

Technology and the Calculus Sequence

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Introduction

Technology is making inroads in the calculus sequence at an ever increasing pace. How useful is it really? A just way to answer this would be to test it against problems from the standard curriculum. At the outset we must say that we are enthusiastic users of technology. For algorithmic and experimental mathematics and as a labour saving device, technology is truly marvellous. What we want to bring in focus is the uncritical and untested acceptance of technology in the curriculum.

This note has two sections. In the first section we give some test problems and try to understand them using software like Maple and Mathematica. It turns out that the software is really not of much use in solving the test problems. In the second section, we outline a teaching module and illustrate it by solving the test problems using classical methods.

1. Some test problems

Here are some problems from the standard curriculum and attempts to understand them using software.

Problem 1. Find the volume of the region inside the cylinder $x^2 + y^2 = 4$ and bounded by the planes $y + z = 4$ and $z = 0$.

A sketch of solid is given below.



The surfaces here and in the following examples have been drawn using Maple.

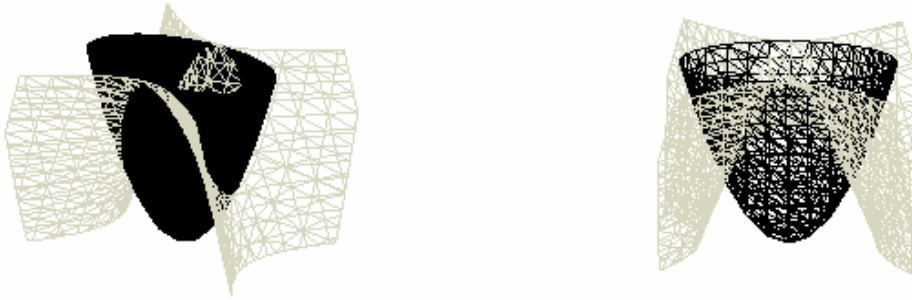
Here it is clear that the volume we want to compute is the volume of the solid above the XY-plane, below the plane $y + z = 4$ and inside the cylinder.

Also the projection of the solid on XY-plane is clearly visible.

Problem 2. Find the volume of the solid bounded by the hyperbolic paraboloid

$$z = 2 + x^2 - y^2 \text{ and the elliptic paraboloid } z = 3x^2 + y^2.$$

Two views of the solid are given below.



The two views seem to indicate that the volume we want to compute is the volume of the solid inside the elliptic paraboloid and below the hyperbolic paraboloid. What is the projection of the solid on the XY -plane?

Problem 3. Find the volume outside the sphere $x^2 + y^2 + z^2 = 1$ and inside the sphere $x^2 + y^2 + (z-1)^2 = 1$.



One sees clearly the solid we are interested in. However the strategy for setting up the limits of integration is not clear from the sketch.

Problem 4. Find the volume inside the cylinders $x^2 + y^2 = R^2$, $y^2 + z^2 = R^2$.
Two views of the solid are given below.



This solid is hard to sketch or imagine as has already been remarked in the archives of Nrich [1]. The interested reader can generate these figures using the following Maple commands. The resulting figure will be an interactive image which can be rotated to get several views of the solid and an idea of the projection on the XY-plane.

```
>with(plots):
>g1:=implicitplot3d( x^2 + y^2 + 0*z^2 = 1,x=-2..2,y=-2..2,
  z=-2..2,grid=[13,13,13],color=black, style=patchnogrid): g1;

>g2:=implicitplot3d( x^2 + 0*y^2 + z^2 = 1,x=-2..2,y=-2..2,
  z=-2..2,grid=[13,13,13],color=wheat, style=wireframe): g2;

>display({g1,g2});
```

Problem 5. Find the volume inside the cylinders $x^2 + y^2 = R^2$, $y^2 + z^2 = R^2$, $z^2 + x^2 = R^2$.
We give two views of the solid.



