



**Lectures on Levi Convexity
and Kahler Manifolds**

By

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PREFACE

These lectures were delivered at the College on Arakelov Geometry which was organized by Professors Amir Assadi, Uwe Jannsen and Norbert Schappacher at the International Centre for Theoretical Physics, Trieste, in September 1992. They cover basic material on Levi convexity and Kähler manifolds.

In view of the varied background of the participants, the organizers had asked me to give as simple proofs as I could of all the basic results. On the other hand, I wanted to reach a certain depth in exposition and for this reason I brought plurisubharmonic functions – though not in their most general form – to the fore and put more emphasis on them than is customary, as their introduction leads very quickly to interesting results. I have added some additional material on group theory which fits well here.

I thank the organizers of the College for their invitation to deliver these lectures.

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Chapter 1

Basic Notions

§ 1. Some notations and definitions

We shall denote by \mathbb{R} the set of real numbers, by \mathbb{C} the set of complex numbers and by \mathbb{R}^n and \mathbb{C}^n their Cartesian products. Recall [GP] that a function defined on an open subset of \mathbb{R}^n is differentiable if its partial derivatives of all orders exist and are continuous. If $f = u + iv$ is a holomorphic function of one variable then the Cauchy–Riemann equations [N]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

can be written as

$$\begin{aligned} 0 &= (u_x - v_y) + i(u_y + v_x) & (u_x = \frac{\partial u}{\partial x}, v_y = \frac{\partial v}{\partial y} \text{ etc.}) \\ &= \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u + iv). \end{aligned}$$

Now for $z = x + iy$ we set $dz = dx + idy$,

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

So $f = u + iv$ is holomorphic iff $\frac{\partial f}{\partial \bar{z}} = 0$.

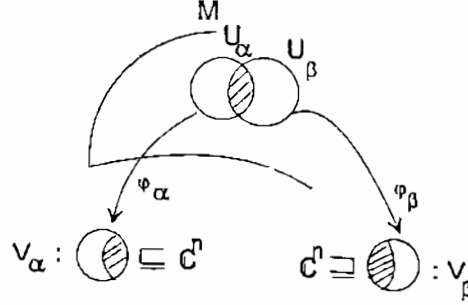
For a differentiable function f defined on an open set U of \mathbb{C}^n , we set

$$\begin{aligned} \frac{\partial f}{\partial z_j} &= \frac{1}{2} \left(\frac{\partial f}{\partial x_j} - i \frac{\partial f}{\partial y_j} \right), \\ \frac{\partial f}{\partial \bar{z}_j} &= \frac{1}{2} \left(\frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} \right) \quad (j = 1, \dots, n), \end{aligned}$$

and say that f is holomorphic if it satisfies the CR equations $\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n$.

We can now define what a complex manifold is.

(1.1) Definition. An n -dimensional complex manifold is a differentiable manifold, which admits an open covering $\{U_\alpha\}$ together with homeomorphisms $\varphi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^n$ open such that for all α, β with $U_\alpha \cap U_\beta \neq \emptyset$ the *transition functions* $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ are *holomorphic*.



(1.2) Remark. The topology of M is completely determined by the maps φ_α . Instead of giving to M first a topology we could simply demand that the maps φ_α should be bijective and the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ should be open. Basic open sets of M are then the inverse images under the φ_α 's of open sets in $\varphi_\alpha(U_\alpha) = V_\alpha$.

Examples of complex manifolds

- 1) \mathbb{C}^n , or any open set of \mathbb{C}^n .
- 2) If f_1, \dots, f_k are holomorphic functions on \mathbb{C}^n and the Jacobian matrix $\left(\frac{\partial f_i}{\partial z_j} \right)_{\substack{i=1, \dots, k \\ j=1, \dots, n}}$ has rank k on the set $M : f_1 = \dots = f_k = 0$, then M is a complex manifold of dimension $n - k$. For example, the equation $z_0^2 + \dots + z_n^2 = 1$ in \mathbb{C}^{n+1} defines an n -dimensional complex manifold.
- 3) $M : \mathbb{C}^n / \Lambda$, Λ a discrete subgroup. Let $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^n / \Lambda$ be the projection, i.e. $\pi(z) = z + \Lambda$. We give to M the quotient topology. For $p \in \mathbb{C}^n$ choose a ball B_p of sufficiently small radius so that $\pi|_{B_p}$ is injective. We take these sets as the open sets of our covering. The transition functions are the translations

$z \mapsto z + \gamma, \quad \gamma \in \Lambda$. Hence M is a complex manifold.

The next example is a fundamental example of a compact complex manifold.

- 4) $\mathbb{P}^n(\mathbb{C})$: The complex n -dimensional projective space. As a set $\mathbb{P}^n(\mathbb{C})$ is the set of all lines in \mathbb{C}^{n+1} passing through 0, i.e. it is the set of all 1-dimensional complex subspaces of \mathbb{C}^{n+1} passing through 0. For $v, w \in \mathbb{C}^{n+1}$, the line $\mathbb{C}v = \mathbb{C}w$ if and only if $v = \lambda w$, $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus 0$. On $\mathbb{C}^{n+1} \setminus 0$ we introduce the equivalence relation $v \sim w$ iff $v = \lambda w$ for $\lambda \in \mathbb{C}^*$, and we denote the equivalence class of v by $[v]$. So $\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^{n+1} \setminus 0 / \sim$. Let $\pi : \mathbb{C}^{n+1} \setminus 0 \rightarrow \mathbb{P}^n(\mathbb{C})$ be the map $\pi(v) = [v]$. If $v = (z_0, \dots, z_n)$ we also write $[v] = [z_0 : z_1 : \dots : z_n]$. To $\mathbb{P}^n(\mathbb{C})$ we give the *quotient topology* determined by the map $\pi : \mathbb{C}^{n+1} \setminus 0 \rightarrow \mathbb{P}^n(\mathbb{C})$, namely the set $U \subset \mathbb{P}^n(\mathbb{C})$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}^{n+1} \setminus 0$.

Let $U_i = \{[z_0 : \dots : z_n] : z_i \neq 0\}$. The sets U_i ($i = 0, \dots, n$) are open and cover $\mathbb{P}^n(\mathbb{C})$. We define $\varphi_i : U_i \rightarrow \mathbb{C}^n$ by $\varphi_i[z_0 : z_1 : \dots : z_n] = (z_0/z_i, \dots, z_{i-1}/z_i, z_{i+1}/z_i, \dots, z_n/z_i)$ and $\psi_i : \mathbb{C}^n \rightarrow U_i$ by $\psi_i(z_1, \dots, z_n) = [z_1 : \dots : z_{i-1} : 1 : z_i : \dots : z_n]$. The map ψ_i is the inverse of φ_i so $\varphi_i : U_i \rightarrow \mathbb{C}^n = V_i$ are homeomorphisms. Moreover, the transition functions $\varphi_{\beta} \circ \varphi_{\alpha}^{-1} : \varphi_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow \varphi_{\beta}(U_{\alpha} \cap U_{\beta})$ are holomorphic because, assuming $\alpha < \beta$, we have $(\varphi_{\beta} \circ \varphi_{\alpha}^{-1})(z_1, \dots, z_n) = (z_1/z_{\beta}, \dots, z_{\alpha-1}/z_{\beta}, 1/z_{\beta}, z_{\alpha}/z_{\beta}, \dots, z_{\beta-1}/z_{\beta}, z_{\beta+1}/z_{\beta}, \dots, z_n/z_{\beta})$. Hence $\mathbb{P}^n(\mathbb{C})$ is a complex n -dimensional manifold. Since every line in \mathbb{C}^{n+1} cuts the unit sphere S^{2n+1} , we see that $\mathbb{P}^n(\mathbb{C})$ is also a continuous image of S^{2n+1} , hence it is also compact. For $n = 1$, $\mathbb{P}^1(\mathbb{C}) = U_0 \cup \{[0 : 1]\}$, where $U_0 = \{[z_0 : z_1] \in \mathbb{P}^1(\mathbb{C}) : z_0 \neq 0\} \cong \mathbb{C}$. So $\mathbb{P}^1(\mathbb{C})$ is the 1-point compactification of the complex plane \mathbb{C} , i.e. it is diffeomorphic to the two spheres S^2 .

§ 2. Subharmonic functions; the maximum principle

Complex manifolds have many properties which are different from those of real analytic manifolds. In contrast to such manifolds, which can always be embedded in some suitable \mathbb{R}^n real analytically [GP] a compact complex manifold cannot be embedded complex analytically in any \mathbb{C}^n . The basic reason for this is that the modulus of a holomorphic function is subharmonic – we define this term presently – and the maximum principle holds for such functions, namely, if a subharmonic function defined on an open connected set $U \subset \mathbb{R}^2$ achieves its maximum value on U , then it must be constant. Subharmonic functions are of fundamental importance in complex variables and complex geometry.

(2.1) Definition. A differentiable function φ defined on an open set $U \subset \mathbb{R}^2$ is subharmonic if $\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} \geq 0$.

(2.2) Example. If $f(z)$ is a holomorphic function then $|f(z)|^2$ is subharmonic. To see this, notice that in terms of the operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ the operator $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ factorizes as $\nabla = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = 4\frac{\partial}{\partial z} \left(\frac{\partial}{\partial \bar{z}}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ and therefore we have:

$$\begin{aligned}\nabla|f(z)|^2 &= 4\frac{\partial}{\partial z} \left(\frac{\partial}{\partial \bar{z}}\right) (f(z)\overline{f(z)}) = 4\frac{\partial}{\partial z} (f(z)\overline{f'(z)}) \\ &= 4f'(z)\overline{f'(z)} = 4|f'(z)|^2 \geq 0.\end{aligned}$$

(2.3) Lemma. Let $f(x, y)$ be a differentiable function defined on the disc $\Delta(a, R)$ (i.e. the disc with center a and radius R). Let $F(r) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$, $(0 < r < R)$. Then $F'(r) = \frac{1}{2\pi} \iint_{\Delta(a, r)} (\nabla f) dx dy$.

Proof. We have

$$\begin{aligned}F'(r) &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial f}{\partial x}(a + re^{i\theta}) r \cos \theta + \frac{\partial f}{\partial y}(a + re^{i\theta}) r \sin \theta \right) d\theta \\ &= \frac{1}{2\pi} \iint_{\Delta(a, r)} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx dy\end{aligned}$$

as $\oint_{\partial\Delta(a,r)} Pdx + Qdy = \iint_{\Delta(a,r)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$ (by Green's theorem). \square

In particular this gives the following important corollary.

(2.4) Corollary. *If f is subharmonic on an open set $U \subset \mathbb{C}$ and $\Delta(a, r) \subset U$ then*

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta.$$

Proof. By (2.3) if $F(t) = \frac{1}{2\pi} \int_0^{2\pi} f(a + te^{i\theta}) d\theta$ then $F'(t) = \iint_{\Delta(a,r)} \nabla f \geq 0$. Hence F is an increasing function of t and so $F(0) = f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$. \square

(2.5) Corollary. *If a subharmonic function f defined on an open connected set $U \subset \mathbb{C}$ achieves its maximum value at a point $a \in U$ then $f(z) = f(a)$.*

Proof. Choose a disc $\Delta(a, R) \subset U$. Now for all $r \leq R$ we have $f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta \leq f(a)$. Therefore $\frac{1}{2\pi} [f(a) - f(a + re^{i\theta})] d\theta = 0$ and so $f(a) = f(a + re^{i\theta})$ for all $0 \leq \theta \leq 2\pi$. Since this holds also for all $r \leq R$ we see that $f(z) = f(a)$ on $\Delta(a, R)$. Therefore the set where $f(z) = f(a)$ is open. It is also closed and as U is connected we must have $f(z) = f(a)$. \square

From Cor. (2.4) we see that if f is a holomorphic function defined on an open connected set $U \subset \mathbb{C}^n$ and $|f(z)| \leq |f(a)|$ for some $a \in U$ then $|f(z)| = |f(a)|$. For, if $a = (a_1, \dots, a_n)$ then in a suitable multidisc $|z_1 - a_1| < \epsilon_1, \dots, |z_n - a_n| < \epsilon_n$ we have by (2.4) $|f(a_1, a_2, \dots, a_n)| = |f(z_1, a_2, \dots, a_n)| = |f(z_1, z_2, a_3, \dots, a_n)| \dots = |f(z_1, z_2, \dots, z_n)|$. Hence $|f(z)| = |f(a)|$ on U i.e. $f(z)\overline{f(z)} = f(a)\overline{f(a)}$. This means that $\overline{f(z)}$ is also holomorphic and therefore $f(z)$ must be constant. This in turn implies that a compact complex manifold does not have any nonconstant holomorphic functions. In particular, such a manifold cannot be embedded holomorphically into any \mathbb{C}^n , because the coordinate functions on \mathbb{C}^n restricted to the manifold in question would be constant and the manifold would be just a point.

