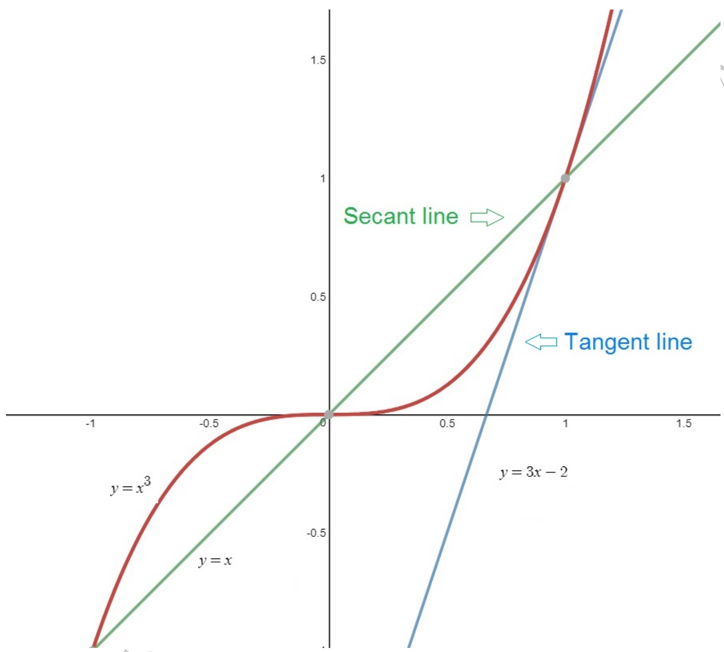
2 Limits

2.1 The Tangent Problems

The word tangent is derived from the Latin word tangens, which means "touching."

A **tangent line** to a curve is a line that touches the curve and a **secant line** is a line that cuts (intersects) a curve more than once.



How to find the slope of a line?

The slope of a line $y = mx + b$ is m which is basically equals to $\dfrac{\text{change on } y}{\text{change on } x}$.
For example, in the previous figure, the slope of the secant line (green) is 1 which can be evaluated by $\dfrac{\text{change on } y}{\text{change on } x} = \dfrac{1-0}{1-0} = 1$.

The Velocity Problem

Read page 84 from the book about the velocity problem. You need to know the difference between the average velocity and the instantaneous velocity.

2.2 The Limit of a Function

To find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

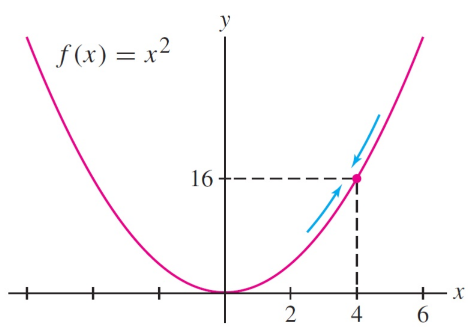
Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.)
Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

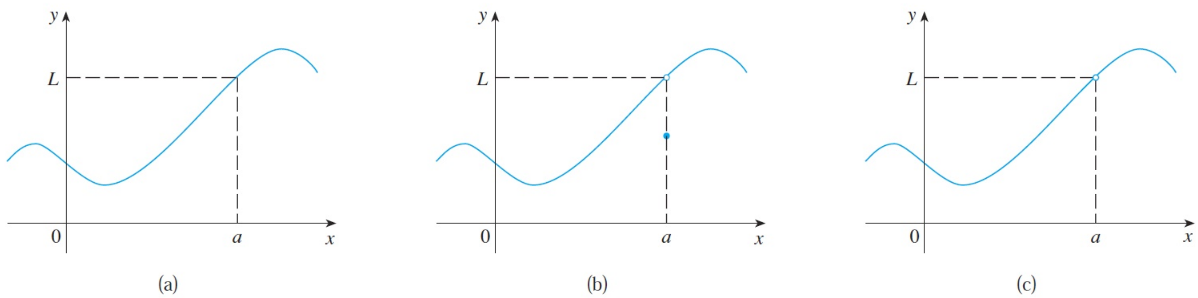
if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Example 2.1. Find $\lim_{x \to 4} x^2$? see the figure below.