Here the shape of the solid is not clear (at least to us). Consequently the strategy for finding its volume is also not clear. An interactive image of the solid can be obtained using the following Maple commands.

```
> with(plots):
> g1:=implicitplot3d( x^2 + y^2 + 0*z^2 = 1,x=-2..2,y=-2..2,
  z=-3..3,grid=[13,13,13],color=black, style=patchnogrid): g1;

> g2:=implicitplot3d( x^2 + 0*y^2 + z^2 = 1,x=-2..2,y=-2..2,
  z=-2..2,grid=[13,13,13],color=grey, style=patchnogrid): g2;

> g3:=implicitplot3d( 0*x^2 + y^2 + z^2 = 1,x=-2..2,y=-2..2,
  z=-2..2,grid=[13,13,13],color=wheat, style=patchnogrid): g3;

> display({g1,g2,g3});
```

Comments:

In our opinion, to generate such figures, introductory level tutorials of at least 3 lectures would be needed. In all, the student would have to spend about 12 to 15 hours to get a good grip on these graphing techniques.

2. A module for teaching multiple integration

The students do not have access to advanced software during the examinations so the ability to sketch simple figures by hand remains essential.

The problem of finding the volume for solid bounded by three cylinders of equal radius whose axes are mutually perpendicular is given as a Discovery Project in Stewart's Calculus [3]. It is also mentioned as an inscrutable problem in the archives of Nrich [1]. It is one of those problems in which a sketch – even a computer-generated sketch as seen above – is not helpful. This difficulty forces one to look for easier approaches. Fortunately, there is a rather straightforward method available for dealing with problems of this type and which was published in [2]. Here we want to explain the ideas involved in more detail.

The idea of the method is this: *If one cannot draw an illuminating sketch, one should describe the region of integration by inequalities. These inequalities will suggest projections to suitable planes, which reduce triple integrals to double integrals.*

The description by inequalities is facilitated by drawing the boundaries separately and then superimposing them.

Let us see how this works in practice. The main points to keep in mind are the following:

1. *The projection of a subset E of \mathbb{R}^3 in the plane $z=0$ is the set of all points (x, y) for which there is a z with (x, y, z) in E .*
In general, if H is a plane and p a point, the projection of p in H is the point in H of least distance from p : we denote it by p_s . So the projection of a set E on H is the shadow of E cast by rays which are perpendicular to H .
2. *If a surface is defined by an equation $f(x, y, z) = 0$, then for all the points on one side of the surface, the function f does not change its sign and the sign is*

determined by choosing a test point: here, and in what follows, it is assumed that f is continuous.

3. The region E bounded by the surfaces $z = f(x, y)$ and $z = g(x, y)$ is the union of the sub-regions

$$E_1 = \{(x, y, z) : f(x, y) \leq z \leq g(x, y)\}$$

and

$$E_2 = \{(x, y, z) : g(x, y) \leq z \leq f(x, y)\}.$$

The projection of, say, E_1 on the XY -plane is the plane region R_1 given by the inequality $f(x, y) \leq g(x, y)$. The volume of E_1 is then

$$\iint_{R_1} [g(x, y) - f(x, y)] dA,$$

where dA is the area element of the projection R_1 .

We now give solutions to examples discussed above.

Example 1. Find the volume of the region inside the cylinder $x^2 + y^2 = 4$ and bounded by the planes $y + z = 4$ and $z = 0$.

As we saw above by graphing the bounded region, the top face of the solid is on the plane $y + z = 4$ and its base is on the plane $z = 0$. Let us see how we can reach the same conclusion by working with inequalities.

The points inside the cylinder $x^2 + y^2 = 4$ are given by the inequality $x^2 + y^2 \leq 4$. The region bounded by the planes $z = 4 - y$ and $z = 0$ is a union of the regions

$$E_1 : 0 \leq z \leq 4 - y$$

and

$$E_2 : 4 - y \leq z \leq 0.$$

For points in E_1 , $y \leq 4$ and for points in E_2 , $y \geq 4$. Since points inside the cylinder are given by $x^2 + y^2 \leq 4$, we have $|y| \leq 2$ for such points. So our solid is indeed given by $x^2 + y^2 \leq 4$, $0 \leq z \leq 4 - y$.