Exercise. Show that the real projective plane $\mathbb{P}^2(\mathbb{R})$ embeds in \mathbb{R}^4 by using the map $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$.

§ 3. The Levi-form of a function; plurisubharmonic functions

Another important geometric concept relating to complex manifolds is that of the Levi-form of a real-valued function defined on such a manifold.

Let $U \subset \mathbb{C}^n$ be an open set and $f : U \rightarrow \mathbb{R}$ a differentiable function. Let $z_\alpha = x_\alpha + iy_\alpha$ ($\alpha = 1, \dots, n$) be the coordinates in \mathbb{C}^n . Denoting the operators $\frac{1}{4} \left(\frac{\partial}{\partial x_\alpha} - i \frac{\partial}{\partial y_\alpha} \right) \left(\frac{\partial}{\partial x_\beta} + i \frac{\partial}{\partial y_\beta} \right)$ by $\frac{\partial}{\partial z_\alpha} \left(\frac{\partial}{\partial \bar{z}_\beta} \right)$ we see that matrix $\left[\left(\frac{\partial^2 f}{\partial z_\alpha \partial \bar{z}_\beta} \right) \right]$ is Hermitian matrix, called the Levi-matrix of f . The associated Hermitian form, denoted by $L(f)$, is the Levi-form of f . So, for $P \in U$, $u, v \in \mathbb{C}^n$ we have $L(f)_P(u, v) = u \left[\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}(P) \right] v^{-i}$. The Levi-form has the following important invariance property.

(3.1) Proposition. *If f is a real-valued differentiable function defined on an open set $U \subset \mathbb{C}^n$, and $V \subset \mathbb{C}^m$ is open and $h : V \rightarrow U$, $h(z_1, \dots, z_m) = (h_1(z_1, \dots, z_m), \dots, h_n(z_1, \dots, z_m))$ is holomorphic then $L(f \circ h)_P(u, v) = L(f)_{h(P)}(h_*(P)(u), h_*(P)(v))$; here $h_*(P)$ is the complex Jacobian of h at P , namely $h_*(P) = \left[\left(\frac{\partial h_i}{\partial z_j}(P) \right) \right]_{\substack{i=1, \dots, n \\ j=1, \dots, m}}$.*

Proof. We have $(f \circ h)(z_1, \dots, z_m) = f(h_1, \dots, h_n)$ so, by the chain rule,

$$\begin{aligned} \frac{\partial(f \circ h)}{\partial z_\alpha} &= \frac{\partial f}{\partial h_i} \frac{\partial h_i}{\partial z_\alpha} + \frac{\partial f}{\partial \bar{h}_i} \frac{\partial \bar{h}_i}{\partial z_\alpha} \quad (\text{using summation convention}) \\ &= \frac{\partial f}{\partial h_i} \frac{\partial h_i}{\partial z_\alpha} \quad \left(\frac{\partial \bar{h}_i}{\partial z_i} = 0 \text{ because } h_i \text{ is holomorphic} \right). \end{aligned}$$

Hence

$$\frac{\partial^2}{\partial \bar{z}_\beta \partial z_\alpha} (f \circ h) = \frac{\partial^2 f}{\partial \bar{h}_k \partial h_i} \frac{\partial \bar{h}_k}{\partial \bar{z}_\beta} \frac{\partial h_i}{\partial z_\alpha}$$

i.e.

$$\frac{\partial^2(f \circ h)}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{\partial h_i}{\partial z_\alpha} \frac{\partial^2 f}{\partial h_i \partial \bar{h}_k} \frac{\partial \bar{h}_k}{\partial \bar{z}_\beta}$$

which is equivalent to the assertion in the proposition. \square

This proposition shows that given a *complex* manifold M and a real-valued differentiable function f we can associate to f an intrinsic Hermitian form namely the Levi-form $L(f)$ of f , whose matrix in a system of local holomorphic coordinates z_1, \dots, z_n is $\left[\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right) \right]$.

(3.2) Definition. If the Levi-form of f is positive definite, we say that f is strictly plurisubharmonic (s.p.s.h); if the Levi-form of f is positive semi-definite then we say that f is plurisubharmonic (p.s.h.).

(3.3) Remarks

1. If f is p.s.h. on a complex manifold M and $\gamma : U \rightarrow M$ ($U \subset \mathbb{C}$) is a holomorphic map then by (3.1) $\frac{\partial^2}{\partial z \partial \bar{z}}(f(\gamma(z))) \geq 0$. In other words, the function f is subharmonic along every complex curve. Hence the word “plurisubharmonic”.
2. Strictly plurisubharmonic functions are intimately related to *Kählerian metrics*, as we shall see in § 2.2, Chapter 2.

(3.4) Example. As $|z|^2$ is subharmonic we see that if f is a holomorphic function on \mathbb{C}^n then $|f|^2$ is p.s.h. If f_1, \dots, f_m are holomorphic functions on an open set $U \subset \mathbb{C}^n$ then $|f_1|^2 + \dots + |f_m|^2 + \epsilon(|z_1|^2 + \dots + |z_n|^2)$ is s.p.s.h.

Chapter 2

Kähler manifolds

§1. Geometric aspects

We assume familiarity with basic terminology related to Riemannian metrics and differential forms, as explained in, e.g., [DC] and [GP].

Let M be a real manifold on which there exists an almost complex structure, say J (i.e. $\forall p \in M \quad J_p \in \text{End}(T_p(M))$ with $J_p^2 = -\text{Id}$ and J maps smooth vector fields to smooth vector fields). So $\dim M$ must be even, say $\dim M = 2n$. We say that M is an almost complex manifold.

A Riemannian metric g on M such that $g(JX, JY) = g(X, Y)$ for all smooth vector fields X, Y is said to be a Hermitian metric. The tensor ω defined by $\omega(X, Y) = g(JX, Y)$ is skew-symmetric and it is called the fundamental form of g . Since $\omega(X, JX) = g(X, X)$ we see that ω_p is nondegenerate for all $p \in M$. By linear algebra we can find a basis of $T_p(M)$, say $e_1, f_1, \dots, e_n, f_n$ such that $\omega_p(e_i, f_i) = 1$, $\omega_p(e_i, f_j) = 0$ if $i \neq j$ and $\omega_p(e_i, e_i) = \omega_p(f_i, f_i) = 0 \quad \forall i, j$; that is the matrix of ω_p in this basis is

$$\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}.$$

If $\xi_1, \eta_1, \dots, \xi_n, \eta_n$ is the dual basis then $\omega_p = \xi_1 \wedge \eta_1 + \dots + \xi_n \wedge \eta_n$ and therefore

$$\begin{aligned} \omega_p^n &= \sum_{\{i_1, \dots, i_n\} \subset \{1, 2, \dots, n\}} \xi_{i_1} \wedge \eta_{i_1} \wedge \xi_{i_2} \wedge \eta_{i_2} \wedge \dots \wedge \xi_{i_n} \wedge \eta_{i_n} \\ &= \sum \xi_{i_1} \eta_{i_1} \wedge \dots \wedge \xi_{i_n} \eta_{i_n} \quad (\text{the sum being over permutations } \{i_1, \dots, i_n\} \\ &\quad \text{of } \{1, \dots, n\}) \\ &= n! \xi_1 \wedge \eta_1 \wedge \dots \wedge \xi_n \wedge \eta_n. \end{aligned}$$

So $\omega^n \neq 0$ and therefore M is orientable and ω^n is a volume form. In particular, if M is compact (without boundary) then ω^n is not an exact form.

If the exterior derivative $d\omega$ of the fundamental 2-form of g is zero we say that g is a Kählerian metric. This has interesting implications for the homology of M .

(1.1) Proposition. *If M is a compact $2n$ -dimensional Kähler manifold then*

$$H^{2k}(M, \mathbb{R}) \neq 0, \quad 0 \leq k \leq n.$$

Proof. Let ω be the fundamental form of a given Hermitian metric with $d\omega = 0$. As ω^n is a volume form for M we have, by Stokes theorem [GP], that ω^n is not exact (i.e. $\omega^n \neq d\eta$ for any $(2n-1)$ form η). Now if $\omega^k = d\xi$ for some $k < n$ then $d(\xi \wedge \omega) = d\xi \wedge \omega + (-1)^{2k-1}\xi \wedge d\omega = d\xi \wedge \omega = \omega^{k+1}$. Continuing, we would have $\omega^n = d\eta$ for a suitable η , a contradiction. As $d\omega^k = 0$ we see that ω^k are closed forms which are not exact. This is what had to be shown. \square

(1.2) Proposition. *If M is a complex Kähler manifold then every complex submanifold of M is also Kählerian.*

Proof. Let J be the complex structure tensor of M , g a J -invariant Riemannian metric and ω the associated fundamental form. If N is a complex submanifold of M and $p \in N$ then J_p maps $T_p(N)$ into $T_p(M)$. Moreover if $i : N \rightarrow M$ is the inclusion map then $i^*(d\omega) = d(i^*\omega) = 0$. Hence N is Kählerian.

(1.3) Examples

- 1) \mathbb{C}^n with the flat metric $g = (dx_1)^2 + (dy_1)^2 + \dots + (dx_n)^2 + (dy_n)^2$, where $z_a = x_a + iy_a$ are the coordinates of $z \in \mathbb{C}^n$, is a Kähler metric. If $v = (v_1, v'_1, \dots, v_n, v'_n) \in T_p(\mathbb{C}^n)$ then $J_v = (-v'_1, v_1, \dots, -v'_n, v_n)$ so g is J -invariant. If ω is the fundamental form

of g then as $J\left(\frac{\partial}{\partial x_a}\right) = \frac{\partial}{\partial y_a}$ and $J\left(\frac{\partial}{\partial y_a}\right) = -\frac{\partial}{\partial x_a}$, we have

$$\begin{aligned}\omega\left(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial y_a}\right) &= \omega\left(\frac{\partial}{\partial x_a}, J\frac{\partial}{\partial x_a}\right) \\ &= g\left(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial x_a}\right) = 1\end{aligned}$$

and $\omega\left(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial x_b}\right) = 0 = \omega\left(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial y_b}\right) \quad (b \neq a)$. So $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ and $d\omega = 0$.

