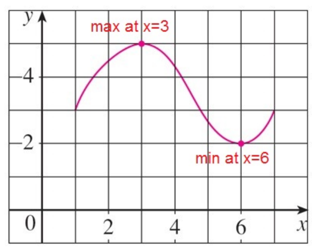
4 Application of Differentiation

4.1 Maximum and Minimum Values

One important application for differentiation is to find the maximum or minimum values of a function. Let's first explain exactly what we mean by maximum and minimum values. We see that the highest point on the graph of the function f, shown in the below figure, is the point (3, 5).



So, the largest value of f is f (3) = 5 and the smallest value is f (6) = 2. We say that $f(3) = 5$ is the **absolute maximum** of f and $f(6) = 2$ is the **absolute minimum**. In general, we use the following definition.

Definition 4.1.

Let c be a number in the domain D of a function f. Then $f(c)$ is the

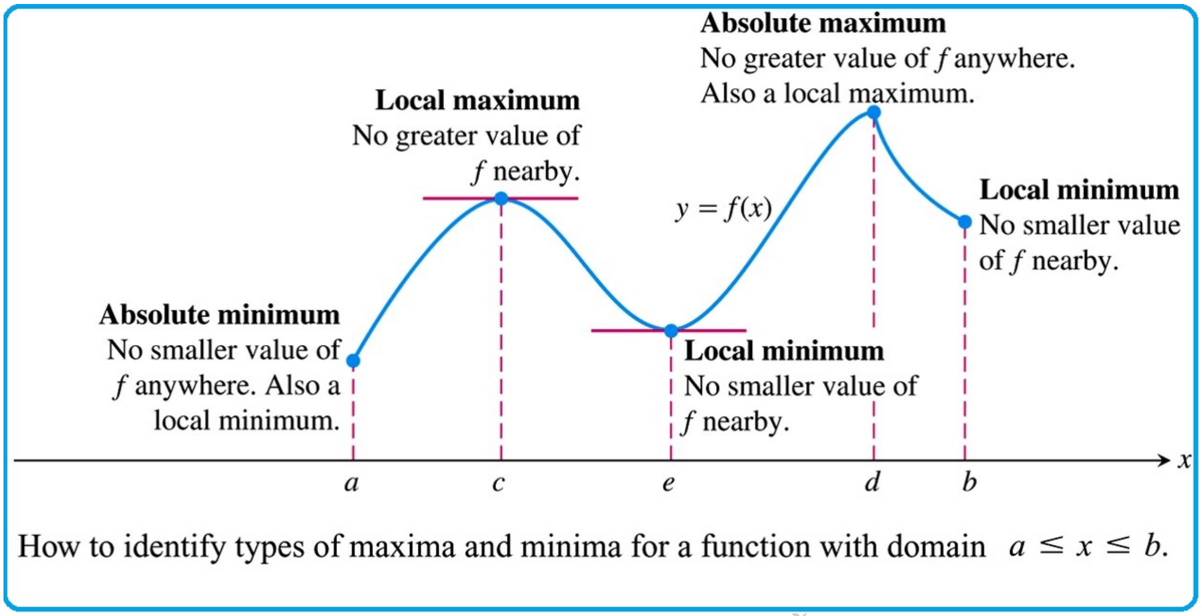
- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

The maximum and minimum values of f are called **extreme values** of f.

Definition 4.2.

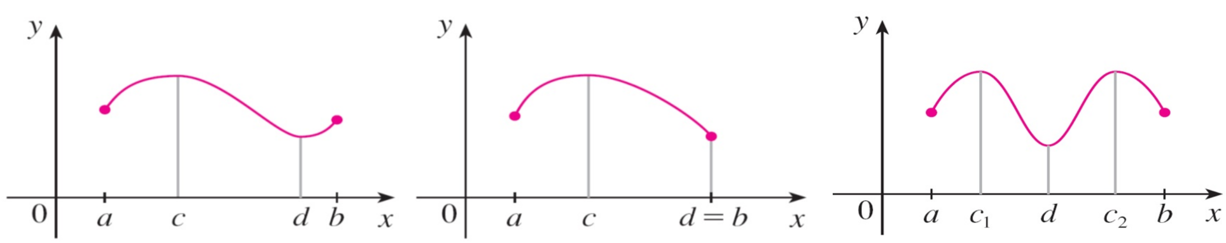
The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.



The following theorem gives conditions under which a function is guaranteed to possess extreme values.

Theorem 4.1. ***The Extreme Value Theorem*** *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

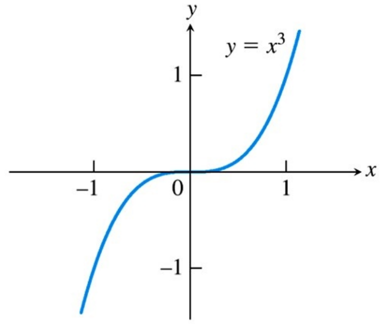


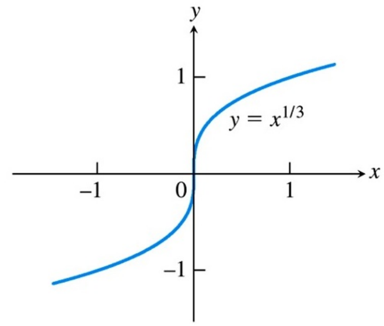
Note 15. The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values.

Theorem 4.2. ***Fermat's Theorem*** *If f has a local maximum or minimum at c, and if $f'(c)$ exisits, then $f'(c) = 0$.*

Question: Is the converse of Fermat's Theorem correct? No.

- If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$. But, f has no maximum or minimum at 0, as you can see from its graph.
- If $f(x) = x^{\frac{1}{3}}$ then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ is undefined at $x = 0$.





Fermat's Theorem does suggest that we should at least start looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

Definition 4.3. A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Note 16. By that, Fermat's Theorem is simply: If f has a local maximum or minimum at c, then c is a critical number of f.

The Closed Interval Method

Steps for searching for th absolute maximum and th absolute minimum of a function f on a closed interval $[a, b]$:

1) Find the critical number of f in the open interval (a, b).
2) Find the value of f at each critical number found in step 1.
3) Find the values of f at the endpoints of the interval.
4) The largest of the values from step 2 and step 3 is the absolute maximum; and the smallest is absolute minimum.

