve the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ to get the eigenvalues;
2. Then, for each eigenvalue $\lambda$, solve the linear system $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$ to find the eigenvectors associated with $\lambda$.

If $\lambda$ is an eigenvalue of $\mathbf{A}$, then the set of all eigenvectors associated with $\lambda$ is the set of all nonzero solutions of the system

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}.$$

The solution space of this system is called the *eigenspace* associated with the eigenvalue $\lambda$. Remember that the zero vector does not count as an eigenvalue (even though it is an element of the eigenspace).

### 4.5 Diagonalisable matrices

The $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are called *similar* if there is an invertible matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$.

An $n \times n$ matrix $\mathbf{A}$ is called *diagonalisable* or $\mathbf{A}$ is similar to a diagonal matrix $\mathbf{D}$.

The $n \times n$ matrix $\mathbf{A}$ is diagonalisable if and only if it has $n$ linearly independent eigenvectors.

If $\mathbf{A}$ has $n$ distinct eigenvalues, then these give rise to $n$ linearly independent eigenvectors and so $\mathbf{A}$ is diagonalisable.

To diagonalise the matrix $\mathbf{A}$, find the $n$ distinct eigenvalues $\lambda_1, \dots, \lambda_n$ and the $n$ associated linearly independent eigenvectors $\vec{v_1}, \dots, \vec{v_n}$ and let $\mathbf{P}$ be the matrix

$$\mathbf{P} = [\vec{v_1}\vec{v_2}\dots\vec{v_n}]$$

and then $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix whose diagonal entries are the eigenvalues $\lambda_1, \dots, \lambda_n$.

If $\mathbf{A}$ has $k < n$ distinct eigenvalues, then

1. For each eigenvalue $\lambda$ find a basis for the associated eigenspace;
2. If you have found at least $n$ eigenvalues then you can pick $n$ eigenvectors $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ and their associated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and let

   $$\mathbf{P} = [\vec{v_1}\vec{v_2}\dots\vec{v_n}]$$

   and $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ will be a diagonal matrix whose diagonal entries are the eigenvalues $\lambda_1, \dots, \lambda_n$.

#### 4.5.1 Powers of matrices

The important point here is to note that, if $\mathbf{A} = \mathbf{PDP}^{-1}$ where $\mathbf{D}$ is a diagonal matrix, then

$$A^k = \mathbf{PD}^k\mathbf{P}^{-1}.$$

#### 4.5.2 The Cayley-Hamilton Theorem

If the $n \times n$ matrix $\mathbf{A}$ has the characteristic polynomial

$$p(\lambda) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_2\lambda^2 + c_1\lambda + c_0$$

then

$$p(\mathbf{A}) = (-1)^n\mathbf{A}^n + C_{n-1}\mathbf{A}^{n-1} + \dots + c_2\mathbf{A}^2 + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0}.$$

You can use this to compute the inverse of the matrix $\mathbf{A}$. For an example of this see pp.392–393 of the textbook.

## 5 Linear systems of differential equations

First, note that a system of differential equations can be transformed into a system of first-order equations and vice versa (see pp.396–400 of the textbook).

A *linear* system of first order equation has the form

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \dots + p_{1n}(t)x_n + f_1(t)\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \dots + p_{2n}(t)x_n + f_2(t)\\
&\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \dots + p_{nn}(t)x_n + f_n(t).
\end{aligned}$$

Such a system is called *homogeneous* if the functions $F_1, f_2, \dots, f_n$ are all identically zero.

The *coefficient* matrix is the matrix $\mathbf{P}(t) = [p_{ij}(t)]$. If we write $\vec{x} = [x_i]$ and $\vec{f}(t) = [f_i(t)]$ then we can write the system as a matrix equation

$$\frac{d\vec{x}}{dt} = \mathbf{P}(t)\vec{x} + \vec{f}(t).$$

A solution of this system on the open interval $I$ is a column vector function $\vec{x}(t) = [x_i(t)]$ such that the component functions satisfy the system.