$x \to 4^-$ (left)	x^2	$x \to 4^+$ (right)	x^2
3.9	15.21	4.1	16.81
3.99	**15.9**201	4.01	**16.0**801
3.999	**15.99**2001	4.001	**16.00**8001
3.9999	**15.999**20001	4.0001	**16.000**80001

Example 2.2. What are the differences between the cases (a), (b), and (c) in the figure?



(a) the limit and the value of the function equal to L

(b) the limit equals to L but the value of the function is not equal to L

(c) the limit equals to L but the function is not defined at a.

Example 2.3. Find

a) $\lim_{x\to 2}(3x-2)$

b) $\lim_{x\to 3}\dfrac{x^2-9}{x-3}$

c) $\lim_{x\to 1}\dfrac{x^2-1}{(x-1)^2}$

One-Sided Limits

Definition 2.1. We write:

(i) $\lim_{x\to a^-} f(x) = L$, and say the left-hand limit of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the left) is equal to L if we can make the values of arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

(ii) $\lim_{x\to a^+} f(x) = L$, and say the right-hand limit of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the right) is equal to L if we can make the values of arbitrarily close to L by taking x to be sufficiently close to a and x greater than a.

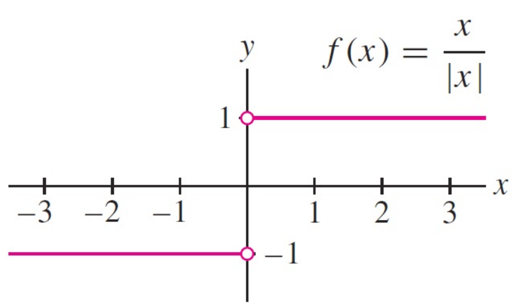
Note 1. If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ does not exist or simply DNE.

Example 2.4. Let $f(x) = \dfrac{x}{|x|}$. Find

a) $\lim_{x\to 0^+} f(x)$

b) $\lim_{x\to 0^-} f(x)$

c) $\lim_{x\to 0} f(x)$



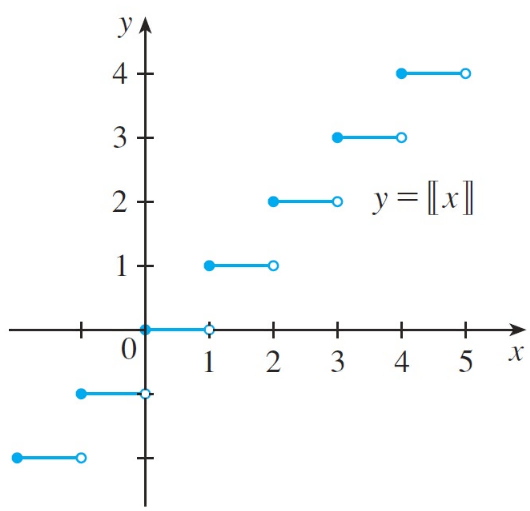
Example 2.5. Let $[\![x]\!]$ denotes the greatest integer less than or equal to x. Find

a) $\lim_{x \to 1^-} [\![x]\!]$

b) $\lim_{x \to 1^+} [\![x]\!]$

c) $\lim_{x \to 1} [\![x]\!]$

d) $\lim_{x \to 3} [\![\frac{x}{3}]\!]$



Note 2. We reach to this result:

$$\boxed{\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L}$$

Infinite Limits (Vertical Asymptote)

Definition 2.2.