Its projection on the XY -plane is therefore the region R given by

$$\begin{aligned} R &= \{(x, y) : x^2 + y^2 \leq 4, 0 \leq 4 - y\} \\ &= \{(x, y) : x^2 + y^2 \leq 4\} \end{aligned}$$

For any such (x, y) in R , z varies from $z = 0$ to $z = 4 - y$. Therefore the volume of E is given by

$$\text{Vol}(E) = \iint_R [4 - y - 0] dA.$$

Example 2. Find the volume of the solid bounded by the hyperbolic paraboloid $z = 2 + x^2 - y^2$ and the elliptic paraboloid $z = 3x^2 + y^2$.

From the graph of the solid drawn above, we see that the base of the solid is the elliptic paraboloid $z = 3x^2 + y^2$ whereas its top is the hyperbolic paraboloid $z = 2 + x^2 - y^2$. This can be checked very easily by working with inequalities. The region above the surface $z = 3x^2 + y^2$ and below the surface $z = 2 + x^2 - y^2$ is given by

$$3x^2 + y^2 \leq z \leq 2 + x^2 - y^2.$$

Its projection on the XY-plane is given by

$$3x^2 + y^2 \leq 2 + x^2 - y^2.$$

The projection is therefore the disc $D: x^2 + y^2 \leq 1$.

The region above the surface $z = 2 + x^2 - y^2$ and below the surface $z = 3x^2 + y^2$ projects on the XY-plane to the unbounded region $x^2 + y^2 \geq 1$. Therefore, if we want a solid with a finite volume bounded by the given surfaces, we should pick the one whose projection is bounded. So, the solid is indeed described by the inequalities

$$3x^2 + y^2 \leq z \leq 2 + x^2 - y^2$$

and its volume is

$$\iint_D [2 + x^2 - y^2 - (3x^2 + y^2)] dA.$$

Notice that we are not working with boundaries of the type $z = e^{-(x^2+y^2)}$, for which the above argument would fail.

Example 3. Find the volume outside the sphere $x^2 + y^2 + z^2 = 1$ and inside the sphere

$$x^2 + y^2 + (z-1)^2 = 1.$$

The solid is completely described by the inequalities

$$x^2 + y^2 + z^2 \geq 1 \text{ and } x^2 + y^2 + (z-1)^2 \leq 1.$$

Changing to spherical coordinates (ρ, θ, φ) , we see that the solid is given by

$$\rho \geq 1 \text{ and } \rho^2 - 2\rho \cos \varphi \leq 0.$$

So $\rho \geq 1$ and $\rho \leq 2 \cos \varphi$. Hence the solid is given by the inequalities

$$1 \leq \rho \leq 2 \cos \varphi.$$

Therefore, for points in the solid we have

$$1 \leq 2 \cos \varphi.$$

Since φ takes the values between 0 and π and $\cos \varphi$ is decreasing in this range, the

inequality $\frac{1}{2} \leq \cos \varphi$ forces φ for the given solid to take values between 0 and $\frac{\pi}{3}$. As

there are no constraints on θ , it can take any value between 0 and 2π . Therefore, the solid is completely described by the inequalities

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \frac{\pi}{3}$$

and for any such φ and θ , ρ varies from $1 \leq \rho \leq 2 \cos \varphi$. Therefore, its volume is

$$\int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/3} \int_{\rho=1}^{2\cos\varphi} \rho^2 \sin \varphi d\rho d\varphi d\theta.$$

Example 4. Find the volume inside the cylinders $x^2 + y^2 = R^2$, $y^2 + z^2 = R^2$.