- 2) Any complex submanifold of \mathbb{C}^n is Kähler.
- 3) On the negative side the spheres S^{2n} ($n \neq 1$) are, by Prop. 4.1, not Kähler as $H^k(S^n) = 0$ if $k \neq 0, n$.
- 4) The Complex Projective Space $\mathbb{P}^n(\mathbb{C})$ is Kähler. In fact, up to a constant, it has a unique $U(n+1)$ -invariant Hermitian metric which is automatically Kähler. This is the Fubini-Study metric. To see this, notice that $U(n+1)$ operates transitively on $\mathbb{P}^n(\mathbb{C})$ and the stabilizer of $p_0 = [1 : 0 : \cdots : 0]$ is $U(1) \times U(n)$. For $z \in \mathbb{C}^n$ the map $z \mapsto [1 : z] = [1 : z_1 : \cdots : z_n]$ maps \mathbb{C}^n into an open neighborhood of p_0 . Hence if $h = (e^{i\theta} \times g) \in H$ then the differential h_x of h maps $v \in T_o(\mathbb{C}^n) \cong T_{p_0}(\mathbb{P}^n)$ to $g(v/e^{i\theta})$ (*). Now a $U(n+1)$ invariant Hermitian metric is completely determined by choosing an H -invariant Hermitian metric on $T_{p_0}(\mathbb{P}^n)$ and from (*) we see that such a metric is unique up to a positive constant. From (*) we also see that $\exists g \in H$ such that $g_*(v) = -v, \quad \forall v \in T_{p_0}(\mathbb{P}^n)$. Therefore if ω is the fundamental 2-form of a $U(n+1)$ -invariant Hermitian metric on \mathbb{P}^n then $\eta = d\omega$ is a $U(n+1)$ -invariant 3-form. Choosing $g \in H$ such that $g_*v = -v$ for $v \in T_{p_0}(\mathbb{P}^n)$ we see that $(g_*\eta)_{p_0} = (\eta)_{p_0}$ implies that $\eta_{p_0} = 0$ and therefore $\eta = 0$ everywhere. \square
- 5) In view of Proposition 1.2 and Example 5 we see that any complex submanifold of $\mathbb{P}^n(\mathbb{C})$ is a Kähler manifold.

We now come to some finer properties of complex manifolds.

(1.4) Wirtinger's Theorem. *Let (M, J) be an almost complex manifold, g a Hermitian metric on M and ω the associated fundamental form. Let $N \subset M$ be a $2m$ -dimensional oriented real submanifold and dV the canonical volume form of N . Then $\omega^m/m!|_{T_p(N)} \leq dV_p$ with equality if and only if $T_p N$ is a J -invariant subspace of $T_p(N)$ with its canonical orientation.*

To prove this we need the following lemma from linear algebra

(1.5) Lemma. *If ω is a skew symmetric $2n \times 2n$ matrix then there exists $A \in O(2n, \mathbb{R})$ such that $A\omega A^{-1} = A\omega A^t = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix}$.*

Proof. Let \langle, \rangle be the standard inner product on \mathbb{R}^{2n} . For $u, v \in \mathbb{R}^{2n}$ we have $\langle \omega(u), v \rangle = -\langle u, \omega(v) \rangle$ so if ω leaves a subspace W invariant then it also leaves its orthogonal complement W^\perp invariant. Therefore \mathbb{R}^{2n} is a direct sum of ω -invariant orthogonal planes. Choosing orthonormal bases in these planes we see that the matrix of ω in this basis is of the form $\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix}$. Therefore $\exists A \in O(2n, \mathbb{R})$ such that $A\omega A^{-1} = A\omega A^t$ is of the desired form. Moreover, by interchanging a pair of basis vectors we can also arrange that $A \in SO(2n, \mathbb{R})$. Hence an equivalent formulation of this lemma is the following.

If ω is a skew-symmetric bilinear form on an oriented $2n$ -dimensional real vector space V then there is an oriented orthonormal basis of V in which the matrix of ω is of the form $\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix}$.

Proof of the Theorem 1.4. There exists an oriented orthonormal basis v_1, \dots, v_{2m} of $T_p(N)$ in which ω is represented by a matrix of the form

$$\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix}, \text{ where } \lambda_h = \omega(v_{2h-1}, v_{2h}).$$

Let $\omega_1, \dots, \omega_{2m}$ be the 1-forms dual to v_1, \dots, v_{2m} . So

$$\omega = \sum_{k=1}^m \lambda_k \omega_{2k-1} \wedge \omega_{2k} \quad \text{and} \quad \omega^m = m! \lambda_1 \cdots \lambda_m \omega_1 \wedge \cdots \wedge \omega_{2m} = m! \lambda_1 \cdots \lambda_m dv_0 \quad (*)$$

For unit vectors $X, Y \in T_p(N)$ we have $|\omega(X, Y)| = |g(JX, Y)| \leq |JX| |Y| \leq 1$ with equality if and only if $JX = \pm Y \Leftrightarrow$ the subspace spanned by X, Y is J -invariant.

From (*) we therefore have $|\omega^m/m!| = |\lambda_1 \cdots \lambda_m| dV_0 \leq dV_0$ as $|\lambda_k| = |\omega(v_{2k-1}, v_{2k})| \leq 1$, and $|\omega^m/m!| = dV_0$ if and only if $T_p(N)$ is a J -invariant subspace $T_p(M)$. Since $\omega(X, JX) > 0 \forall X$ we see that we can find an orthonormal basis of $T_p(N)$ of the form $e_1, \tau e_1, \dots, e_m, \tau e_m$ and $\omega^m/m! (e_1, \tau e_1, \dots, e_m, \tau e_m) = 1$. Hence $\omega^m/m!$ is the canonical volume form of M . \square

(1.6) **Corollary.** *If M is an arbitrary oriented $2m$ -dimensional manifold embedded in a Hermitian manifold \tilde{M} with fundamental form ω then*

$$\frac{1}{m!} \int_M \omega^m \leq \text{vol}_{2m}(M)$$

and in case the volume is finite, equality holds $\Leftrightarrow M$ is a complex submanifold of \tilde{M} with canonical orientation.

This in turn follows from the theorem of Levi-civita:

(Levi-civita). *If M is a submanifold of a complex manifold \tilde{M} then M is complex if and only if $T_p(M)$ is a complex subspace of $T_p(\tilde{M}) \forall p \in M$.*

Proof. The idea of the proof is contained in the special case: M is a surface in \mathbb{C}^2 given by $z = f(w)$, where f is a smooth function. Suppose for all $p = (a, a') \in M$, $T_p(M)$ is a complex subspace of $T_p(\mathbb{C}^2)$. Now if $v = \alpha \frac{\partial}{\partial z} + \bar{\alpha} \frac{\partial}{\partial \bar{z}} + \beta \frac{\partial}{\partial w} + \bar{\beta} \frac{\partial}{\partial \bar{w}} \in T_p(M)$, then

$$\alpha = \beta \frac{\partial f}{\partial w}(a') + \bar{\beta} \frac{\partial f}{\partial \bar{w}}(a'). \quad (*)$$

Since $T_p(M)$ is complex we also have

$$i\alpha = i\beta \frac{\partial f}{\partial w}(a') - i\bar{\beta} \frac{\partial f}{\partial \bar{w}}(a').$$

Therefore $\bar{\beta} = \frac{\partial f}{\partial \bar{w}}(a') = 0$.

Since β is arbitrary we see that $\frac{\partial f}{\partial \bar{w}}(a') = 0 \forall (a, a') \ni a = f(a')$. Hence $\frac{\partial f}{\partial \bar{w}} = 0$ and f is holomorphic.

It suffices to prove the general statement in a neighborhood of a fixed point, which we may take to be zero. If $T_0(M)$ is a complex subspace of \mathbb{C}^n then by a unitary transformation it can be mapped onto the coordinate plane $z' = (z_1, \dots, z_p)$, $2p = \dim_{\mathbb{R}} M$. By the implicit function theorem, in some neighborhood $U = U' \times U''$ of 0 the manifold M is given by the equation $z'' = g(z')$, $z = (z', z'')$, where $g : U' \rightarrow U''$ is a smooth function. If $a = (a', a'')$ and $T_a(M)$ is complex and $(v', v'') \in T_a(M)$, then (iv', iv'') is also in $T_a(M)$. So exactly the same argument as before shows that $\frac{\partial g}{\partial z'}(a') \cdot \bar{v}' = 0$ on $T_a(M)$. Since $T_a(M)$ projects onto $\mathbb{C}_{a'}^p$, we see that $\frac{\partial g}{\partial z'}(a') = 0 \forall a'$. Hence $\frac{\partial g}{\partial z'} \equiv 0$ in U' , i.e. g is holomorphic in U' and so $M \cap U$ is a complex submanifold in \mathbb{C}^n .

Proof of Corollary 4.4. We have $\frac{1}{m!} \int_M \omega^m \leq \int_M dv_0 = \text{vol}_{2m}(M)$, and equality holds \Leftrightarrow for almost all $p \in M$ the planes $T_p(M)$ are complex. By continuity all $T_p(M)$ are complex and so by Levi-Civita M is a complex manifold.

(1.7) Corollary. *If \bar{M} is a Kähler manifold and $M \subset \bar{M}$ a compact complex submanifold then $\text{Vol}(M) \leq \text{Vol. of any other } 2m\text{-dimensional submanifold } M_1 \text{ which is homologous to } M \text{ in } \bar{M}$.*

Proof. Let $[M] - [M_1] = [\partial Z]$ and let ω be the fundamental form of the given Kählerian metric on \bar{M} . As $d(\omega^k) = 0$ we have $\int_M \omega^m/m! - \int_{M_1} \omega^m/m! = \int_{\partial Z} \omega^m/m! = \int_Z d(\omega^m)/m! = 0$ (by Stokes). Hence $\text{Vol}(M) = \int_M \omega^m/m! = \int_{M_1} \omega^m/m! \leq \text{Vol}(M_1)$. \square

§ 2. Kähler metrics (analytic aspects)

So far we have not made any essential use of complex local coordinates. We shall now show that on a complex manifold a Kähler metric is generated locally by a plurisubharmonic function and we shall also give a local canonical form for such a function. Before doing this, it is best to get a little bit of linear algebra out of the way.

(2.1) Positive (1,1)-forms. Let V be a real vector space, $J \in \text{End}(V)$ with $J^2 = -1$. Let g be a symmetric J -invariant bilinear form on V . Extend g to a complex bilinear form on $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. The form $h(X, Y) = g(X, \bar{Y})$ ($X, Y \in V_{\mathbb{C}}$) is a Hermitian form. Now on $V_{\mathbb{C}}$, the endomorphism J splits and we set $V^{1,0}$ to be the i -eigenspace of V and $V^{0,1}$ to be the $(-i)$ -eigenspace of V . We have

$$V^{1,0} = \langle u - iJu : u \in V \rangle$$

$$V^{0,1} = \langle u + iJu : u \in V \rangle.$$

For $u \in V$ if we set $\xi_u = (u - iJu)$ then $u = \frac{1}{2}(\xi_u + \bar{\xi}_u)$, the map $u \mapsto \xi_u$ is bijective and $\xi_{Ju} = i\xi_u$. Now $h(\xi_u, \xi_v) = g(\xi_u, \bar{\xi}_v) = g(u - iJu, v + iJv) = 2g(u, v) - 2ig(Ju, v)$. If we set $\omega(u, v) = g(Ju, v)$ then ω is skew symmetric and $h(\xi_u, \xi_v) = 2[g(u, v) - i\omega(u, v)]$. Hence $h|_{V^{1,0}}$ is positive definite $\Leftrightarrow g$ is positive definite on $V \Leftrightarrow \omega(u, Ju) > 0 \forall u \neq 0$.

Notice that $h(\xi_u, \xi_v) = \omega(\xi_u, \bar{J}\bar{\xi}_v) = -i\omega(\xi_u, \bar{\xi}_v)$. So $h(\xi_u, \xi_u) > 0 \Leftrightarrow -i\omega(\xi_u, \bar{\xi}_u) > 0$ for all $u \neq 0$.

