Math 102
Quiz#4
Please: Solve only 10 questions.

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$\boxed{1}$

1. Evaluate $\int_1^2 x \sec^{-1} x \, dx \quad = I$

Let $\boxed{u = \sec^{-1} x \quad , dv = x\,dx \quad du = \frac{dx}{x\sqrt{x^2-1}} \quad v = \frac{x^2}{2}}$,

$I = \frac{x^2}{2}\sec^{-1}x \Big]_1^2 - \frac{1}{2}\int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx$, let $\boxed{u = x^2-1 \quad du = 2x\,dx}$

$= \qquad -\frac{1}{4}\int_0^3 \frac{du}{\sqrt{u}}$

$= \left[\frac{(2)^2}{2}\sec^{-1}(2) - \frac{1}{2}\sec^{-1}(1)\right] - \frac{1}{4}\left(2\sqrt{u}\right)_0^3$

$= \left[2\left(\frac{\pi}{3}\right) - 0\right] - \frac{1}{4}\left[2\sqrt{3} - 0\right]$

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

2. Evaluate $\int \frac{\sqrt{x^2-9}}{x}dx \quad = I$

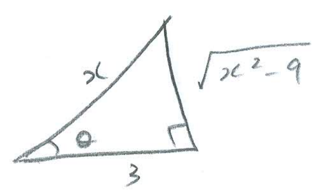
Let $\boxed{x = 3\sec\theta \quad dx = 3\sec\theta\tan\theta\,d\theta}$

$I = \int \frac{3\tan\theta \; \cancel{3\sec\theta}\tan\theta\,d\theta}{\cancel{3\sec\theta}}$

$= 3\int \tan^2\theta\,d\theta$

$= 3\int (\sec^2\theta - 1)\,d\theta$

$= 3(\tan\theta - \theta) + c$



$I = 3\left(\frac{\sqrt{x^2-9}}{3} - \sec^{-1}\frac{x}{3}\right) + c$.

3. Evaluate $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} \quad = I$

Let $\boxed{u = \sqrt[3]{x}}$, $u^3 = x$

$\boxed{3u^2\,du = dx}$.

$I = \int \frac{3u^2\,du}{u^{\frac{3}{2}} + u}$

$= 3\int \frac{u}{u^{\frac{1}{2}}+1}\,du$

let $\boxed{z^2 = u}$, $\boxed{2z\,dz = du}$

$I = 3\int \frac{z^2\,(2z\,dz)}{z+1}$

$= 6\int \frac{z^3}{z+1}$

$= 6\int \left((z^2 - z + 1) - \frac{1}{z+1}\right)dz$

$= 6\left[\frac{z^3}{3} - \frac{z^2}{2} + z - \ln|z+1|\right] + c$

$= 6\left[\frac{(x^{\frac{3}{2}})^3}{3} - \frac{(x^{\frac{3}{2}})^2}{2} + (x^{\frac{3}{2}}) - \ln\left|x^{\frac{3}{2}}+1\right|\right] + c$

$$\begin{array}{r|l} & z^2 - z + 1 \\ z+1 & z^3 \\ & z^3 + z^2 \\ \hline & -z^2 \\ & -z^2 - z \\ \hline & +z \\ & z+1 \\ \hline & -1 \end{array}$$

4. Write down the partial fraction decomposition of $\frac{1}{x(x^2-1)^2(x^2+1)^2}$ $= \frac{1}{x(x-1)^2(x+1)^2(x^2+1)^2}$

[Do not calculate the coefficients]

$$= \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2} + \frac{A_4}{x+1} + \frac{A_5}{(x+1)^2} + \frac{A_6x+A_7}{x^2+1} + \frac{A_8x+A_9}{(x^2+1)^2}.$$

where $A_1, A_2, .., A_9$ are real numbers.

5. Evaluate $\int \sin 2x \cos 3x \, dx$ $= I$

Use $\boxed{\sin 2x \cos(3)x = \frac{1}{2}\left[\sin(2-3)x + \sin(2+3)x\right]}$,

$$I = \frac{1}{2}\int \sin(-x) + \sin(5x)\, dx$$

$$= \frac{1}{2}\left[\cos x - \frac{1}{5}\cos 5x\right] + c.$$

6. Find $\int (x^3+3x)e^{x^2}\,dx.$ $= I$

$$I = \int x^3 e^{x^2} dx + 3\int x e^{x^2} dx$$

let $\boxed{u = x^2, \quad du = 2x\,dx}$

$$I = \int \frac{\cancel{x} u e^u}{2\cancel{x}} du + 3\int \frac{\cancel{x} e^u}{2\cancel{x}} du$$

$$= \frac{1}{2}\int u e^u \, du + \frac{3}{2}\int e^u du$$

Let $\boxed{\begin{array}{ll} z = u & dv = e^u du \\ dz = du & v = e^u \end{array}}$

$$I = \frac{1}{2}\left[u e^u - \int e^u du\right] + \frac{3}{2}\int e^u du$$

$$= u e^u - \frac{1}{2}e^u + \frac{3}{2}e^u + c$$

$$= u e^u + e^u + c$$

$$= e^u(u+1) + c.$$

7. Evaluate $\int_0^{\pi/2} \sin^5 x \; dx$.

$$= \int_0^{\frac{\pi}{2}} \sin x \; \sin^4 x \; dx = \int_0^{\frac{\pi}{2}} \sin x \left(1-\cos^2 x\right)^2 dx$$

$$= -\int_1^0 \left(1-u^2\right)^2 du = \int_0^1 \left(1-2u^2+u^4\right) du$$

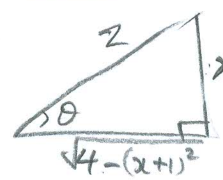
Let $\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array}}$

$$= u - \frac{2}{3}u^3 + \frac{u^5}{5}\Big]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{15-10+3}{15} = \frac{8}{15}.$$