Example 4.1. Given the function $f(x) = x^3 + x^2 - x$.

a) Find the critical numbers of f

b) Find the absolute maximum and the absolute minimum values of f on $[-2, 1]$

Example 4.2. Given the function $f(x) = e^{-x} - e^{-2x}$.

a) Find the critical numbers of f

b) Find the absolute maximum and the absolute minimum values of f on $[0, 1]$

Example 4.3. Find the critical numbers for the following functions:

a) $f(x) = 2\cos x + \sin^2 x$

b) $g(x) = x^{\frac{1}{3}} - x^{-\frac{2}{3}}$

Exercise

1) Let $f(x) = \frac{3}{8}x^{8/3} - \frac{6}{5}x^{5/3} + \frac{3}{2}x^{2/3}$. If

$$C = \text{the number of critical numbers of } f(x)$$

$$m = \text{the number of local minimum of } f(x)$$

$$M = \text{ the number of local maximum of } f(x)$$

then $2C + m - M =$

a) 5
b) 0
c) -1
d) 4
e) 3

2) The **sum** of the critical numbers of $f(x) = \sqrt[3]{x+1} \cdot e^x$ on $[-2, 0]$ is equal to

a) $-\frac{7}{3}$
b) -3
c) -1
d) $-\frac{5}{3}$
e) 0

3) The function $f(x) = x^{4/3} - x^{1/3}$ has

a) No absolute maximum
b) No local maximum
c) One local maximum and one local minimum
d) No vertical tangents
e) Two critical numbers

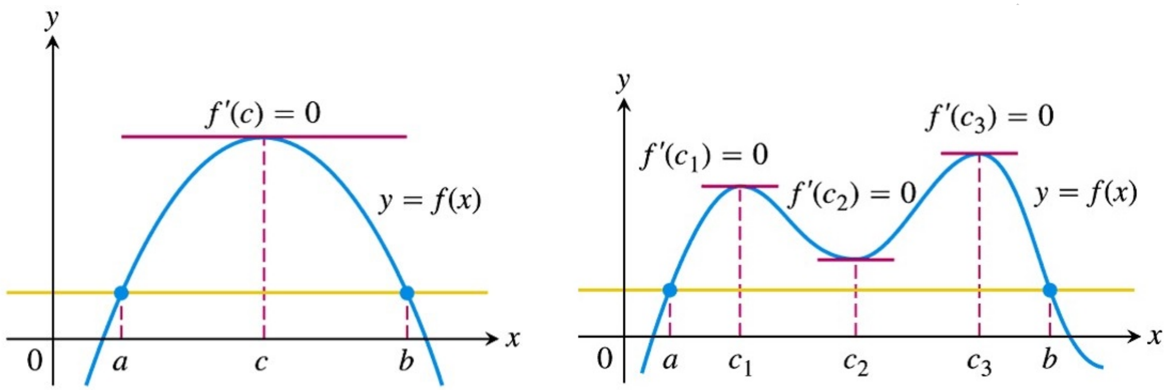
4) Given that $f(x) = \dfrac{x^2+1}{\sqrt{2x+1}}$, then the number of critical points of f is

a) 2
b) 4
c) 1
d) 3
e) 0

4.2 The Mean Value Theorem

We will see that many of the results depend on the Mean Value Theorem. First, we introduce the following result.

Theorem 4.3. *(Rolle's Theorem) If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists a c in the open interval (a, b) such that $f'(c) = 0$.*



Note 17. From the above figure, there is at least one point $(c, f(c))$ on the graph where the tangent is horizontal and therefore $f'(c) = 0$.

Example 4.4. Do the following functions satisfy the hypothesis of Rolle's Theorem, if yes find the value(s) of c?

(a) $f(x) = x - x^{\frac{1}{3}}$ on $[-1, 1]$

(b) $f(x) = \sqrt{x} - \frac{1}{3}x$ on $[0, 9]$

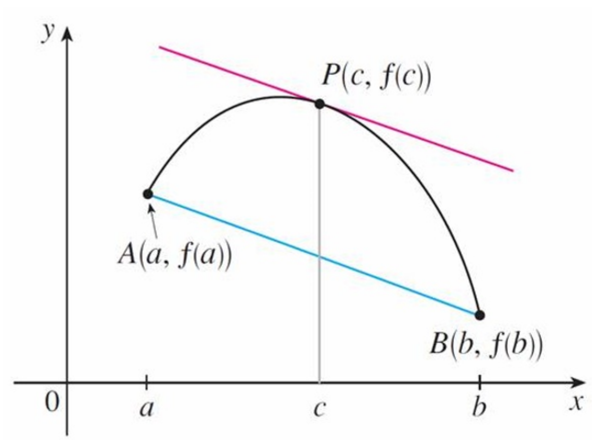
Example 4.5. Prove that the equation $x^3 + x–1 = 0$ has exactly one real root.

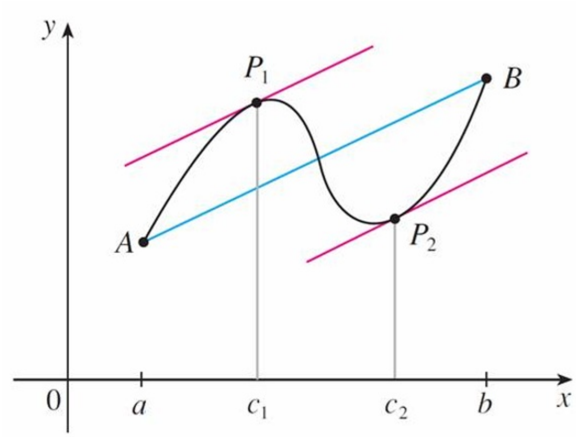
See the book for the detail.

Theorem 4.4. *(The Mean Value Theorem - MVT) If $f(x)$ is defined and continuous on the interval $[a,b]$ and differentiable on (a,b), then there is at least one number c in the interval (a,b) such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

It means, graphically as in the figure below, the tangent line passes through c (a point on the graph f) is parallel to the secant line AB.





Example 4.6. Let $f(x) = \frac{1}{4}x^3 + 1$ in $[0,2]$. Does the function f satisfies the hypothesis of The Mean Value Theorem? Find the value(s) of c that satsify the conclusion of the MVT?