Summarizing, we see that a real J -invariant bilinear form g gives rise to a J -invariant 2-form ω with $\omega(X, Y) = g(JX, Y)$ and conversely a J -invariant 2-form ω gives rise to a J -invariant bilinear form $g(X, Y) = \omega(X, JY)$ and we call ω positive iff the associated bilinear form g is positive definite. This is so if and only if $-i\omega(\xi_u, \bar{\xi}_u) > 0 \forall u \in V, u \neq 0$.

(2.2) Definition. J -invariant 2-forms are said to be of type (1,1). If ω is such a

skew-symmetric form and $\omega(X, JX) > 0 \forall X \neq 0$, then we say that ω is a positive (1,1)-form.

The reason for this terminology is explained in (2.3) below.

(2.3) Decomposition into Types. Let (V, J) be as in (2.2). In $V \otimes \mathbb{C}$ we have a conjugation σ defined by $\sigma(v + iw) = v - iw$, $v, w \in V$. If η is a complex r -linear form on $V_{\mathbb{C}} = V \bigotimes_{\mathbb{R}} \mathbb{C}$ then $\eta^{\sigma}(v_1, \dots, v_r) = (\eta(\sigma v_1, \dots, \sigma v_r))^{-}$ is also complex multilinear. On the other hand if ξ is a real multilinear form on C then ξ extends uniquely to a complex multilinear form on $V_{\mathbb{C}}$ – we denote this extension also by ξ – and $\xi^{\sigma} = \xi$. Thus as $(V_{\mathbb{C}})_{\sigma} = V$ we see that $\eta^{\sigma} = \eta \Leftrightarrow \eta$ takes real values on V . So if η is a complex multilinear form on $V_{\mathbb{C}}$ with $\eta^{\sigma} = \eta$, we call η a real form.

In particular, let us look at bilinear forms on $(V_{\mathbb{C}}) = V^{1,0} \oplus V^{0,1}$. We have $V_{\mathbb{C}} \otimes V_{\mathbb{C}} = (V^{1,0} \otimes V^{1,0}) \oplus (V^{1,0} \otimes V^{0,1}) \oplus (V^{0,1} \otimes V^{0,1}) \equiv_{\text{defn.}} V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$ and J operates on $V^{2,0}$ and $V^{0,2}$ as multiplication by -1 and on $V^{1,1}$ as the identity. Therefore a bilinear form ω on $V_{\mathbb{C}}$ is J -invariant if and only if $\omega \in V^{1,1}$, i.e. ω is of type (1,1). Hence ω is a real J -invariant form if and only if $\omega^{\sigma} = \omega$ and ω is of type (1, 1).

In general, if $e_1, Je_1, \dots, e_n, Je_n$ is a basis of V then $\frac{e_1 - iJe_1}{2} = \xi_1, \dots, \frac{e_n - iJe_n}{2} = \xi_n, \bar{\xi}_1, \dots, \bar{\xi}_n$ is a complex basis of $V_{\mathbb{C}}$. If $\omega_1, \dots, \omega_n, \bar{\omega}_1, \dots, \bar{\omega}_n$ is the dual basis of $V_{\mathbb{C}}^*$ then a basis for $\bigwedge^N V_{\mathbb{C}}^*$ is $\omega_{i_1} \wedge \dots \wedge \omega_{i_r} \wedge \bar{\omega}_{j_1} \wedge \dots \wedge \bar{\omega}_{j_s}$, where $i_1 < \dots < i_r$, $j_1 < \dots < j_s$ and $r + s = N$.

Now if L is a complex linear transformation of $V_{\mathbb{C}}$ then $L^{\sigma}(v) = (L(\bar{v}))^{-}$ is also complex linear and L is induced from a real linear transformation of V if and only if $L^{\sigma} = L$. Hence $L^{\sigma} = L$ and L maps $V^{1,0}$ into $V^{1,0}$ if and only if L is induced from a real linear transformation of V which commutes with J . Therefore the decomposition into types is preserved under such transformations.

Finally elements of $(\wedge^r(V^{1,0})) \wedge (\wedge^s(V^{0,1}))$ are called forms of type (r, s) .

(2.4) **p, q -forms on Manifolds.** Now let M be a complex manifold. Choose holomorphic local coordinates $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$. We have $dz_\alpha = dx_\alpha + idy_\alpha, \frac{\partial}{\partial z_\alpha} = \frac{1}{2} \left(\frac{\partial}{\partial x_\alpha} - i \frac{\partial}{\partial y_\alpha} \right)$, $d\bar{z}_\alpha = dx_\alpha - idy_\alpha, \frac{\partial}{\partial \bar{z}_\alpha} = \frac{1}{2} \left(\frac{\partial}{\partial x_\alpha} + i \frac{\partial}{\partial y_\alpha} \right)$ and $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}$ is a local basis of complex valued C^∞ -vector fields with $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$ the dual basis of complex valued C^∞ -1 forms. For $i_1 < \dots < i_p, j_1 < \dots < j_q$ let us set $dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q} = dz_I \wedge d\bar{z}_J$ where $I = \{i_1, \dots, i_p\}, J = \{j_1, \dots, j_q\}$. The (p, q) -forms $dz_I \wedge d\bar{z}_J$ with $|I| = p, |J| = q$ form a local basis of (p, q) forms.

An n form ω can locally then be written uniquely as $\omega = \sum_{\substack{|I|=p \\ |J|=q}} a_{I,J} dz_I \wedge d\bar{z}_J$, the $a_{I,J}$ being C^∞ -functions. Let $\pi_{p,q}(\omega)$ be the sum of components of ω of type (p, q) . Now for a function f we have $df = \sum_{\alpha=1}^n \left(\frac{\partial f}{\partial z_\alpha} dz_\alpha + \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \right) = \sum_{\alpha=1}^n \left(\frac{\partial f}{\partial z_\alpha} dz_\alpha + \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \right)$ and so

$$d(a_{I,J} dz_I \wedge d\bar{z}_J) = \sum_{\alpha=1}^n \left(\frac{\partial a_{I,J}}{\partial z_\alpha} dz_\alpha \wedge dz_I \wedge d\bar{z}_J + \frac{\partial a_{I,J}}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge dz_I \wedge d\bar{z}_J \right).$$

$$\begin{aligned} \text{We set } \partial(a_{I,J} dz_I) &= \sum_{\alpha=1}^n \frac{\partial a_{I,J}}{\partial z_\alpha} dz_\alpha \wedge dz_I \wedge d\bar{z}_J \text{ and } \bar{\partial}(a_{I,J} dz_I \wedge d\bar{z}_J) \\ &= \sum_{\alpha=1}^n \frac{\partial a_{I,J}}{\partial \bar{z}_\alpha} d\bar{z}_\alpha \wedge dz_I \wedge d\bar{z}_J. \end{aligned}$$

Intrinsically, if η is a form of type (p, q) then $d\eta = \prod_{p+1,q} d\eta + \prod_{p,q+1} d\eta$ and we set

$$\bar{\partial}\eta = \prod_{p+1,q} d\eta, \bar{\partial}\eta = \prod_{p,q+1} d\eta.$$

We extend the definition of ∂ and $\bar{\partial}$ to an arbitrary n -form in the obvious way. So now we have $d = \partial + \bar{\partial}$. As $d^2 = 0$ we have $\partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = 0$ and as ∂ maps a form of type (p, q) to a form of type $(p+1, q)$ and $\bar{\partial}$ maps a form of type (p, q) to $(p, q+1)$, comparing types we see that $\partial^2 = 0, \partial\bar{\partial} = -\bar{\partial}\partial$. The operator $d = \partial + \bar{\partial}$ is a real operator. Set $d^{\mathbb{C}} = \frac{\partial - \bar{\partial}}{2i}$. We then have the important identity $dd^{\mathbb{C}} = i\partial\bar{\partial}$.

(2.5) **Examples** (i) Let φ be a real-valued function defined on a complex manifold. Let (z_1, \dots, z_n) be a system of local holomorphic coordinates. Now $\partial\bar{\partial}\varphi = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$, so the real (1,1)-form $i\partial\bar{\partial}\varphi$ is positive if and only if $\partial\bar{\partial}\varphi(\xi, \bar{\xi}) > 0 \forall \xi \in T^{1,0}(M)$. As $\partial\bar{\partial}\varphi \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) = \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}$ we see that $i\partial\bar{\partial}\varphi$ is positive if and only if φ is strictly plurisubharmonic, in the sense of § 3, Chapter 1. As $i\partial\bar{\partial}\varphi = dd^c\varphi$ we see that $d\omega = 0$.

(ii) Let $M = \mathbb{C}^n$ and f a radially symmetric function, say $f(z) = g(r)$, $r^2 = |z|^2$. When is the form $i\partial\bar{\partial}f$ positive? Answer: Exactly when $\ddot{g}(r) + \dot{g}(r)/r > 0$, $\dot{g}(r)/r > 0$ ($r \neq 0$) and $\ddot{g}(0) > 0$. To see this, recall that the form $h(\xi, \eta) = \partial\bar{\partial}\varphi(\xi, \bar{\eta})$, $\xi, \eta \in T^{1,0}$ is the Levi form of φ . As f is $U(n)$ -invariant the signature of the Levi-form $L(f)$ [see §3, Ch. 1] is determined by its signature on a set of representatives of $U(n)$ -orbits, say on the half line $(r, 0, \dots, 0)$ ($r \geq 0$). At such points we find that $\left[\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right) \right]$ is diagonal with entries $g''(r) + g'(r)/r$ and $g'(r)/r$, the latter with multiplicity 1. Since $g(r) = g(-r)$ these entries at 0 become $g''(0)$.

(iii) Let us determine now those $U(n+1)$ -invariant functions f on $\mathbb{C}^{n+1} \setminus 0$ such that the form $i\partial\bar{\partial}f$ is the pull-back of a $U(n+1)$ -invariant positive form on \mathbb{P}^n . Recall that the Fubini metric on \mathbb{P}^n is Kählerian and if ω is its fundamental 2-form then we are asking if $\pi^*\omega = i\partial\bar{\partial}f$ for a $U(n+1)$ -invariant function, $\pi : \mathbb{C}^{n+1} \setminus 0 \rightarrow \mathbb{P}^n$ being the map $\pi(z_0, \dots, z_n) = [z_0 : \dots : z_n]$. Since $d\pi$ is surjective with 1-dimensional kernel, the Levi-form of f must be positive semi-definite and it must have zero as an eigenvalue. By (ii) we must have $f''(r) + f'(r)/r = 0$ or $f'(r)/r = 0$; since $f'(r)/r$ occurs with multiplicity n , the only possibility is that $f''(r) + f'(r)/r = 0$. So as $f'(r) > 0$ we have $f''(r)/f'(r) = -1/r$ whose solution is $f(r) = A \ln r + B$, $A > 0$ i.e. $f(r) = A \ln r^2 + B$. Since $\text{Ker } d\pi_{\xi_0} = \mathbb{C} \cdot \xi_0$, $\xi_0 = [1 : 0 : \dots : 0] = [1 : 0]$ we see that $i\partial\bar{\partial}f$ is positive on $(\text{Ker } \pi_{\xi_0})^\perp$.

Let us check that the form $\omega = i\partial\bar{\partial} \log \|z\|^2$ descends to \mathbb{P}^n . But this is obvious as under the homothetics $z \mapsto \lambda z$ ($\lambda \in \mathbb{C}^*$), then form ω is invariant. We call $\log \|z\|^2$ a

quasi-potential for the Fubini metric. Of course as \mathbb{P}^n is compact we cannot find any nonconstant plurisubharmonic function on \mathbb{P}^n .

Let us now show that locally a Kählerian metric is of the form $i\partial\bar{\partial}\varphi$. For this we need two basic results, the Poincare Lemma and the Dolbeault Lemma.

(2.6) Poincare Lemma. *Let $I \subset \mathbb{R}^n$ be an open rectangular box. If ω is an r -form on I with $d\omega = 0$ then $\omega = d\eta$ for an $(r-1)$ -form η .*

The idea of the proof is completely contained in the following example.