Let f be a function defined on both sides of a, except possibly at a itself. Then

I. $\lim_{x \to a} f(x) = \infty$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

II. $\lim_{x \to a} f(x) = -\infty$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Definition 2.3.

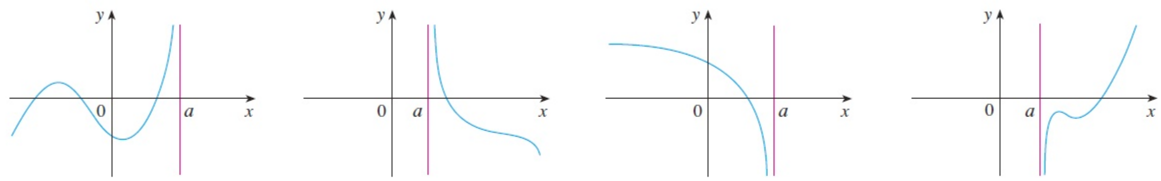
The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a} f(x) = -\infty$$

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Note 3. One of the above statements is enough to say that we have a vertical asymptote.



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

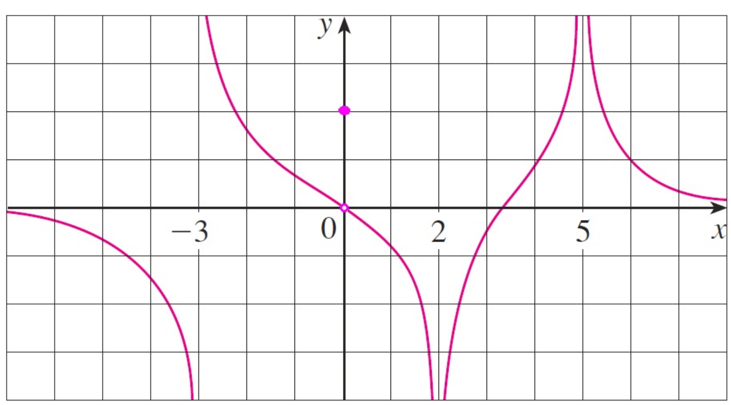
Example 2.6. The graph of the curve $y = f(x)$ is given below. Find:

a) $\lim\limits_{x \to 0} f(x)$

b) $\lim\limits_{x \to -3^-} f(x)$

c) $\lim\limits_{x \to 5^+} f(x)$

d) $\lim\limits_{x \to \infty} f(x)$



Example 2.7. a) $\lim\limits_{x \to 2^+} \dfrac{5x+6}{2-x}$

b) $\lim_{x \to \frac{\pi}{2}} \tan x$

c) $\lim_{x \to 0^+} \ln x$

d) $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} x^2 + x & \text{for} \quad x < 1 \\ 3e^{x-1} - x & \text{for} \quad x \geq 1 \end{cases}$ (piecewise function)

*************************** Go over the examples in the book ***************************

2.3 Calculating Limits Using the Limit Laws

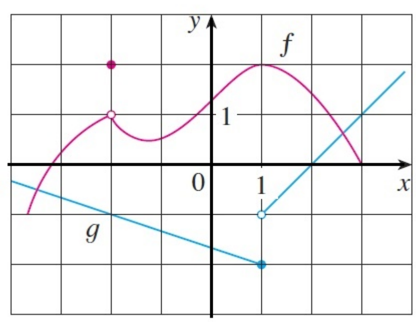
Limit Laws I

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ Sum Law
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ Difference Law
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$ Constant Multiple Law
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ Product Law
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$ Quotient Law

Example 2.8. The graphs of f and g are given below:



Evaluate:

a) $\lim_{x\to -2} f(x)\, g(x)$

b) $\lim_{x\to 2} \dfrac{g(x)}{f(x)}$

c) $\lim_{x \to 1^+} (2f(x) + 3g(x))$

Limit Laws II

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer — Power Law

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
(If n is even, we assume that $a > 0$.)

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer — Root Law
[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

Direct Substitution Property

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Theorem 2.1. *We introduced it before.*

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Theorem 2.2.

If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

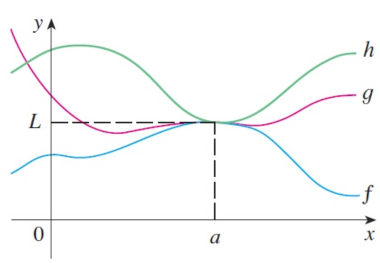
Theorem 2.3. *The Squeeze Theorem (or the Sandwich Theorem)*

If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$



Go over the Examples 1-11 in the book.