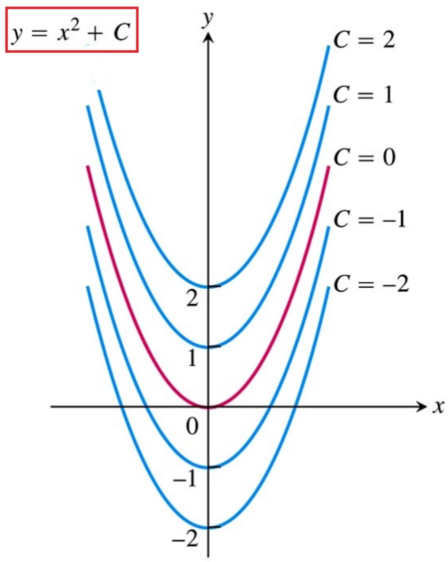
Example 4.7. If $f(1) = 10$ and $f' \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ be?

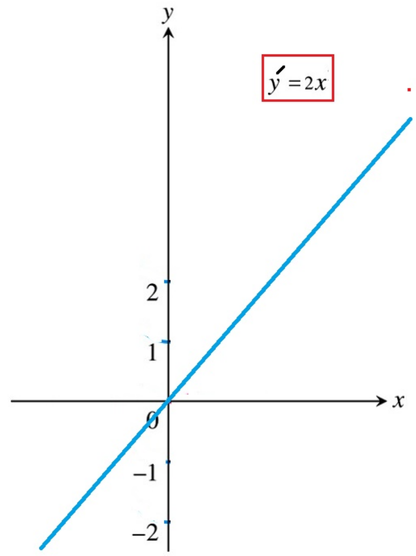
Example 4.8. Let $f(x) = x^{\frac{2}{3}}$ on $[-8, 27]$. Why the conclusion from MVT fails to have such c?

The coming theorems are direct results from the MVT.

Theorem 4.5. *If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).*

Theorem 4.6. *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b). i.e. $f(x) = g(x) + c$, where c is a constant.*





Example 4.9. Use the above results to prove the following identities:

(a) $2\sin^{-1} x = \cos^{-1}(1 - 2x^2), \;\; x \geq 0.$

(b) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

Exercise

1) If c is a number which satisfies the conclusion of the Mean Value Theorem for the function $f(x) = \sin^{-1}\left(\frac{x}{2}\right)$ on the interval $[0, 2]$, then $\pi^2 c^2 + 16 =$

   a) π^2
   b) $4\,\pi^2$
   c) $2\pi^2$
   d) 4
   e) 0

2) Let $f(x) = \alpha x^2 + \beta x + \gamma$, where $\alpha \neq 0, \beta, \gamma$ are constants. The value of c that satisfies the conclusion of the Mean Value Theorem for f on the interval $[3, 7]$ is

   a) 3.5
   b) 6
   c) 4.5
   d) 5
   e) 4

3) If $f(x) = x^4 - x$ on $[-1, 1]$, then the value of c that satisfies the Mean Value Theorem is

   a) $-\frac{1}{4}$
   b) $\frac{1}{4}$
   c) 2
   d) 1
   e) 0

4) If $f(x) = \begin{cases} C & \text{if} \quad x = 0 \\ Ae^x & \text{if} \quad 0 < x < 1 \\ \ln x + B & \text{if} \quad 1 \leq x \leq 2 \end{cases}$ satisfies the conditions of the Mean Value Theorem, then $A + B - C =$

   a) $\frac{1}{e}$
   b) $1 + e$
   c) 1
   d) $\frac{1}{e^2}$
   e) e

5) How many points in $(-2, 2)$ satisfy the conclusion of the Mean Value Theorem for $f(x) = x^2\, e^{-x^2}$ on $[-2, 2]$?

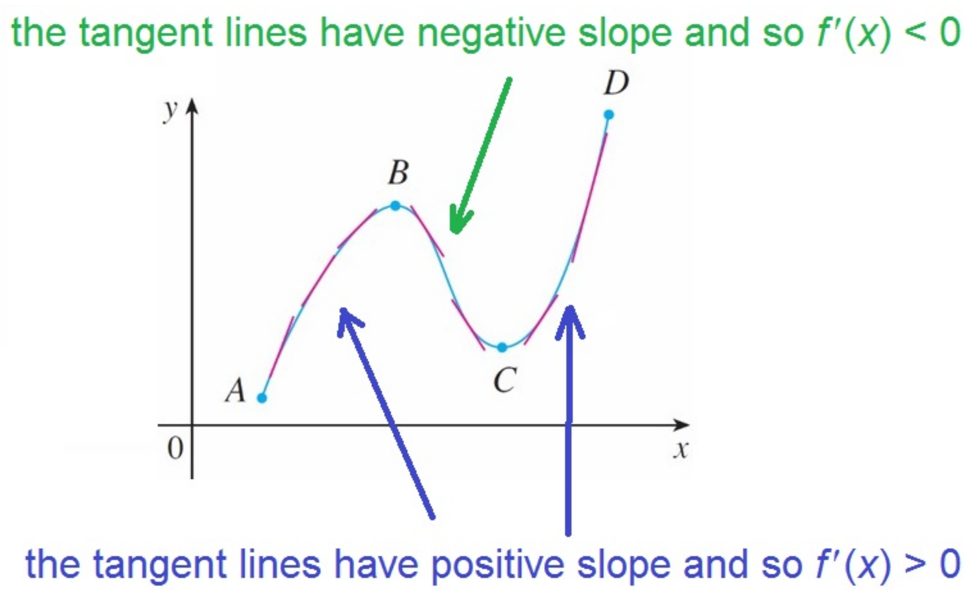
a) 2
b) 0
c) 4
d) 3
e) 1

6) Given a function $f(x) = x + \cos x$ and an interval $\left[0, \frac{\pi}{2}\right]$. The Mean Value Theorem tells us that there is a number c between 0 and $\frac{\pi}{2}$ such that

a) $f'(c) = 0$
b) $f'(c) = \dfrac{1}{1 - \frac{2}{\pi}}$
c) $f'(c) = 1 - \dfrac{2}{\pi}$
d) $f'(c) = 1$
e) $f'(c) = \dfrac{2}{\pi}$

4.3 How Derivatives Affect the Shape of a Graph

What Does f' Say About f ?