(2.7) Example (a) Let ω be a 3-form on \mathbb{R}^3 , $\omega = f dx dy dz$. If we set $\eta = \left(\int_0^x f(t, y, z) dt \right) dy dz$ then $d\eta = \omega$ as $dy \cdot dy = 0 = dz dz$.

(b) Let ω be a closed 2-form on \mathbb{R}^3 which involves only $dx dy$, say $\omega = f dx dy$. As $d\omega = 0$ we see that $\partial f / \partial z = 0$ so $f = f(x, y)$ and we are in a case analogous to (a).

(c) Let ω be an arbitrary closed 2-form on \mathbb{R}^3 , say

$$\begin{aligned} \omega &= f dx dy + g dx dz + h dy dz \\ &= f dx dy + (g dx + h dy) dz. \end{aligned}$$

If we set $\alpha = \left(\int_0^z g(x, y, t) dt \right) dx + \left(\int_0^z h(x, y, t) dt \right) dy$ then $d\alpha = g(x, y, z) dz dx + h(x, y, z) dz dy + \text{terms involving only } dx \text{ and } dy$. Hence $\omega = \beta - d\alpha$, where β is a 2-form which only involves $dx dy$. As $d^2 = 0$ and $d\omega = 0$, we have $d\beta = 0$. By (b) we have $\beta = d\xi$, $\xi = \xi(x, y)$ and so $\omega = d(\xi - \alpha)$.

Proof of the Poincare Lemma. The idea of the proof is to show that a closed r -form on I is cohomologous to a closed r -form which is supported by dx^1, \dots, dx^{n-1} and obtain the result by iteration.

For $i_1 < i_2 < \dots < i_k$ set $dx^I = dx^{i_1} \dots dx^{i_k}$. Let ω be an r -form with $\omega = \alpha + \beta dx^n$, where α is supported by dx^1, \dots, dx^{n-1} and $\beta = \beta_I dx^I$, where $I \subset \{1, \dots, n-1\}$ and $\#I = r-1$ (using the summation convention). Let $\gamma = \left(\int_0^{x^n} \beta_I(x^1, \dots, x^{n-1}, t) dt \right) dx^I$. So $d\gamma = \beta_I dx^n dx^I + \delta$, where δ is supported by (dx^1, \dots, dx^{n-1}) . So $\omega = \alpha + (-1)^{r-1}(d\gamma - \delta) = \tilde{\alpha} + (-1)^{r-1}d\gamma$, where $\tilde{\alpha}$ is supported by dx^1, \dots, dx^{n-1} and $\tilde{\alpha}$ is an r -form. As $d\omega = 0$, we have $d\tilde{\alpha} = 0$. Now $\tilde{\alpha} = \tilde{\alpha}_J dx^J$ where $J \subset \{1, 2, \dots, n-1\}$, $\#J = r$. Hence $d\tilde{\alpha} = 0$ implies $\partial \tilde{\alpha}_J / \partial x_n = 0$, so $\tilde{\alpha}_J = \tilde{\alpha}_J(x^1, \dots, x^{n-1})$. Continuing this process we see that $\omega = f dx^1 \dots dx^r + d\xi$ where $f = f(x^1, \dots, x^r)$. Finally $d \left(\left(\int_0^{x^1} f(t, x^2, \dots, x^r) dt \right) dx^2 \dots dx^r \right) = f dx^1 \dots dx^r$, which shows that $\omega = d\eta$ for a suitable $(r-1)$ -form η . \square

Dolbeaut Lemma. Let $P_r = \{z \in \mathbb{C}^n : |z_i| < r\}$ $i = 1, \dots, n$ be the polydisc of radius r in \mathbb{C}^n . If ω is a (p, q) -form on $P_{r+\epsilon}$ ($q > 0$) with $\bar{\partial}\omega = 0$ then there exists a $(p, q-1)$ form on P_r with $\bar{\partial}\eta = \omega$.

The proof is formally the same as that of the Poincare lemma once the following result has been proved.

(2.7) Lemma For a C^∞ -function f defined in an open neighborhood U of a compact subset $K \subset \mathbb{C}^1$ there exists a C^∞ function g defined in an open neighborhood $V \subseteq U$ of K such that $\partial g / \partial \bar{z} = f$ on V .

Proof. For any open neighborhood U of K there exists a C^∞ -function ϕ on \mathbb{C} such that $\phi \equiv 1$ in an open neighborhood $V \subseteq U$ of K and $\phi \equiv 0$ in open neighborhood of $\mathbb{C} - U$ [see N. p. 101], i.e. $\exists V \subset \bar{V}_1 \subset U$ with $K \subset V$, \bar{V}_1 compact with $\phi \equiv 1$ on V , $\phi \equiv 0$ on $\mathbb{C} \setminus \bar{V}_1$. The function $\psi = \phi f$ on U , $\psi = 0$ on $\mathbb{C} \setminus \bar{V}_1$ is C^∞ with compact support and $\psi = f$ on V . Replacing f by ψ we may assume that $f = 0$ on $\mathbb{C} \setminus \bar{V}_1$.

Fix $z \in \mathbb{C}$. By the generalized Cauchy integral formula we have

$$\begin{aligned}
2\pi i f(z) &= \int_{\partial\Delta(z,R)} \frac{f(w)dw}{w-z} + \iint_{\Delta(z,R)} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} \\
&= \iint_{\Delta(z,R)} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{(w-z)} \quad (\text{for sufficiently large } R) \\
&= \iint_{\mathbb{C}} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{(w-z)}.
\end{aligned}$$

So, if we set $g(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f(w)}{(w-z)} dw \wedge d\bar{w}$ then

$$\begin{aligned}
g(z) &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f(\xi+z)}{\xi} d\xi \wedge d\bar{\xi} \\
&= \frac{1}{2\pi i} \int_0^{2\pi} \int_0^\infty \frac{f(re^{i\theta}+z)}{e^{i\theta}} dr d\theta.
\end{aligned}$$

So g is C^∞ . Moreover we have

$$\begin{aligned}
\frac{\partial g}{\partial \bar{z}} &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} \left(\frac{f(\xi+z)}{\xi} \right) d\xi \wedge d\bar{\xi} \\
&= \frac{1}{2\pi i} \iint_{\mathbb{C}} \left(\frac{\partial f}{\partial \bar{\xi}}(\xi+z) \right) \frac{\partial \xi \wedge d\bar{\xi}}{\xi} = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial f}{\partial \bar{\xi}}(\xi) \frac{d\xi \wedge d\bar{\xi}}{\xi-z} = f(z)
\end{aligned}$$

This completes the proof of the lemma. \square

Now to complete the proof of Dolbeaut lemma we simply replace, in the proof of the Poincare lemma, dx^j by $d\bar{z}_j$.

Some Immediate Consequences

- (1) *If ω is a real closed J -invariant form on a complex manifold M , then locally ω has a potential ϕ in the sense that $\omega = i\partial\bar{\partial}\phi$.*

Proof. As $d\omega = 0$ we have, by Poincare, that $\omega = d\alpha$ (locally) for a real form α . Let $\alpha = \alpha^{1,0} + \alpha^{0,1}$ ($\alpha^{0,1} = \overline{\alpha^{1,0}}$) be the decomposition of α as a sum of $(1,0)$ and $(0,1)$ forms. Now $d = \partial + \bar{\partial}$ and so $\omega = d\alpha = d\alpha^{1,0} + \bar{\partial}\alpha^{1,0} + \partial\alpha^{0,1} + \bar{\partial}\alpha^{0,1}$.

Since ω is a $(1, 1)$ form we have $\partial\alpha^{1,0} = 0 = \bar{\partial}\alpha^{0,1}$. By Dolbeaut we have, locally, $\alpha^{0,1} = \bar{\partial}\eta\psi$ for a suitable complex valued function ψ . Hence $\omega = \partial\bar{\partial}\eta\psi + \bar{\partial}\partial\eta\bar{\psi} = i\partial\bar{\partial}((\psi - \bar{\psi})/i) = i\partial\bar{\partial}\varphi$ (locally), φ being a real-valued function.

- (2) *If $\partial\bar{\partial}\varphi = 0$ then locally φ is a sum of a holomorphic function and an antiholomorphic function.*

Proof. Let $\alpha = \bar{\partial}\varphi$. Now $d\alpha = (\partial + \bar{\partial})(\bar{\partial}\varphi) = 0$, so by Poincaré we have $\bar{\partial}\varphi = d\xi = \partial\xi + \bar{\partial}\xi$. Hence $\bar{\partial}(\varphi - \xi) = 0 = \partial\xi$, so $\varphi - \xi$ is holomorphic and ξ is antiholomorphic.

- (3) *Existence of complex geodesic coordinates.* If g is a Hermitian metric on a complex manifold M and $(z_j)_{1 \leq j \leq n}$ is a system of local holomorphic coordinates, we set $g_{i\bar{j}} = g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$. We say that $(z_j)_{1 \leq j \leq n}$ is a system of complex geodesic coordinates at $p \in M$ if $g_{i\bar{j}}(p) = \delta_{ij}$ and $dg_{i\bar{j}}(p) = 0$.

Now let (M, g) be Kählerian and φ a local potential for g (defined near p) and $(z_j)_{1 \leq j \leq n}$ a system of local holomorphic coordinates with $z(p) = 0$. We have $\left[\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(0)\right] = [a_{i\bar{j}}] = A$ is a positive definite hermitian matrix, so by a unitary transformation we may assume that A is the identity matrix, i.e. $\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(0) = \delta_j^i$.

Now we make a holomorphic change of coordinates of the form $z_i = w_i + c_{jk}^i w_j w_k$ such that $c_{jk}^i = c_{kj}^i$. If $\psi(w_1, \dots, w_n) = \varphi(z_1, \dots, z_n)$, then $\frac{\partial^2 \psi}{\partial w_i \partial \bar{w}_j} = \sum_{\alpha, \beta} \frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta} \frac{\partial z_\alpha}{\partial w_i} \times \frac{\partial \bar{z}_\beta}{\partial \bar{w}_j}$. We have $\frac{\partial^2 \psi}{\partial w_i \partial \bar{w}_j}(0) = \delta_j^i$ and $\frac{\partial}{\partial w_k} \left(\frac{\partial^2 \psi}{\partial w_i \partial \bar{w}_j} \right) (0) = \frac{\partial^3 \varphi(0)}{\partial z_k \partial z_i \partial \bar{z}_j} + c_{ki}^j + c_{ik}^j = \frac{\partial^3 \varphi(0)}{\partial z_k \partial z_i \partial \bar{z}_j} + 2c_{ik}^j$. So $-2c_{ik}^j = \frac{\partial^3 \varphi(0)}{\partial z_k \partial z_i \partial \bar{z}_j}$ are the required coefficients with $c_{ik}^j = c_{ki}^j$. \square

Chapter 3

Line bundles and analytic significance of signature of Levi form

§ 1. Line bundles

A complex line bundle over a manifold M is a manifold \mathcal{L} together with a map $\pi : \mathcal{L} \rightarrow M$ such that

(a) π is surjective.

(b) M can be covered by open sets U_α such that $V_\alpha = \pi^{-1}(U_\alpha) \stackrel{\cong}{\text{diffeo}} U_\alpha \times \mathbb{C}$ by a map φ_α and on V_α we have $\pi = pr_1 \circ \varphi_\alpha$, pr_1 being the projection of $U_\alpha \times \mathbb{C}$ on U_α .

(c) $\forall \alpha, \beta$ with $U_\alpha \cap U_\beta$ nonempty we have $\varphi_{\beta\alpha} \varphi_\alpha^{-1} : U_\alpha \cap U_\beta \times \mathbb{C} \rightarrow U_\alpha \cap U_\beta \times \mathbb{C}$ is of the form $(p, z) \mapsto (p, g_{\beta\alpha}(p)(z))$.