Example 2.9. Evaluate:

a) $\lim_{x \to -1} (x^2 + 1)^3 (x + 3)^5$

b) $\lim_{t \to -2} \sqrt{t^4 + 3t + 6}$

c) $\lim_{y \to -4} \left(\frac{y^2 + 5y + 4}{y^2 + 3y - 4} \right)$

d) $\lim_{x \to -4} \left(\frac{\sqrt{x^2 + 9} - 5}{x + 4} \right)$

e) $\lim_{x \to 0} \left(\sqrt{x^5 + x^4} \ \sin\left(\frac{\pi}{x}\right) \right)$ using the Squeeze Theorem

f) $\lim_{x \to 1} \left(\frac{x^2 - 1}{|x - 1|} \right)$

*************************** Go over the examples in the book ***************************

2.4 The Precise Definition of a Limit

Definition 2.4.

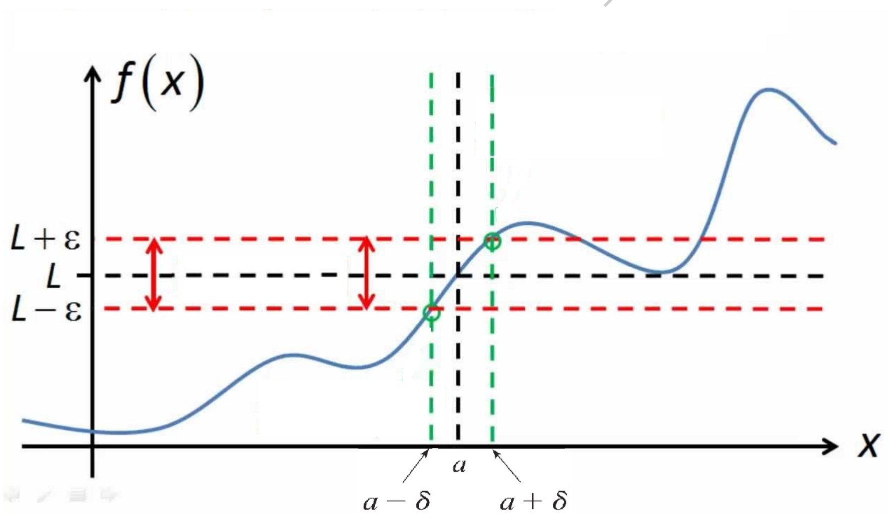
Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

- ϵ: Epsilon
- δ: Delta.



There are three possible type of questions you may be asked:

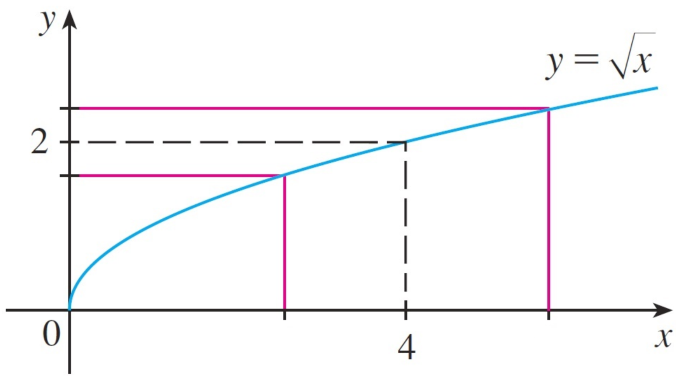
i) Given f, a, L, ϵ, use the graph to find δ.

ii) Given f, a, L, use the $\epsilon - \delta$ definition to find a relation between them.

iii) Given f, a, L, use the $\epsilon - \delta$ definition to show the function has a limit at a.

Note 4. If the function $f(x)$ is not linear, we get two deltas δ_1 and δ_2. We choose the minimum of δ_1 and δ_2 to be δ. i.e. $\delta = \min(\delta_1, \delta_2)$.