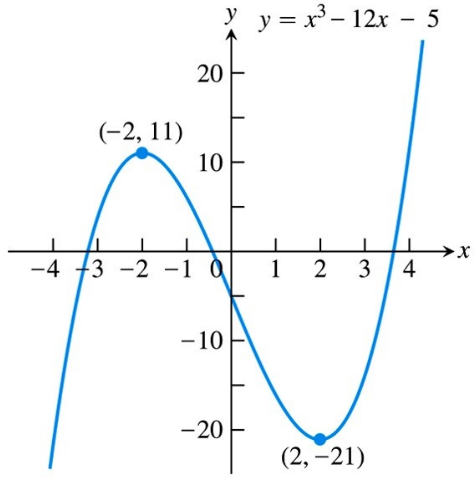
4.3.1 Increasing and Decreasing Test

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 4.10. Find where the function $f(x) = x^3 – 12x – 5$ is increasing and where it is decreasing.



Example 4.11. Find the interval on which the function $f(x) = \sin x + \cos x$ is increasing or decreasing.

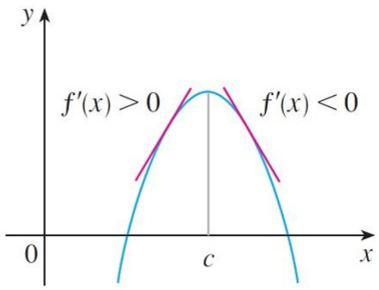
4.3.2 The First Derivative Test

The First Derivative Test Suppose that c is a critical number of a continuous function f.

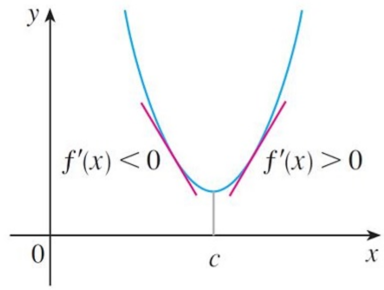
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

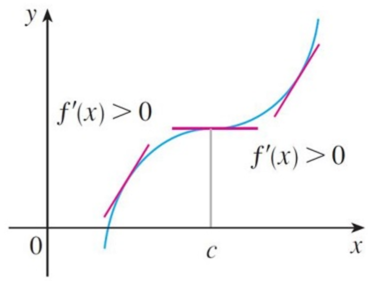
(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.



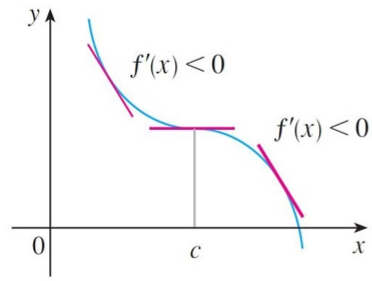
a) Local maximum



b) Local minimum



c) No maximum or minimum



d) No maximum or minimum

Example 4.12. Let

$$f(x) = \begin{cases} -x^2 + 8 & \text{if} \quad x \leq 2 \\ x^2 & \text{if} \quad x > 2 \end{cases}$$

a) Find the interval on which the function f is increasing or decreasing?

b) Find the local maximum and minimum values of f?

Example 4.13. Let $g(x) = x + 2\sin x$ where $0 \le x \le 2\pi$.

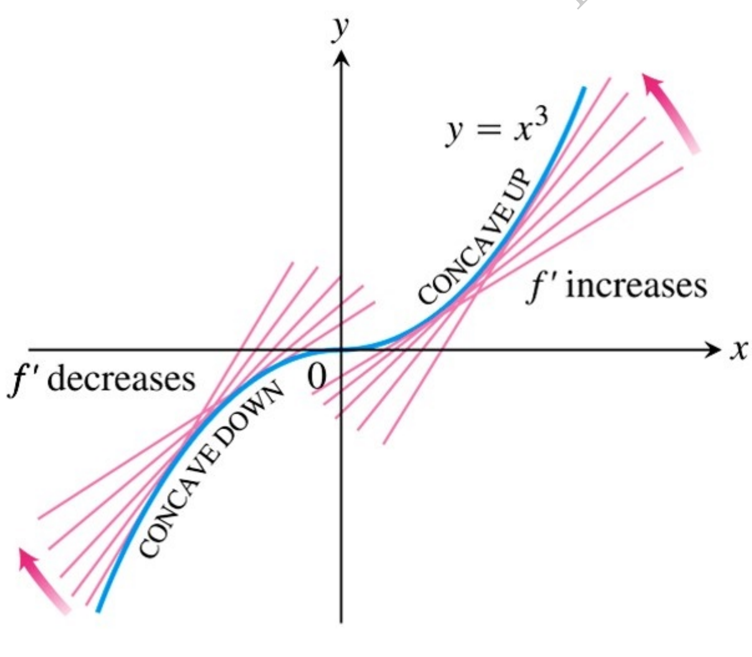
a) Find the interval on which the function g is increasing or decreasing?

b) Find the local maximum and minimum values of g?

What Does f'' Say About f ?

Definition 4.4. (Concavity)

- If the graph of f lies above all of its tangents on an interval I, the it is called **concave upward** on I.
- If the graph of f lies below all of its tangents on an interval I, the it is called **concave downward** on I.



4.3.3 Concavity Test

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition 4.5. (Inflection Points)
A point on a curve $y = f(x)$ is called an **inflection point** if f is a continuous there and the curve changes from concave upward to downward (or from downward to upward).

Note 18. From the above definition, we can say that if $f''(c)$ exists and $(c, f(c))$ is an inflection point, then $f''(c) = 0$. How about the converse? (take $f(x) = x^4$)

Example 4.14. Discuss the concavity and inflection points of the curve $y = \frac{1}{30}x^6 - \frac{1}{12}x^4$

4.3.4 The second Derivative Test

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

$f' = 0, f'' > 0 \Rightarrow$ local min $\qquad$ $f' = 0, f'' < 0 \Rightarrow$ local max

Example 4.15. Discuss the following curves with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

a) $y = \frac{1}{5}x^5 - \frac{1}{3}x^3$

b) $y = (x-1)^{\frac{1}{3}}(x+2)^{\frac{2}{3}}$

c) $y = e^{2x} + e^{-x}$

Example 4.16. Sketch a graph of a function that satisfies the following conditions:

- $f'(1) = f'(-1) = 0$,
- $f'(x) < 0$ if $|x| < 1$,
- $f'(x) > 0$ if $1 < |x| < 2$,
- $f'(x) = -1$ if $|x| > 2$,
- $f''(x) < 0$ if $-2 < x < 0$,
- $(0, 1)$ is inflection point.