The function $g_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow \mathbb{C}^*$ are called the transition functions of the line bundle \mathcal{L} and satisfy $g_{\beta\alpha}g_{\alpha\beta} = 1$ on $U_\alpha \cap U_\beta$, $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Conversely, given an open covering $\{U_k\}$ of M and functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{C}^*$ such that $g_{\alpha\beta}g_{\beta\alpha} = 1$ on $U_\alpha \cap U_\beta$ and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$, there is a line bundle which has the functions $\{g_{\alpha\beta}\}$ as transition functions.

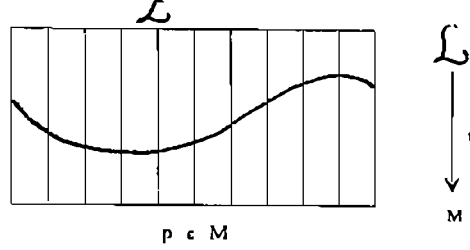
If $p \in U_\alpha$ and $\xi_1, \xi_2 \in \pi^{-1}(p)$ we define

$$\begin{aligned}\xi_1 + \xi_2 &= \varphi_\alpha^{-1}(\varphi_\alpha(\xi_1) + \varphi_\alpha(\xi_2)) \\ \lambda \xi_1 &= \varphi_\alpha^{-1}(\lambda \varphi_\alpha(\xi_1)) \quad (\lambda \in \mathbb{C}) : \end{aligned}$$

here each fiber $p \times \mathbb{C}$ is identified with \mathbb{C} .

The conditions (a), (b) and (c) ensure that addition and multiplication in the fiber $\pi^{-1}(p)$ is independent of the open set U_α in which the point p lies.

A section of \mathcal{L} is a map $s : M \rightarrow \mathcal{L}$ such that $\pi \circ s = id_M$



If s is a section then on U_α we have $\varphi_\alpha(s(p)) = (p, s_\alpha(p))$ for a suitable function s_α and therefore on $U_\alpha \cap U_\beta$ we have $(p, s_\beta(p)) = (p, g_{\beta\alpha}(p)s_\alpha(p))$. Hence a section s corresponds to a family of functions $s_\alpha : U_\alpha \rightarrow \mathbb{C}$ such that on $U_\alpha \cap U_\beta$ we have

$$g_{\beta\alpha}(p)s_\alpha(p) = s_\beta(p).$$

(1.1) Examples

- 1) Take functions f_α on U_α such that on $U_\alpha \cap U_\beta$ $f_\beta/f_\alpha = g_{\beta\alpha}$ is nonvanishing. Then the $\{g_{\beta\alpha}\}$ are transition functions for a line bundle and the family $\{f_\alpha\}$ is a section of this line bundle.
- 2) On $\mathbb{P}^n(\mathbb{C})$ consider the hyperplane defined by the equation $z_0 = 0$. On the open set $U_i = \{[z] \in \mathbb{P}_n : z_i \neq 0\}$ it is defined by the function $f_i = z_0/z_i$. On $U_i \cap U_j$ we have $f_i/f_j = z_j/z_i = g_{ij}$.

The line bundle on \mathbb{P}_n with transition functions $g_{ij} = z_j/z_i$ is called the hyperplane bundle.

From now on we assume that M is a complex manifold and the transition functions $\{g_{\alpha\beta}\}$ are holomorphic.

(1.2) **Exercise.** Show that the line bundle on \mathbb{P}^n with transition functions $g_{ij}([z]) = z_i/z_j$ has no sections.

The existence of sections is related to the (Levi) curvature of the bundle.

(1.3) **Definition** A norm on a line bundle \mathcal{L} is a function $N : \mathcal{L} \rightarrow \mathbb{R}^{\geq 0}$ such that for all $p \in M$ and $v \in \pi^{-1}(p)$ $N(v) = 0 \Leftrightarrow v = 0$ and $N(zv) = |z|^2 N(v)$ ($z \in \mathbb{C}$).

If we have a local trivialization $\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{C}$ (of \mathcal{L} over U) and we set $N_U(\xi, z) = N(\varphi_U^{-1}(\xi, z))$, then in any other local trivialization $\varphi_V : \pi^{-1}(V) \rightarrow V \times \mathbb{C}$ the functions N_V and N_U are related as follows:

For $\xi \in U \cap V$ we have

$$\begin{aligned} N_V(\xi, z) &= N(\varphi_V^{-1}(\xi, z)) \\ &= N(\varphi_U^{-1} \varphi_U \varphi_V^{-1}(\xi, z)) = N(\varphi_U^{-1}(\xi, g_{UV}(\xi)(z))) \\ &= N(g_{UV}(\xi) \varphi_U^{-1}(\xi, z)) \\ &= |g_{UV}(\xi)|^2 N(\varphi_U^{-1}(\xi, z)) \\ &= |g_{UV}(\xi)|^2 N_U(\xi, z). \end{aligned}$$

So $|z|^2 N_V(\xi, 1) = |g_{UV}(\xi)|^2 |z|^2 N_U(\xi, 1)$, i.e., $N_V(\xi, 1) = |g_{UV}(\xi)|^2 N_U(\xi, 1)$.

Conversely, given a covering $\{U_\alpha\}$ of M and transition functions $\{g_{\alpha\beta}\}$ and functions $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R} > 0$ such that $|g_{\alpha\beta}(\xi)|^2 \varphi_\alpha(\xi) = \varphi_\beta(\xi)$ we can define a norm on each $U_\alpha \times \mathbb{C}$ by $N_\alpha(\xi, z) = |z|^2 \varphi_\alpha(\xi)$. These functions give a well-defined norm on the line bundle defined by the transition functions $\{g_{\alpha\beta}\}$.

The existence of norms on line bundles is entirely analogous to that for proving the existence of Riemannian metrics [DC], using partitions of unity [N]. In the following example we give an explicit norm on line bundles over \mathbb{P}^1 . It is instructive here to use

group theory and we assume familiarity with the concept of homogeneous line bundles, as explained in e.g. [WW].

(1.4) **Example.** Let $G = SL(2, \mathbb{C})$, B the group of upper triangular matrices and χ_n the character of B defined by $\chi_n \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix} = t^n \quad (n \in \mathbb{Z})$. The homogeneous line bundles $G \times \mathbb{C}$ give up to isomorphism all the line bundles over $G/B = \mathbb{P}^1$. Take $\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in SL(2, \mathbb{C})$. By the Gram-Schmidt process applied to the column vectors and then applying a dilation the columns become orthonormal, i.e. the matrix is in $SU(2)$. Therefore,

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = k \begin{pmatrix} (|x|^2 + |z|^2)^{1/2} & 0 \\ 0 & (|y|^2 + |t|^2)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix},$$

where $\eta = (y\bar{x} + t\bar{z})/(|x|^2 + |z|^2)^{1/2}$

with $k \in SU(2)$. Writing $\chi = \chi_n$ we have $G \times_{\chi} \mathbb{C} \xrightarrow{\cong} K \times_{\chi|S^1} \mathbb{C}$, $K = SU(2)$.

On $K \times_{\chi|S^1} \mathbb{C}$ we set $\|k \times z\| = z\bar{z}$ and define $N(g \times_{\chi} z) = \|\theta(g \times_{\chi} z)\|$. Explicitly, $N\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times_{\chi} z\right) = (|a|^2 + |c|^2)^n z\bar{z}$. A computation shows that the Levi-form of N on a level set is of the same parity as n . If $n > 0$ then the level sets $N = c$ ($c > 0$) therefore have positive Levi-curvature. This implies (see Prop. 3.3) that L_{χ} has no non-zero section if $n > 0$. \square

§ 2 The Chern class of a line bundle

Let \mathcal{L} be a line bundle over a complex manifold M and N a norm on \mathcal{L} . Given $p \in M$ and a local nonvanishing section s , the form $\partial\bar{\partial} \log N(s)$ is a well-defined form: for if t is another nonvanishing section defined near p then $s = ht$ for a nonvanishing holomorphic function h and $\log N(s) = \log |h|^2 + \log N(t)$. Since $\partial\bar{\partial} \log |h|^2 = 0$ we see that $\partial\bar{\partial} \log N(s) = \partial\bar{\partial} \log N(t)$.

The Chern form of \mathcal{L} is, by definition,

$$C_1(\mathcal{L}) = \frac{i}{2\pi} \partial \bar{\partial} \log(N(s))^{-2} = \frac{1}{2\pi} dd^{\mathbb{C}} \log(N(s))^{-2}$$

(recall that $d = \frac{\partial + \bar{\partial}}{2}$, $d^{\mathbb{C}} = \frac{\partial - \bar{\partial}}{2i}$)

The reason for this choice of signs is explained by a theorem (Theorem 2.3) which we shall prove presently.

(2.1) Remarks

1. If N_1 is another norm on \mathcal{L} then for any local nonvanishing section s the function $\varphi(p) = N(s(p))/N_1(s(p))$ is a well-defined function (independent of the choice of s) and therefore $\partial \bar{\partial} \log N(s) - \partial \bar{\partial} \log N_1(s) = \partial \bar{\partial} \varphi$, i.e. $dd^{\mathbb{C}} \log N(s) = dd^{\mathbb{C}} \log N_1(s) + dd^{\mathbb{C}} \varphi$, φ a globally defined function. Therefore $[C_1(\mathcal{L})]$ is a well-defined cohomology class of M .
2. If M is compact and N, N_1 are norms with the same Chern forms then $N_1 = rN$ for some $r > 0$.

By (1), we know that if $\varphi(p) = N(s(p))/N_1(s(p))$ (s a local nonvanishing section) then by assumption $\partial \bar{\partial} \log \varphi = 0$, i.e., $\log \varphi$ is pluriharmonic on the compact manifold M and therefore constant. Hence $N_1(s(p)) = rN(s(p)) \forall p \in M$.

(2.2) The Chern Class of a Divisor. Let D be a hypersurface in a compact complex manifold M given locally by holomorphic functions $f_\alpha = 0$ on U_α with $g_{\alpha\beta} = f_\alpha/f_\beta \neq 0$ on $U_\alpha \cap U_\beta$. As $g_{\alpha\beta} f_\beta = f_\alpha$, the functions $\{f_\alpha\}$ represent a section of the line bundle \mathcal{L} defined by the transition functions $\{g_{\alpha\beta}\}$. Call this section t ; so t vanishes exactly on D . Let Z be a closed complex one-dimensional submanifold which intersects D in only a finite number of points.

(2.3) **Theorem.** $\int_Z C(\mathcal{L}) = \text{number of points of intersection of } D \text{ with } Z \text{ counted with multiplicity.}$

Proof. Without loss of generality we may assume that z intersects D in just one point, which in a suitable system of coordinates we may take to be $z = 0$. Let s be a local nonvanishing section defined near $z = 0$. Take a small disc $\Delta(\epsilon, 0) \subset Z$. We have

$$\begin{aligned}
2\pi \int_Z C(\mathcal{L}) &= 2\pi \left(\int_{Z \setminus \Delta(\epsilon, 0)} C(\mathcal{L}) + \int_{\Delta(\epsilon, 0)} C(\mathcal{L}) \right) \\
&= \int_{Z \setminus \Delta(\epsilon, 0)} dd^{\mathbb{C}} \log \|t\|^{-2} + \int_{\Delta(\epsilon, 0)} dd^{\mathbb{C}} \log \|s\|^{-2} \\
&\quad \text{(because } t \text{ vanishes on } z \text{ only at } 0) \\
&= \int_{\partial(Z \setminus \Delta(\epsilon, 0))} d^{\mathbb{C}} \log \|t\|^{-2} + \int_{\partial \Delta(\epsilon, 0)} d^{\mathbb{C}} \log \|s\|^{-2} \quad (*)
\end{aligned}$$

(denoting by $\| \cdot \|$ the given norm so that $\|zt\| = |z|^2 \|t\|$, $z \in \mathbb{C}$)

In a neighborhood of $\Delta(\epsilon, 0)$ in Z (ϵ sufficiently small) we have $t/s = z^k \psi$ in this neighborhood, ψ holomorphic and nonvanishing, k being the multiplicity of intersection at 0 of D with Z .