8. Find $\int \frac{\sqrt{3-2x-x^2}}{x+1} dx. = I$

$\boxed{\begin{array}{l} 3-2x-x^2 = -(x^2+2x+1)+3+1 \\ = -(x+1)^2+4 \end{array}}$

$$I = \int \frac{\sqrt{4-(x+1)^2}}{x+1} dx$$



$\boxed{x+1 = 2\sin\theta}$ $\boxed{dx = 2\cos\theta \, d\theta}$

$$I = \int \frac{\cancel{2}\cos\theta}{\cancel{2}\sin\theta} \cdot 2\cos\theta \, d\theta = 2\int \frac{\cos^2\theta \, d\theta}{\sin\theta} = 2\int \frac{(1-\sin^2\theta)}{\sin\theta} d\theta$$

$$= 2\left[\int \csc\theta \, d\theta - \int \sin\theta \, d\theta\right] = 2\left(-\ln|\csc\theta + \cot\theta| + \cos\theta\right) + C$$

$$= 2\left[-\ln\left|\frac{2}{x+1} + \frac{\sqrt{4-(x+1)^2}}{x+1}\right| + \frac{\sqrt{4-(x+1)^2}}{2}\right] + C.$$

9. Find $\int \frac{x^3-x+1}{x^3+x} dx. = I$

$$I = \int \left(1 + \frac{-2x+1}{x^3+x}\right) dx$$

$$= \int \left[1 - \frac{2}{x^2+1} + \frac{1}{x^3+x}\right] dx$$

$$= \int \left[1 - \frac{2}{x^2+1} + \frac{1}{x} - \frac{x}{x^2+1}\right] dx$$

$$= x - 2\tan^{-1}x + \ln|x| - \frac{1}{2}\ln|x^2+1| + C$$

$$\begin{array}{r|l} & 1 \\ x^3+x & x^3-x+1 \\ & x^3+x \\ \hline & -2x+1 \end{array}$$

$$\frac{1}{x^3+x} = \frac{A_1}{x} + \frac{A_2x+A_3}{x^2+1}$$

$$A_1(x^2+1) + (A_2x+A_3)x = 1$$

When $\boxed{x=0} \Rightarrow \boxed{A_1 = 1}$

coeff x^2: $A_1 + A_2 = 0 \Rightarrow \boxed{A_2 = -1}$

coeff x: $\boxed{A_3 = 0}$

10. Find $\int \sqrt{1+\sqrt{x}}\ dx.$ $= I$

Let $\boxed{u = \sqrt{1+\sqrt{x}}}$

$\Leftrightarrow u^2 = 1+\sqrt{x}$, $\Leftrightarrow \boxed{(u^2-1)^2 = x}$

$\Rightarrow \boxed{2(u^2-1)(2u)\,du = dx}$

then

$$I = \int u \cdot 2(u^2-1)(2u)\, du$$

$$= 4\int u^4 - u^2\, du = 4\left[\frac{u^5}{5} - \frac{u^3}{3}\right] + c$$

$$= 4\left[\frac{(\sqrt{1+\sqrt{x}})^5}{5} - \frac{(\sqrt{1+\sqrt{x}})^3}{3}\right] + c.$$

11. Find $\int \frac{\ln(x-1)}{x^3} dx.$ $= I$

Let $\boxed{u = \ln(x-1), \quad dv = \frac{1}{x^3}dx \\ du = \frac{1}{x-1}dx, \quad v = \frac{x^{-2}}{-2}}$

$$I = -\frac{1}{2}\cdot\frac{\ln(x-1)}{x^2} + \frac{1}{2}\boxed{\int \frac{1}{(x-1)x^2}dx}$$

$$= \frac{1}{2}\frac{\ln(x-1)}{x^2} + \frac{1}{2}\left[\int \frac{1}{x-1} + \left(\frac{-1}{x}\right) + \left(\frac{-1}{x^2}\right) dx\right]$$

$$= \frac{1}{2}\frac{\ln(x-1)}{x^2} + \frac{1}{2}\left[\ln|x-1| - \ln|x| + \frac{1}{x}\right] + c.$$

$$\frac{1}{(x-1)x^2} = \frac{A_1}{x-1} + \frac{A_2}{x} + \frac{A_3}{x^2}$$

$A_1 x^2 + A_2(x-1)x + A_3(x-1) = 1$

when $\boxed{x=0} \Rightarrow \boxed{A_3 = -1}$

when $\boxed{x=1} \Rightarrow \boxed{A_1 = 1}$

coeff x: $-A_2 + A_3 = 0$

$\Rightarrow \boxed{A_2 = -1}$

12. Evaluate $\int \tanh^2 x\ \mathrm{sech}^4 x\ dx$ $= I$

$$I = \int \tanh^2 x\ \boxed{\mathrm{sech}^2 x}\ \mathrm{sech}^2 x\, dx$$

use $\boxed{\mathrm{sech}^2 x = 1 - \tanh^2 x}$

$$= \int \tanh^2 x\,(1-\tanh^2 x)\ \mathrm{sech}^2 x\, dx$$

Let $\boxed{u = \tanh x}$, $\boxed{du = \mathrm{sech}^2 x\, dx}$

then

$$I = \int u^2(1-u^2)\,du = \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{\tanh^3 x}{3} + \frac{\tanh^5 x}{5} + c.$$