The solid inside the two cylinders is completely described by the inequalities

$$x^2 + y^2 \leq R^2, \quad y^2 + z^2 \leq R^2.$$

In other words, the solid is given by

$$x^2 + y^2 \leq R^2, \quad -(R^2 - y^2)^{1/2} \leq z \leq (R^2 - y^2)^{1/2}.$$

Therefore, its projection on the XY-plane is the disc $D = \{(x, y) : x^2 + y^2 \leq R^2\}$. Hence its

volume – review point 3 given on page 6 now - is given by

$$\iint_D \left[(R^2 - y^2)^{\frac{1}{2}} - \left(-(R^2 - y^2)^{\frac{1}{2}} \right) \right] dA = \iint_D 2(R^2 - y^2)^{\frac{1}{2}} dA.$$

Thinking of D as a Type II region in the terminology of [3, p. 856], we see that the volume is given by

$$\int_{-R}^R \left(\int_{x=-(R^2-y^2)^{\frac{1}{2}}}^{(R^2-y^2)^{\frac{1}{2}}} (R^2 - y^2)^{\frac{1}{2}} dx \right) dy.$$

Example 5. Find the volume inside the cylinders $x^2 + y^2 = R^2$, $y^2 + z^2 = R^2$, $z^2 + x^2 = R^2$.

This solid is very hard to imagine, as can be seen from sketches above or in [3, p.901]. However, its volume can be computed with surprising ease and it is a variant of the previous example.

The solid E is described by the inequalities

$$x^2 + y^2 \leq R^2, \quad y^2 + z^2 \leq R^2, \quad z^2 + x^2 \leq R^2.$$

Its projection on the XY-plane is therefore the disc $D = \{(x, y) : x^2 + y^2 \leq R^2\}$. The reason is that given any such (x, y) , there is a z with $y^2 + z^2 \leq R^2$ and $z^2 + x^2 \leq R^2$. Therefore, $|z| \leq (R^2 - y^2)^{\frac{1}{2}}$ and $|z| \leq (R^2 - x^2)^{\frac{1}{2}}$. This means that for (x, y, z) in the solid, $|z| \leq \min \left\{ (R^2 - y^2)^{\frac{1}{2}}, (R^2 - x^2)^{\frac{1}{2}} \right\}$. In the region D_1 where $|y| \leq |x|$, we have $(R^2 - y^2)^{\frac{1}{2}} \geq (R^2 - x^2)^{\frac{1}{2}}$ and in the region D_2 where $|y| \geq |x|$, we have $(R^2 - y^2)^{\frac{1}{2}} \leq (R^2 - x^2)^{\frac{1}{2}}$. Therefore, for (x, y) in D_1 , the range for z is given by $|z| \leq (R^2 - x^2)^{\frac{1}{2}}$ and for (x, y) in D_2 , the range for z is given by $|z| \leq (R^2 - y^2)^{\frac{1}{2}}$. So,

$$\text{Vol}(E) = \iint_{D_1 \cap D} 2(R^2 - x^2)^{\frac{1}{2}} dA + \iint_{D_2 \cap D} 2(R^2 - y^2)^{\frac{1}{2}} dA.$$

Introducing polar coordinates, the lines $y^2 - x^2 = 0$ are given by the rays $\theta = -\frac{\pi}{4}$,

$\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$ and $\theta = \frac{5\pi}{4}$. The volume then works out to be

$$\text{Vol}(E) = 16R^3 \left(1 - \frac{1}{\sqrt{2}} \right);$$

(see [2] for details).

Conclusion:

As these examples show, the ability to sketch simple figures is absolutely essential for using inequalities effectively and their use supplements and enhances geometric understanding.

An uncritical acceptance of technology would be harmful for the transmission of necessary skills which are needed for a mastery of the subject and its applications in the development of technology itself.

It is clear that technology is here to stay. Should there be separate courses in technology? How should it be integrated into the curriculum so that the teachers and students do not become simple consumers? These are questions which need to be debated thoroughly.

References

- [1] Asked NRIC, Shape and Geometry-Volume, area, perimeter, missing length-Intersection of two cylinders.
(<http://nrich.maths.org/askedNRICH/edited/1369.html>)
- [2] H. Azad, The projection method in evaluating multiple integrals, Int. J. Math. Ed. Sci. Tech., 2001, Vol. 32, No.3, 444-446.
- [3] J. Stewart, Calculus and Concepts, 1997, Brooks Cole Publishing Company, California.