Now on $\partial \Delta(\epsilon, 0)$ we have

$$\log \|t\|^{-2} = \log \|s\|^{-2} - k \log |z|^2 + \log |\psi|^{-2}$$

so

$$\begin{aligned}
\int_{\partial \Delta(\epsilon, 0)} d^{\mathbb{C}} \log \|s\|^{-2} &= \int_{\partial \Delta(\epsilon, 0)} d^{\mathbb{C}} \log \|t\|^{-2} + k \int_{\partial \Delta(\epsilon, 0)} d^{\mathbb{C}} \log |z|^2 - \int_{\partial \Delta(\epsilon, 0)} d^{\mathbb{C}} \log |\psi|^{-2} \\
&= \int_{\partial \Delta(\epsilon, 0)} d^{\mathbb{C}} \log \|t\|^{-2} + k \int_{\partial \Delta(\epsilon, 0)} d^{\mathbb{C}} \log |z|^2 - \int_{\Delta(\epsilon, 0)} dd^{\mathbb{C}} \log |\psi|^{-2} \\
&= \int_{\partial \Delta(\epsilon, 0)} d^{\mathbb{C}} \log \|t\|^{-2} + 2\pi k - 0 \quad (**)
\end{aligned}$$

(because $dd^{\mathbb{C}} \log |\psi|^{-2} = 0$ on $\Delta(\epsilon, 0)$).

So from (*), we have:

$$\begin{aligned} 2\pi \int_Z C(\mathcal{L}) &= \int_{\partial Z \setminus \Delta(\epsilon, 0)} d^{\mathbb{C}} \log \|t\|^{-2} + \int_{\partial \Delta(\epsilon, 0)} d^{\mathbb{C}} \log \|t\|^{-2} + 2\pi k \\ &= 2\pi k \end{aligned}$$

as the boundaries of $Z \setminus \Delta(\epsilon, 0)$ and $\Delta(\epsilon, 0)$ are oppositely oriented. Hence

$$\int_Z C_1(\mathcal{L}) = k = Z \cdot D$$

Notice that $\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_{Z \setminus \cup \Delta(\epsilon, p)} dd^{\mathbb{C}} \log \|t\|^{-2} = Z \cdot D$ and in this sense we write

$$\frac{1}{2\pi} \int_Z dd^{\mathbb{C}} \log \|t\|^{-2} = Z \cdot D. \quad \square$$

Also if $\omega_0 = i\partial\bar{\partial} \log \|\sigma\|^{-2}$ is positive and $\omega_1 = \omega_0 + i\partial\bar{\partial}\varphi$, φ a real function, then changing $\|\cdot\|$ to $\|\sim\| = e^a \|\cdot\|$, $i\partial\bar{\partial} \log \|\sigma\|^{-2} = i\partial\bar{\partial}(-2a) + i\partial\bar{\partial} \log \|\sigma\|^2$, so change in representative is given by metric scaling.

§ 3. Analytic significance of signature of Levi-form of a function

Let φ be a real-valued function defined on a complex manifold M and let $p \in M$ with $(d\varphi)(p) \neq 0$. Choose local coordinates $(z_j)_{1 \leq j \leq n}$ so that $z(p) = 0$ and $\frac{\partial \varphi}{\partial z_1}(p) \neq 0$. Now $\varphi(z) = a + \sum a_i z_i + \sum \bar{a}_i \bar{z}_i + \sum a_{ij} z_i z_j + \sum \bar{a}_{ij} \bar{z}_i \bar{z}_j + \sum a_{i\bar{j}} z_i \bar{z}_j + 0(\|z\|^2)$. If we set $u_1 = \sum a_i z_i + \sum \bar{a}_{ij} z_i \bar{z}_j$, $u_2 = z_2, \dots, u_n = z_n$ then $(du_1 \wedge \dots \wedge du_n)(0) = a_1(dz_1 \wedge \dots \wedge dz_n)(0)$, so we can solve for u_1, \dots, u_n in terms of z_1, \dots, z_n , say $z = \theta(u)$ and $\varphi(\theta(u)) = a + u_1 + \bar{u}_1 + \sum b_{i\bar{j}} u_i \bar{u}_j + 0(\|u\|^2)$. So in the coordinates u the function φ is: $\varphi(u) = a + u_1 + \bar{u}_1 + \sum b_{i\bar{j}} u_i \bar{u}_j + 0(\|u\|^2)$ and $\varphi(0) = a$.

Now if $v = a_j \frac{\partial}{\partial z_j} + \bar{a}_j \frac{\partial}{\partial \bar{z}_j}$ is a tangent vector to the level set $\varphi = c$, then $a_j \frac{\partial \varphi}{\partial z_j} + \bar{a}_j \frac{\partial \varphi}{\partial \bar{z}_j} = 0$. We have $Jv = ia_j \frac{\partial}{\partial z_j} - i\bar{a}_j \frac{\partial}{\partial \bar{z}_j}$ if Jv is also tangent to the level set $\varphi = c$ then $ia_j \frac{\partial \varphi}{\partial z_j} - i\bar{a}_j \frac{\partial \varphi}{\partial \bar{z}_j} = 0$; therefore $a_i \frac{\partial \varphi}{\partial z_i} = 0$.

Conversely if $a_i \frac{\partial \varphi}{\partial z_i} = 0$ then as $\varphi = \bar{\varphi}$, $a_i \frac{\partial \varphi}{\partial \bar{z}_i} = 0$ and so v and Jv are both tangent to the level set $\varphi = c$. So the maximal complex subspace of $\varphi \equiv C$ is given by $\partial \varphi(v) = 0$, $v = a_i \frac{\partial}{\partial z_i}$ (recall that every vector $\xi = \eta + \bar{\eta}$, $\eta \in T^{1,0}$). So the holomorphic tangent space to $\varphi = a$ at $z = 0$ is given by $z_1 = 0$. If φ is defined on a neighborhood U of 0 then $\varphi|U \cap T_0^{1,0}(\varphi = a)$ is given by:

$$\varphi(0, z_2, \dots, z_n) = a + \sum_{i,j \geq 2} b_{i\bar{j}} z_i \bar{z}_j + 0(\|z\|^2).$$

By a unitary transformation L of z_2, \dots, z_n we have $(\varphi \circ L)(0, z_2, \dots, z_n) = a + (Uz)^t (B\bar{u}\bar{z} + 0(\|z\|^2)) = a + z^t U^t B \bar{U} \bar{z} + 0(\|z\|^2)$ ($B = (b_{i\bar{j}})$) $= a + \lambda_2 |z_2|^2 + \dots + \lambda_p |z_p|^2 - (\lambda_{p+1} |z_{p+1}|^2 + \dots + \lambda_{p+q} |z_{p+q}|^2) + 0(\|z\|^2)$ where $p-1, q$ are the number of positive and negative eigenvalues of $\mathcal{L}(\varphi \circ L)(0)$ restricted to the complex tangent space to the hypersurface $(\varphi \circ L) = a$. So $(\varphi \circ L)|_{z_1=0, z_2=\dots=z_1=0} - a$ is $-\lambda_{p+1} |z_{p+1}|^2 - \dots + 0(\|z\|^2)$ and $\frac{(\varphi \circ L) - a}{|z|^2} \leq -\lambda \frac{\|z\|^2}{\|z\|^2} + 0(\|z\|^2)/\|z\|^2$ on this subspace, where $\lambda = \min\{\lambda_{p+1}, \dots, \lambda_{p+q}\}$, so on $H : z_1 = \dots = z_p = 0$ and for $\|z\|$ sufficiently small, we see that $(\varphi \circ L)(z) < a$ if $z \neq 0$ and $(\varphi \circ L)(0) = a$. In other words the q -disc $z_1 = \dots = z_p = 0$, $|z_{p+1}| < \epsilon, \dots, |z_{p+q}| < \epsilon$ meets the level set $\varphi = a$ at just the point p , and the punctured disc lies in the region $\varphi < a$. Similarly if $\mu = \min\{\lambda_2, \dots, \lambda_p\}$ then

$$\begin{aligned} (\varphi \circ L)(z) - a &= \lambda_2 |z_2|^2 + \dots + \lambda_p |z_p|^2 + 0(\|z\|^2) \\ &\geq \lambda(|z_2|^2 + \dots + |z_p|^2) + 0(\|z\|^2) \end{aligned}$$

So $\frac{(\varphi \circ L)(z) - a}{\|z\|^2} \geq \lambda + 0(\|z\|^2)/\|z\|^2$. Hence if $z_1 = 0$, $z_{p+1} = \dots = z_{p+q} = 0$ and $\|z_2\|^2 < \epsilon, \dots, \|z_p\|^2 < \epsilon$, then this disc touches the level set $\varphi = a$ only at $z = 0$ and the punctured disc lies in the region $\varphi > a$.

An immediate consequence is the following result:

(3.1) Proposition. *Let M be a complex manifold, $\varphi : M \rightarrow \mathbb{R}$ a differentiable function such that for some regular value c the sublevel set $\varphi \leq c$ is compact with nonempty interior. If the Levi-form of φ restricted to the level set $\varphi = c$ has one negative eigenvalue at every point, then M has no nonconstant holomorphic functions.*

Proof. Let S be the sublevel set $\varphi \leq c$. Assume that M has a nonconstant holomorphic function, say f . Then $|f|$ restricted to S assumes its maximum value at some point of the boundary $\varphi = c$, say at p . Now there is a one-dimensional disc $\Delta(p, \epsilon)$ which meets the set $\varphi = c$ at just the point p and $\Delta(p, \epsilon) \setminus \{p\} \subset \{z \in M : \varphi(z) < c\}$. But as $|f|$ restricted to $\Delta(p, \epsilon)$ achieves its maximum value at p we see by the maximum principle that $|f|$ is constant on $\Delta(p, \epsilon)$. So $|f|$ achieves its maximum value at an interior point of S . Hence f is constant on S and therefore on M . This is what we had to prove. \square

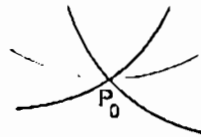
(3.2) Definition. A line bundle \mathcal{L} on M is called positive if its Chern class is positive. Choosing a norm N on \mathcal{L} this is equivalent to Levi form of N being negative definite on the nonzero level sets of N . One defines negative line bundles analogously.

(3.3) Proposition *Let \mathcal{L} be a line bundle on a compact complex manifold M and N a norm on \mathcal{L} . If the Levi-form of N has one positive eigenvalue at any nonzero level set, then \mathcal{L} has no nonzero sections. In particular, a negative line bundle has no nonzero sections.*

Proof. Suppose \mathcal{L} has a nonzero holomorphic section. So the function $(N\sigma)$ has a maximum value $c > 0$ at some point $p_0 \in M$. Now for every holomorphic curve γ through p_0 with $\gamma(0) = p_0$ the function $N((\gamma(z)))$ has a maximum value at 0. Hence

$$(a) \quad dN_{s(p_0)}(s_*(p_0)(\gamma'(0))) = 0$$

and



$$(b) \mathcal{L}(N)_{s(p_0)}(s_*(p_0)(\gamma'(0))) = \frac{\partial^2}{\partial z \partial \bar{z}}(N \circ s \circ \gamma)(0) \leq 0.$$

$$(\text{as } \frac{\partial^2 \psi(0)}{\partial x^2} \leq 0, \frac{\partial^2 \psi(0)}{\partial y^2} \leq 0 \text{ if } \psi(x, y) \text{ has a maximum at } 0)$$

The assertion (a) implies that $s_*(p_0)$ maps $T_0(M)$ into $T_{q_0}^{1,0}(N = C)$ ($q_0 = s(p_0)$); as s is a section, $s_*(p_0)$ is in fact surjective. Now (b) implies that the Levi form of N at $s(p_0)$ is negative definite on $T_{q_0}^{1,0}(N = c)$. This contradiction proves that \mathcal{L} has no nonzero sections.