The vector-valued functions $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}$ are *linearly dependent* on the interval $I$ provided that there exist constants (not all zero) such that

$$c_1\vec{x_1}(t) + c_2\vec{x_2}(t) + \cdots + c_n\vec{x_n}(t) = \vec{0}$$

for all $t$ in $I$. Otherwise they are *linearly independent.*

Their *Wronskian* is

$$W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix}$$

and if $\vec{x_1}, \ldots, \vec{x_n}$ are $n$ solutions (on an open interval $I$) of the homogeneous linear system

$$\frac{d\vec{x}}{dt} = \mathbf{P}(t)\vec{x}$$

and $\mathbf{P}(t)$ is continuous on $I$ then if $\vec{x_1}, \ldots, \vec{x_n}$ are linearly independent on $I$, $W \neq 0$ at each point of $I$.

If $\vec{x_1}, \ldots, \vec{x_n}$ are linearly independent solutions of the homogeneous linear system on $I$ a general solution for the system has the form

$$\vec{x}(t) = c_1\vec{x_1}(t) + c_2\vec{x_2}(t) + \cdots + c_2\vec{x_n}(t).$$

The nonhomogeneous linear system

$$\frac{d\vec{x}}{dt} = \mathbf{P}(t)\vec{x} + \vec{f}(t)$$

has a general solution of the form

$$\vec{x}(t) = \vec{x_c}(t) + \vec{x_p}(t)$$

where $\vec{x_c}(t)$ is a general solution of the associated homogeneous linear system and $\vec{x_p}(t)$ is a particular solution of the nonhomogeneous system.

## 5.1 The eigenvalue method

To solve the $n \times n$ homogeneous constant-coefficient system $\vec{x}' = \mathbf{A}\vec{x}$:

1. Solve the characteristic equation to find eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $\mathbf{A}$.
2. Find $n$ linearly independent eigenvectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ of $\mathbf{A}$ (if possible)
3. You then get the $n$ linearly independent solutions

$$\vec{x_1}(t) = \vec{v_1}e^{\lambda_1 t}, \ldots, \vec{x_n}(t) = \vec{v_n}e^{\lambda_n t}.$$

If you have $n$ distinct real eigenvalues, this is easy.

### 5.1.1 Complex eigenvalues

If you get a complex-conjugate pair $\lambda$ and $\overline{\lambda}$ of eigenvalues: find the real and imaginary parts $\vec{x_1}(t)$ and $\vec{(x_2)}(t)$ of the associated complex solution to get two linearly independent real solutions. (See pp.425–429 of the textbook for examples).

### 5.1.2 Multiple eigenvalues

An eigenvalue is of *multiplicity* $k$ if it is a $k$-fold root of the characteristic equation. If the associated eigenspace has dimension $k$ then we are still OK (we say that the eigenvalue is *complete*, see p446f for an example).

If an eigenvalue $\lambda$ of multiplicity $k$ has only $p < k$ linearly independent eigenvectors we say $\lambda$ is *defective* and call the number $d = k - p$ the *defect* of $\lambda$. This eigenvalue will give us only $p$ solutions. We need to find another $d$ linearly independent solutions to make up for this.

I will describe the method for a single defective multiplicity 2 eigenvalue here. For multiplicity 3 eigenvalue see Section 7.5 of the textbook.

If we have a defective multiplicity two eigenvalue $\lambda$ with associated eigenvector $\vec{v_1}$ we will have a solution

$$\vec{x_1} = \vec{v_1} e^{\lambda t}.$$

To find another solution:

1. Find a nonzero vector $\vec{v_2}$ that is a solution of the equation

$$(\mathbf{A} - \lambda \mathbf{I})^2 \vec{v_2} = \vec{0}$$

   such that

$$(\mathbf{A} - \lambda \mathbf{I}) \vec{v_2} = \vec{v_1}.$$

2. We can then form two linearly independent solutions

$$\vec{x_1}(t) = \vec{v_1} e^{\lambda t}$$

   and

$$\vec{x_2}(t) = (\vec{v_1} t + \vec{v_2}) e^{\lambda t}$$

   of $\vec{x}' = \mathbf{A}\vec{x}$ corresponding to $\lambda$.

See *pp*.$450 - -454$ for an example and a discussion of multiplicity 3 vectors.