Example 2.10. Given that $\lim_{x \to 4} \sqrt{x} = 2$. Use the graph of $f(x) = \sqrt{x}$ to find a number δ if $\epsilon = 0.5$



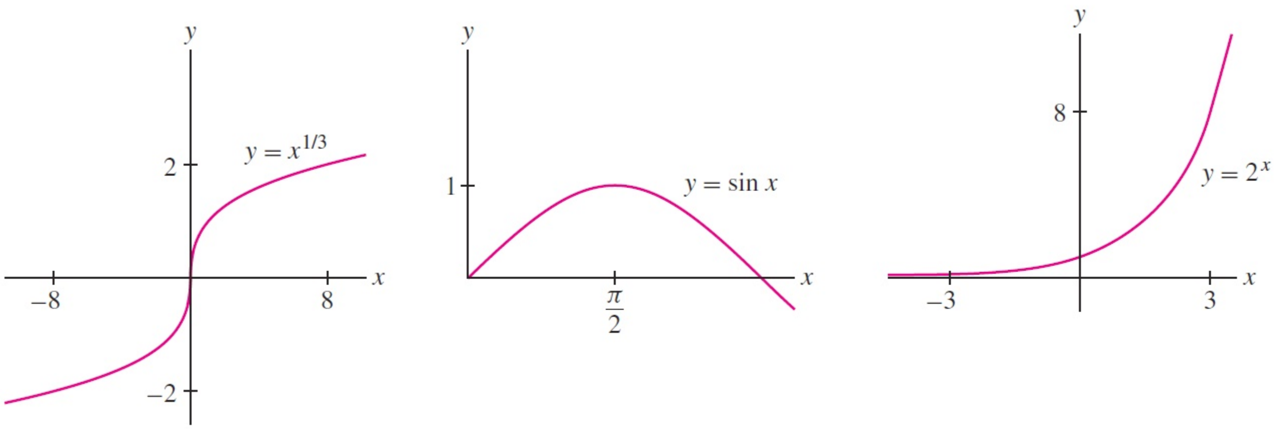
Example 2.11. Find a number δ such that if $|x - 3| < \delta$ then $|4x - 12| < 0.1$.

Example 2.12. Use the $\epsilon - \delta$ definition to prove that $\lim_{x \to 2}(4 - \frac{x}{2}) = 3$.

*************************** Go over the examples in the book ***************************

2.5 Continuity

Examples of continuous functions



Do you have an example of a function which is not continuous at a point? list some!

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Definition 2.5. A function f is continuous at a number a if $\lim_{x\to a} f(x) = f(a)$.

Note 5. f is continuous at a number a implies:

i) $f(a)$ is defined

ii) $\lim_{x\to a} f(x)$ exists (exists mean left-limit = right-limit); and

iii) $\lim_{x\to a} f(x) = f(a)$.

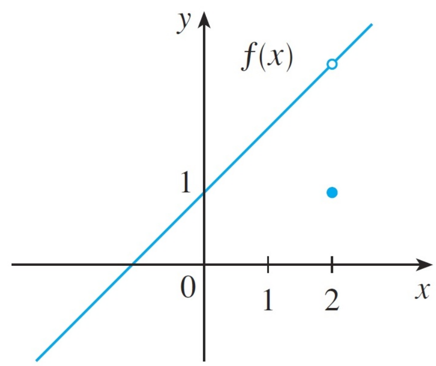
Note 6. If f is NOT continuous at a, then we say that f is discontinuous at a (or has a discontinuity at a).

Note 7. If f is continuous at every number in its domain D, then we say that f is continuous on D.