Exercise

1) The function $f(x) = 5x^3 - 3x^5$ has

 a) 2 local extrema and one point of inflection
 b) 2 local extrema and 3 points of inflection
 c) 3 critical points and no points of inflection
 d) 2 local extrema and 2 points of inflection
 e) 2 critical points and 3 points of inflection

2) Which one of the following statements is **FALSE** about the graph of the function

$$f(x) = -x - \frac{1}{x-1}?$$

 a) The graph of f is concave downward on $(1, \infty)$
 b) The graph of f is concave upward on $(-\infty, 1)$
 c) The graph of f has exactly one inflection point
 d) The line $y = -x$ is an oblique asymptote of the graph of f
 e) The line $x = 1$ is a vertical asymptote to the graph of f

3) Let $f(x) = \dfrac{2x^2}{x^2 - 1}$. Which one of the following statements is **TRUE**

 a) The graph of f is concave down on $(-1, 1) \cup (1, \infty)$
 b) f has two inflection points
 c) The graph of f is concave down on $(-1, 0) \cup (1, \infty)$
 d) The graph of f is concave up on $(-\infty, -1) \cup (1, \infty)$
 e) The graph of f is concave up on $(-\infty, -1) \cup (0, 1)$

4) The number of inflection points for the graph of $f(x) = \dfrac{(x+1)^2}{x^2 + 1}$ is

 a) 3
 b) 0
 c) 2
 d) 1
 e) 4

5) If (α, β) is the point of inflection of the curve
$f(x) = \tan x - 4x$ for $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then $2\alpha - \beta + 2$

a) 0
b) 2
c) $-\frac{\pi}{4}$
d) $\frac{\pi}{4}$
e) $1 - \pi$

6) The function $y = \frac{1}{x^2+3}$ has

a) One absolute maximum and two inflection points
b) One absolute maximum and four inflection points
c) One absolute maximum and one inflection points
d) No maxima or minima and two inflection points
e) One absolute minimum and two inflection points

7) If $f(x) = \frac{x^2}{x^2-1}$, then f is

a) concave down on its domain
b) concave down on $(-1, 1)$ with two inflection points
c) concave up on its domain
d) concave down on $(-\infty, -1) \cup (1, \infty)$ with two inflection points
e) concave down on $(-1, 1)$ with no inflection points

8) The graph of $f(x) = \frac{e^x}{e^x+1}$

a) has two inflection points
b) has no inflection points
c) is concave up on $(0, \infty)$
d) is concave up on $(-\infty, 0)$
e) is concave up on $(-\infty, \infty)$

9) Which one of the following statements is TRUE?

(a) If f and g are increasing on an interval I, then fg is increasing on I.
(b) If $f'(c) = 0$, then f has a local maximum or minimum at c.
(c) If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of the curve $y = f(x)$.
(d) There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.
(e) There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

4.4 Indeterminate Forms and L'Hospital's Rule

Types of indeterminate form:

Type 1: the forms $\frac{0}{0}$ $\quad \frac{\infty}{\infty}$ $\quad 0 \cdot \infty$

Type 2: the form $\infty - \infty$

Type 3: the forms 0^0 $\quad 1^\infty$ $\quad \infty^0$

4.4.1 Type 1: Forms $\frac{0}{0}$ $\frac{\infty}{\infty}$ $0.\infty$

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Note 19. L'Hospital's Rule is also valid for one–sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

Note 20. L'Hospital's Rule can be applied whenever we get the indeterminate form, which means we may use L'Hospital's Rule more than one time.

Example 4.17. Find :

a) $\displaystyle \lim_{x \to 0} \frac{\cos x + 2x - 1}{3x}$

b) $\displaystyle\lim_{x\to(\frac{\pi}{2})^-} \frac{4\tan x}{1+\sec x}$

c) $\displaystyle\lim_{x\to\infty} \frac{e^{3x}}{x^2}$

d) $\displaystyle\lim_{x\to 0^+} (x^2 \ln x)$

4.4.2 Type 2: Form $\infty - \infty$

Example 4.18. Find :

a) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{e^x-1} - \frac{1}{x}\right)$

b) $\displaystyle\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

4.4.3 Type 3: Form 0^0 1^∞ ∞^0

Several indeterminate forms arise from the limit

$$\lim_{x\to a} [f(x)]^{g(x)}$$

1. $\displaystyle\lim_{x\to a} f(x) = 0$ and $\displaystyle\lim_{x\to a} g(x) = 0$ $\Rightarrow$ type 0^0

2. $\displaystyle\lim_{x\to a} f(x) = 1$ and $\displaystyle\lim_{x\to a} g(x) = \pm\infty$ $\Rightarrow$ type 1^∞

3. $\displaystyle\lim_{x\to a} f(x) = \infty$ and $\displaystyle\lim_{x\to a} g(x) = 0$ $\Rightarrow$ type ∞^0

Each of these three cases can be treated by taking the natural logarithm. Let $y = [f(x)]^{g(x)}$ and then

$$\ln y = g(x) \ln f(x).$$

Studying the limit, will led to one of the indeterminate forms $g(x) \ln f(x)$, and use L'Hospital's Rule to find it. We will compute the limit of $\ln y = L$, and hence the final answer is $L = e^{\ln y}$.

Example 4.19. Find :

a) $\lim\limits_{x \to (\frac{\pi}{2})^-} (1 + \cos x)^{\tan x}$

b) $\lim\limits_{x \to 0^+} (1 + \sin 4x)^{\cot x}$

Exercise

1) The value of $\lim\limits_{x\to 0} \dfrac{\sin 3x - 3x - x^2}{1-\cos 2x}$.

   a) $-\frac{1}{2}$
   b) Does not exist
   c) $\frac{2}{3}$
   d) $-\frac{2}{3}$
   e) -2

2) $\lim\limits_{x\to\infty} (1+4x)^{3/\ln x}$.

   a) $e^{3/4}$
   b) e^3
   c) e^{12}
   d) 1
   e) Does not exist.