Remark: (a) For a line bundle $N(\xi, z) = \varphi(\xi)|z|^2$ so if $N(\xi, z) = c > 0$ then $\log \varphi(\xi) + \log |z|^2 = \log c = k$. Now the $T^{1,0}$ tangent space to the level set $N = c$ at (ξ_0, z_0) is given by $\sum a_i \frac{1}{\varphi} \frac{\partial \varphi}{\partial z_i}(\xi_0) + a_0 \bar{z}_0 / z_0 \bar{z}_0 = 0$. So $T^{1,0}(N = c)$ projects onto $T_{\xi_0}^{1,0}(\varphi = c/|z_0|^2)$. Now $\partial \bar{\partial} \log N = \partial \bar{\partial} \log \varphi = \frac{1}{\varphi} \partial \bar{\partial} \varphi - \frac{1}{\varphi^2} \partial \varphi \wedge \bar{\partial} \varphi$. So if $\partial \varphi(\xi_0)(u_1) = 0$, $\partial \varphi(\xi_0)(u_2) = 0$ then $(\partial \bar{\partial} \log N)_{(\xi_0, z_0)}(u, v) = \frac{1}{\varphi} (\partial \bar{\partial} \varphi(\xi_0))(u, v)$. So signature of $\partial \bar{\partial} \log N$ (at $N = c$) = sign. of $\partial \bar{\partial} \log \varphi$ (at $\varphi = k$) = sign. of $\partial \bar{\partial} \varphi$ (at $\varphi = k$) = -sign of $C_1(\mathcal{L})$.

We conclude this chapter by proving the important $(d, d^{\mathbb{C}})$ lemma.

(3.4) The $(d, d^{\mathbb{C}})$ -Lemma. *M is a compact Kähler manifold, $\omega = d\alpha$ a real $(1, 1)$ -form. Then $\omega = i\partial\bar{\partial}\varphi$.*

Recall definitions: For a given metric g on M and v_g the canonical volume form we set, for differential forms ξ, η of same degree $\langle \xi, \eta \rangle = \int_M g(\xi, \eta) v_g$. Then d^* is the adjoint of d . Extend \langle, \rangle to a complex bilinear form on $T_M^{\mathbb{C}}$. It is nondegenerate and ∂^* is the adjoint of ∂ and $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$. $\Delta = dd^* + d^*d = 2(\partial\partial^* + \partial^*\partial) = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$.

Proof of the $d, d^{\mathbb{C}}$ Lemma. Let $\omega = d\alpha$ be a $(1, 1)$ -form. Let $\alpha = \beta^{1,0} + \beta^{0,1}$ be the decomposition into types. We have $\omega = d\alpha = \partial\beta^{0,1} + \bar{\partial}\beta^{1,0}$ and $\partial\beta^{1,0} = \bar{\partial}\beta^{1,0} = 0$. Let $\beta^{1,0} = \beta^0 + \Delta\gamma$ be the Hodge decomposition of β . Now

$$\begin{aligned} 0 = \partial\beta^{1,0} = \partial\beta^0 + \partial\Delta\gamma &= \partial\Delta\gamma \quad (\text{as } \partial\beta^0 = 0) \\ &= \Delta\partial\gamma \end{aligned}$$

Therefore $\partial\gamma$ is harmonic and so $\partial^*(\partial\gamma) = 0$. Hence $\Delta\gamma = (\partial\partial^* + \partial^*\partial)(\gamma) = \partial(\partial^*\gamma)$ and as γ is a $(1,0)$ -form, $\partial^*\gamma$ is a function.

Now

$$\omega = d\alpha = \bar{\partial}\beta^{1,0} + (\bar{\partial}\beta^{1,0})^\perp$$

and

$$\bar{\partial}\beta^{1,0} = \bar{\partial}\beta^0 + \bar{\partial}\Delta\gamma = \bar{\partial}\Delta\gamma = \bar{\partial}\partial(\partial^*\gamma).$$

Therefore

$$\omega = \bar{\partial}\partial(\partial^*\gamma) + \partial\bar{\partial}\overline{\partial^*\gamma} = \partial\bar{\partial}(\bar{\partial^*r} - \partial^*r) = i\bar{\partial}\bar{\partial}\psi. \quad \square$$

Chapter 4

Cohomology of flag manifolds

In this very brief chapter, we explain the relevance of the ideas of the preceding chapters to certain aspects of group theory. More precisely, we give an explicit basis for the 2nd deRham cohomology of flag manifolds, in the framework of the preceding chapters. These results are due originally due to Borel-Hirzebruch [Bo-Hi]. We assume familiarity with the theory of weights and roots of semisimple groups, as set forth in e.g [St.2].

Let G be a complex Lie group and V a linear representation of G . The function $g \mapsto \|g \cdot v\|$ ($g \in G$, v a non-zero vector in V) is plurisubharmonic and the form $i\partial\bar{\partial} \log \|g \cdot v\|$ is a positive semidefinite two form. The reason for the appearance of \log is more or less the same as that for its appearance in the Fubini metric on $\mathbb{P}^n(\mathbb{C})$: see the examples in (2.5), Ch. 2. These degenerate forms are useful in several contexts, e.g. in computations of moments and cohomology of flag manifolds. The 2nd deRham cohomology of flag manifolds is generated by forms which are very close to the above forms. The details are as follows:

Let G be a complex reductive group, B a Borel subgroup of G , T a maximal torus of G contained in B , R the roots of T in G , R^+ the positive system of roots defined by the pair (B, T) and S the corresponding simple system of roots. One knows that for each $\alpha \in R^+$ there exists $X_\alpha, X_{-\alpha} \in \text{Lie}(G)$ such that the map

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_\alpha, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto X_{-\alpha}$$

is an isomorphism of $sl(2, \mathbb{C})$ onto the Lie algebra generated by $X_\alpha, X_{-\alpha}$. Hence there exists a homomorphism ϕ_α from $SL(2, \mathbb{C})$ onto a subgroup L_α of G whose Lie algebra

is generated by $X_\alpha, X_{-\alpha}$. We set, for $\alpha \in R^+$,

$$u_\alpha(z) = \phi_\alpha \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad u_{-\alpha}(z) = \phi_\alpha \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \quad \check{\alpha}(z) = \phi_\alpha \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

By a variant of Bruhat's Lemma [St 1, p. 99], the group

$$K = \langle \phi_\alpha(SU(2)); \alpha \text{ is simple} \rangle.$$

Let $\pi \subset S$ and $P = P_\pi$ the corresponding parabolic subgroup. Let $\xi_0 = eP$. For $\alpha \in S \setminus \pi$ we have: $L_\alpha \cdot \xi_0 \cong \mathbb{P}^1(\mathbb{C})$ where $L_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$: we denote this line by \mathbb{P}_α . For $\alpha \in S$ let ρ_α be the irreducible representation with highest weight α , v a highest weight vector therein and $\omega_\alpha = dd^{\mathbb{C}} \log \|\varphi_\alpha(g) \cdot v\|^2$. For $\alpha \in S \setminus \pi$ the form ω_α is the pull-back of a K -invariant form $\bar{\omega}_\alpha$ on G/P . Namely, if s is a local section of $G \rightarrow G/P$ then $\omega_2(\xi) = dd^{\mathbb{C}} \log \|\rho_\alpha(s(\xi) \cdot v)\|^2$.

(4.1) Proposition. *The $\{\omega_\alpha : \alpha \in S \setminus \pi\}$ form a basis of $H^2(G/P, \mathbb{R})$ and $\frac{1}{2\pi} \int_{\mathbb{P}_\beta} \omega_\alpha = \delta_{\alpha, \beta}$.*

Proof. We may assume that G is simply connected so that $\pi_1(G) = 0 = \pi_2(G)$. From the homotopy exact sequence of the fibration $P \rightarrow G \rightarrow G/P$ we have $\pi_1(G/P) = 0$ and $\pi_2(G/P) \cong \pi_1(P)$. Since

$$\begin{aligned} P &= LR_u(p) = \left\{ \prod_{\alpha \in S \setminus \pi} \check{\alpha}(z_i) : z_i \in \mathbb{C}^* \right\} \cdot L' \cdot R_u(P) \\ &= T_1 \cdot L' \cdot R_u(P) \end{aligned}$$

we have $\pi_1(P) = \pi_1(T_1)$. Using $\pi_1(G/P) = 0$ and the Hurewicz theorem we see that $\pi_2(G/P) \cong H_2(G/P, \mathbb{Z})$ and $\text{rank}(\pi_2(G/P)) = \text{card}(S \setminus \pi)$.

Let us now show that $\frac{1}{2\pi} \int_{\mathbb{P}_\beta} \omega_\alpha = \delta_{\alpha, \beta}$. A local section of $G \rightarrow G/P$ defined in a neighborhood of ξ_0 is $r \cdot \xi_0 \mapsto r$, $r \in R_u(P)^-$. We have $\mathbb{P}_\beta = U_{-\beta} \cdot \xi_0 \cup \omega_\beta \cdot \xi_0$,

$\omega_\beta = \varphi_\beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so $\int_{\mathbb{P}_\beta} \omega_\alpha = \int_{U_{-\beta} \cdot \xi_0} \omega_\alpha$. By the Gram-Schmidt process applied to columns of $SL(2, \mathbb{C})$ we have:

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = k \begin{pmatrix} (1 + z\bar{z})^{1/2} & 0 \\ 0 & (1 + z\bar{z})^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & \bar{z}/1 + z\bar{z} \\ 0 & 1 \end{pmatrix}, \quad k \in SU(2)$$

so that

$$U_{-\beta}(z) = \varphi_\beta(k) \hat{\beta}((1 + z\bar{z})^{1/2}) \cdot U_\beta(\bar{z}/1 + z\bar{z}).$$

Hence

$$\begin{aligned} \|U_{-\beta}(z) \cdot v_\alpha\| &= \|\hat{\beta}(1 + z\bar{z})^{1/2} \cdot v_\alpha\| \\ &= ((1 + z\bar{z})^{1/2})^{\omega_\alpha(\hat{\beta})} = (1 + z\bar{z})^{1/2} \delta_{\alpha, \beta}. \end{aligned}$$

So $\int_{\mathbb{P}_\beta} \omega_\alpha = 0$ if $\alpha \neq \beta$ and $\int_{\mathbb{P}_\alpha} \omega_\alpha = \int_C d\bar{d}^C \log(1 + z\bar{z}) = 2\pi$. So $\{\omega_\alpha/2\pi\}_{\alpha \in S \setminus \pi}$ are independent generators of $H^2(G/P)$ and their duals are the lines $\{\mathbb{P}_\alpha\}_{\alpha \in S \setminus \pi}$. \square

(4.2) Concluding remarks. Plurisubharmonic functions and potentials have many interesting applications in group theory: see, e.g. [Az-Lo], [Lo] and [Ne]. In fact, K.H. Neeb in [Ne] has just obtained very interesting results on plurisubharmonic functions which are invariant under a noncompact real form. An important direction of research could be the investigation of homogeneous complex manifolds G/H which admit an exhaustion function of mixed signature, and which is invariant under a noncompact real form of G . Almost nothing is known about this situation.

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