3) Evaluate the limit $\lim\limits_{x\to-\infty} (x+\sqrt{x^2+2x})$

   a) -1
   b) 0
   c) 1
   d) -2
   e) 2

4) $\lim\limits_{x\to 0^+} (1+\sin 3x)^{\cot 5x} =$

   a) $e^{5/3}$
   b) $e^{3/5}$
   c) 1
   d) e^{15}
   e) 0

5) $\lim\limits_{x\to 1} \left(\dfrac{x}{\ln x} - \dfrac{1}{x\,\ln x}\right) =$

   a) 1
   b) 3
   c) 0
   d) 2

e) ∞

6) $\lim\limits_{x\to 0} \dfrac{\sin(\sin^{-1} x)}{\frac{\pi}{2} - \cos^{-1} x} =$

a) $\dfrac{\pi}{2}$
b) π
c) 0
d) ∞
e) 1

7) If $\lim\limits_{x\to 0} \dfrac{\sin 4x}{kx - x} = 2,$ k is constant, then $k =$

a) $\dfrac{4}{5}$
b) $\dfrac{3}{2}$
c) 5
d) 3
e) 2

8) $\lim\limits_{x\to 1} (2-x)^{\tan \pi x/2} =$

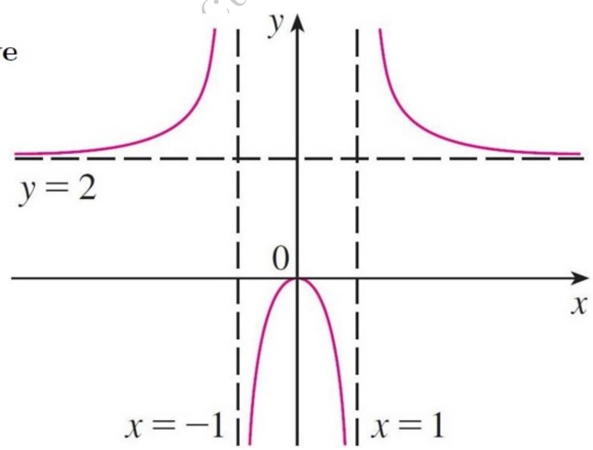
a) 2π
b) $\dfrac{\pi}{2}$
c) $e^{2\,/pi}$
d) $e^{\pi\,/2}$
e) $e^{-2\,/pi}$

4.5 Summary of Curve Sketching

Guidelines for Sketching a Curve

Properties to look for	Explanation	$f(x) = \dfrac{2x^2}{x^2-1}$
1. Domain	**All** x where $f(x)$ is **defined**	$x \neq \pm 1$
2. x- and y-intercepts	**x-intercepts:** $f(x) = 0$ **y-intercepts:** $f(0)$	$x = 0$ $y = 0$
3. Symmetries	**even:** $f(-x) = f(x)$ **odd:** $f(-x) = -f(x)$ **periodic:** $f(x+p) = f(x)$	even
4. Asymptotes	**horizontal:** $y = \lim\limits_{x\to\pm\infty} f(x)$ **vertical:** $x = a$ **if** $\lim\limits_{x\to a^\pm} f(x) = \pm\infty$	$y = 2$ $x = -1$ **and** $x = 1$
5. Increases or Decreases (I/D-Test)	**increases:** $f'(x) > 0$ **decreases:** $f'(x) < 0$	$(-\infty, -1)$ **and** $(-1, 0)$ $(0, 1)$ **and** $(1, \infty)$
6. Local Maxima and Minima (1st or 2nd Derivative Test)	**maximum:** f' from $+$ to $-$ at $x = c$ **minimum:** f' from $-$ to $+$ at $x = c$	$x = 0$ none
7. Concavity and Inflections (Concavity Test)	concave **upward:** $f''(x) > 0$ concave **downward:** $f''(x) < 0$ **inflection** point: f'' changes **sign**	$(-\infty, -1)$ **and** $(1, \infty)$ $(-1, 1)$ none

8. Sketch the Curve



Example 4.20. Sketch the graph of the following:

a) $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

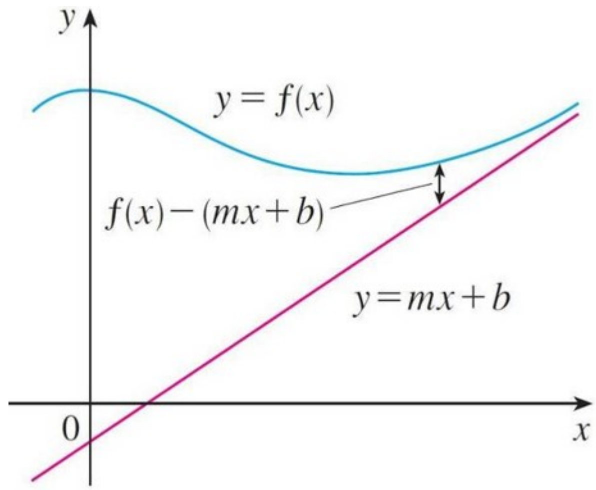
b) $f(x) = \ln(4 - x^2)$

Slant Asymptotes

Some curves have asymptotes that are oblique, that is, neither horizontal nor vertical. If

$$\lim_{x\to\infty(or-\infty)}[f(x)-(mx+b)]=0$$

where $m \neq 0$, then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0, as in the figure below.



Note 21. For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.

Example 4.21. Find the equation of the slant asymptote(s) of

a) $f(x) = \dfrac{x^3}{x^2+1}$

b) $f(x) = x - \tan^{-1} x$

Exercise

1) An equation of the slant asymptote of the graph of $y = \dfrac{5x^4 + x^2 + x}{x^3 - x^2 + 2}$ is given by

   a) $y - 5x + 2 = 0$
   b) $y - 3x + 5 = 0$
   c) $y - 3x + 2 = 0$
   d) $y - 5x - 5 = 0$
   e) $y - 5x + 3 = 0$

2) The function $f(x) = \dfrac{x^2 - 1}{x^2 - x - 2}$ has

   a) one slant asymptote, one vertical asymptote.
   b) one slant asymptote only
   c) one horizontal asymptote, one vertical asymptote
   d) one horizontal asymptote,two vertical asymptotes
   e) one horizontal asymptote, no vertical asymptotes

3) The slant asymptote of $f(x) = 2e^{-x} - 2x + 3$ is

   a) $y = 2x - 3$
   b) $y = -2x$
   c) $y = -2x + 5$
   d) $y = -2x + 3$
   e) $y = x$

4) If $f(x) = \dfrac{x^2}{x^2 + 1}$, then which one of the following statements is **False** about the graph of $f(x)$.

   a) $f(x)$ has an absolute maximum
   b) $f(x)$ has $y = 1$ as a horizontal asymptote
   c) $f(x)$ has no vertical asymptotes
   d) $f(x)$ has an absolute minimum
   e) $f(x)$ is an even function

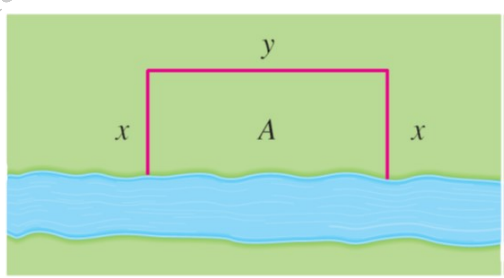
4.6 Graphing with Calculus and Calculator

4.7 Optimization Problems

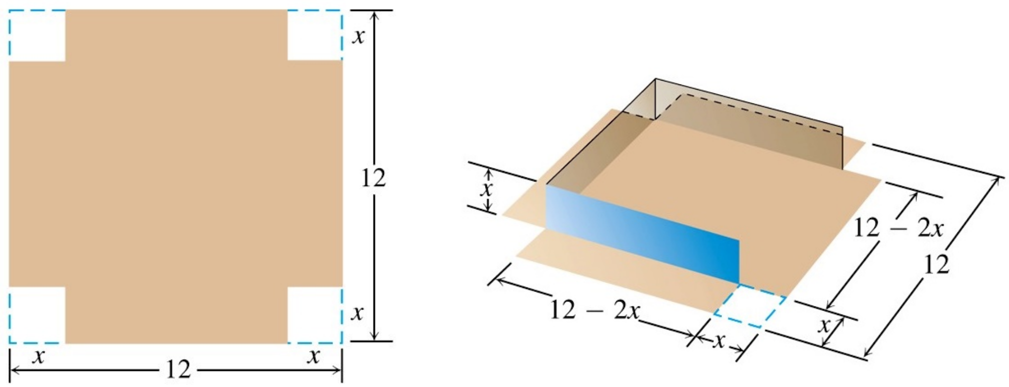
Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

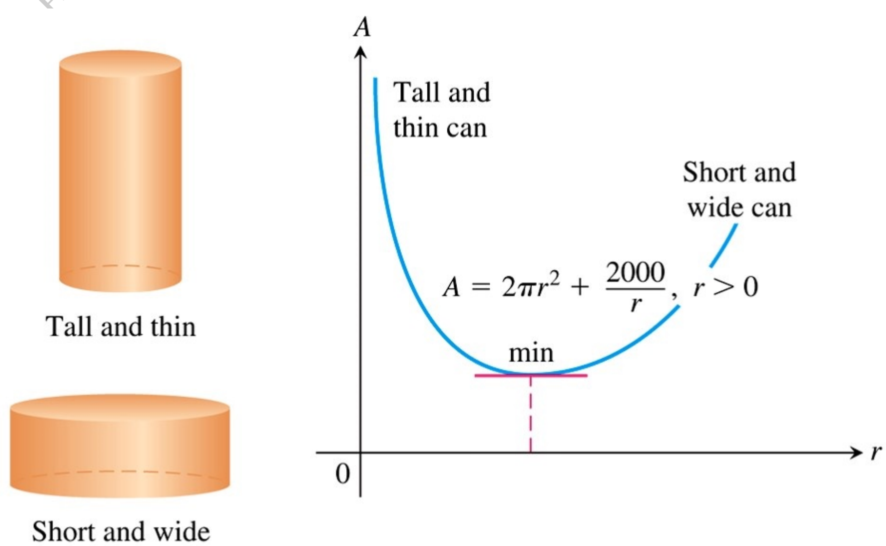
Example 4.22. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



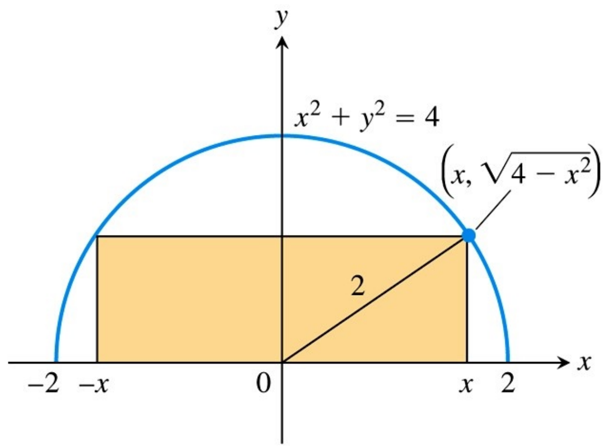
Example 4.23. (Fabricating a Box) An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



Example 4.24. (Designing an Efficient Cylindrical Can) You have been asked to design a 1-liter (1000 cm^3) can shaped like a right circular cylinder. What dimensions will use the least material?



Example 4.25. (Inscribing Rectangles) A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?



Solve Examples 2, 3, and 4 from the book

Exercise

1) A rectangular field will be bounded on one side by a river and on the other three sides by a plastic wire. If the length of the wire used is $100\,m$, then the **maximum area** of the rectangular field is

   a) 1875 m^2
   b) 1200 m^2
   c) 1250 m^2
   d) 2500 m^2
   e) 625 m^2

2) The minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$ is equal to

   a) $\frac{3}{8}$
   b) $\frac{7}{8}$
   c) $\frac{3}{4}$
   d) $\frac{5}{8}$
   e) $\frac{1}{4}$

3) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. The largest possible area of such rectangle is

   a) $18\sqrt{3}$
   b) 16
   c) $8\sqrt{3}$
   d) 30
   e) 32

4) Two nonnegative numbers, whose sum is 9 and the product of one number and the square of the other number is a maximum, are

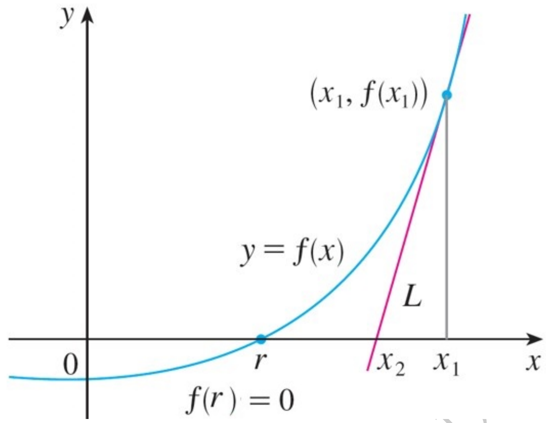
   a) 1 and 8
   b) 2 and 7
   c) 3 and 6
   d) 4 and 5
   e) 0 and 9

5) An open rectangular box with square base is to be made from 48 ft^2 of material. The largest possible volume is

a) 24 ft^3
b) 32 ft^3
c) 48 ft^3
d) 52 ft^3
e) 16 ft^3

4.8 Newton's Method

We know that, a number r is called a root (or a zero) for a function f if $f(r) = 0$. Graphically, it means that the point $(r, f(r))$ is one $x-$intercept of the graph of $y = f(x)$.

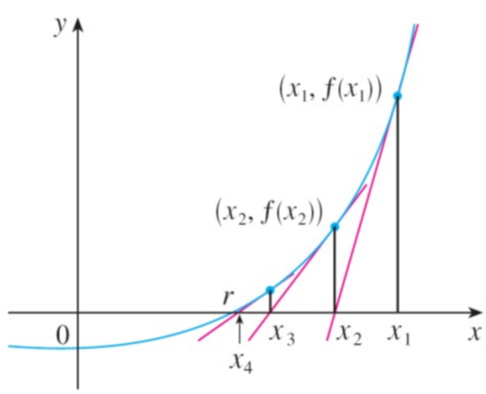


Newton's Method

Let $f(x) = 0$ and f' exists (not equal to zero). Let x_0 be an initial guess. The root(s) of the equitation can be approximated by the following formula:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

The process of finding the roots (number n) will continue until a desire accuracy and then x_n is a good approximation for the root if you chose a large number n.



Example 4.26. Starting with $x_0 = 1$, find the first approximation x_1, given by the Newton's method, of one of the zeros of the function $f(x) = 4x^3 + 7x^2 - 7x - 10$.

Example 4.27. Use Newton's method to find the second approximation x_2 of $\sqrt{2}$ if the first approximation is 1.

Example 4.28. If x_1 is an approximation of the x-coordinate of the point of the intersection of the graphs of $y = x$ and $y = -\ln x$, find x_3.

Exercise

1) Newton's method is used to estimate the x-coordinate of the point of intersection of the curves $y = x^3 + 3x$ and $y = 4x + 1$. If we start with $x_0 = 1$, then $x_1 =$

   a) 1.4
   b) 1.05
   c) 1.75
   d) 1.5
   e) 1.005

2) Newton's method is used to estimate the x-coordinate of the point of intersection of the curves $y = \sqrt{x}$ and
   $y = 1 - x^2$. If we start with $x_0 = 1$, then $x_1 =$

   a) $\frac{-1}{2}$
   b) $\frac{1}{2}$
   c) $\frac{8}{5}$
   d) $\frac{3}{5}$
   e) 0

3) Newton's Method is used to estimate the x-coordinate of the point where the curve of $y = x^3 + 2x$ crosses the horizontal line $y = 2$. If we start with $x_0 = 1$, then $x_1 =$

   a) 1.1
   b) 0.8
   c) 0.9
   d) 1.02
   e) 1.05

4) Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 = 2$. If you start with $x_0 = 1$ to find x_1 and x_2, then $x_1 - x_2 =$

   a) $-\frac{5}{16}$
   b) $\frac{113}{2000}$
   c) $-\frac{113}{2000}$
   d) $\frac{5}{16}$
   e) $\frac{2387}{2000}$

4.9 Antiderivatives

Finding a function F whose derivative is a known function f is basically the relation between the differentiation and integration.

Definition 4.6. A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

For example, if $f(x) = x^2$, then it isn't difficult to discover an antiderivative of f. If $F(x) = x^3$, then $F'(x) = x^2 = f(x)$. Also, $F(x) = x^3 + 1$ is an antiderivative of f.

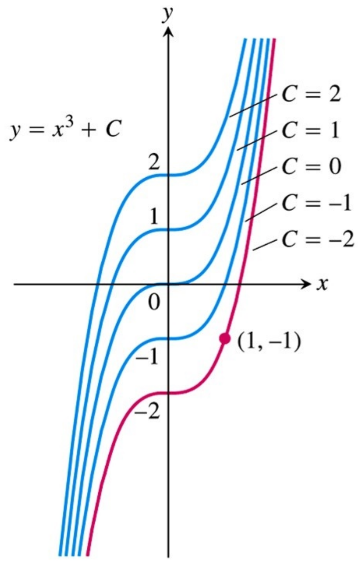
Theorem 4.7. *If F is an aniderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.*

Antiderivative formulas

	Function	General antiderivative		Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	**8.**	e^{kx}	$\frac{1}{k}e^{kx} + C$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$	**9.**	$\frac{1}{x}$	$\ln\|x\| + C, \quad x \neq 0$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$	**10.**	$\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$	**11.**	$\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	**12.**	$\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, \quad kx > 1$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	**13.**	a^{kx}	$\left(\frac{1}{k\ln a}\right)a^{kx} + C, \quad a > 0, \ a \neq 1$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$			

k a nonzero constant

We see that the general antiderivative of f is $x^3/3 + C$.



Example 4.29. Find the most general antiderivative of each of the following functions.

a) $f(x) = \sin 5x$

b) $f(x) = 1/3x$

c) $f(x) = \frac{1}{1+9x^2}$

Example 4.30. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

You see this definition in Math 102:

The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\boxed{\int f(x)\,dx.}$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Exercise

1) The most general antiderivative of $f(t) = \dfrac{te^{2t} + \sqrt[3]{t}}{t}$ is

   a) $\dfrac{1}{2}e^{2t} - \dfrac{3}{2}t^{-5/3} + C$

   b) $2e^{2t} - \dfrac{3}{2}t^{-5/3} + C$

   c) $\dfrac{1}{2}e^{2t} + 3t^{1/3} + C$

   d) $e^{2t} + 3t^{2/3} + C$

   e) $te^{2t} + t^{-2/3} + C$

2) The most general antiderivative of $f(x) = \dfrac{(x+2)^2}{x^{3/2}}$ is $F(x) =$

   a) $\dfrac{2}{5}x^{5/2} + 4x^{3/2} - 4\sqrt{x} + c$

   b) $\dfrac{2}{3}x^{3/2} + 8x^{1/2} - 8x^{-1/2} + c$

   c) $\dfrac{2}{3}\sqrt{x} + x^{3/2} - \dfrac{2}{x} + c$

   d) $2x^{1/2} + 4x^{3/2} - 4x + c$

   e) $\dfrac{(x+2)^3}{3}\sqrt{x} + c$