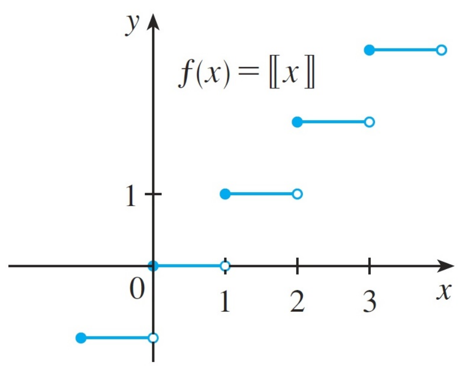
Type of discontinues

i) Removable Discontinues: A function has a removable discontinue at a if $\lim_{x\to a} f(x) = L \neq f(a)$. For example,

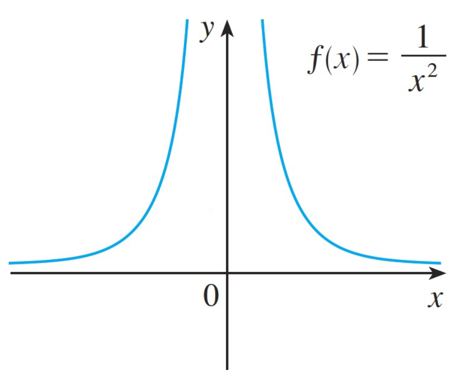
$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$



ii) Jump Discontinues: A function has a jump discontinue at a if $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$. For example,



ii) Infinite Discontinues: A function has an infinite discontinue at a if it has a vertical asymptote. For example,



Definition 2.6. A function is continuous from the right at a number a if $\lim_{x \to a^+} f(x) = f(a)$ and continuous from the left at a number a if $\lim_{x \to a^-} f(x) = f(a)$.

Definition 2.7. A function f is continuous on a closed interval if it is continuous at every number in the interval.

Note 8. Continuous on a closed interval $[a, b]$ means:

i) continuous on the open interval (a, b)

ii) continuous from the right at a

iii) continuous from the left at b

Example 2.13. Discuss the continuity of the function $f(x) = \sqrt{9 - x^2}$.

Theorem 2.4. *If f and g are continuous at a and c is a constant, then the following functions are also continuous at a: $f + g$, $f - g$, $c\,f$, $f\,g$, f/g (where $g(a) \neq 0$).*

Theorem 2.5. .

i) Any polynomial is continuous everywhere

ii) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Example 2.14. Where is the function $f(x) = \dfrac{\ln(x+1) + \sin^{-1}(x+1)}{4x^2 - 1}$ continuous? (Or, similarly, find the domain).

Theorem 2.6.

If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$.

In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

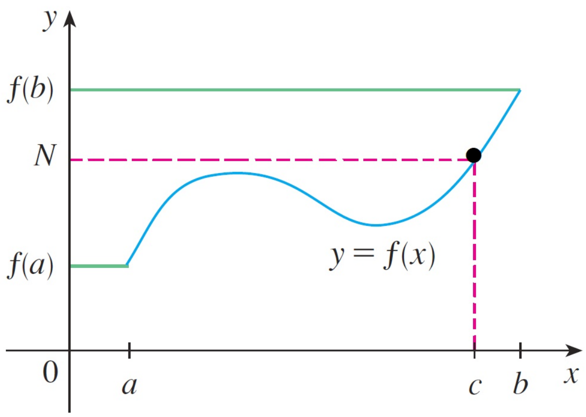
Example 2.15. Evaluate $\lim\limits_{x\to 2} \cos^{-1}\left(\dfrac{x^2-5x+6}{x^2-2x}\right)$.

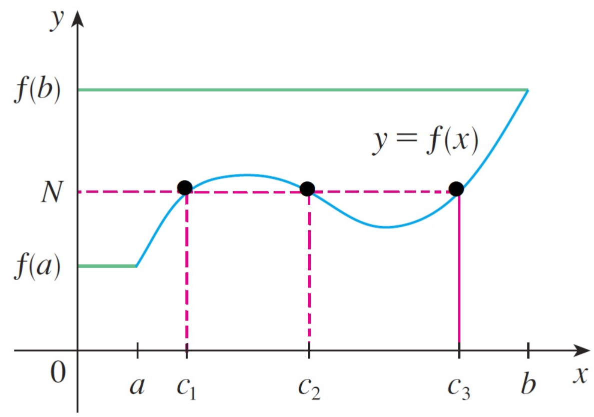
Theorem 2.7.

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example 2.16. Where is the function $f(x) = \ln(1+\cos x)$ continuous?

Theorem 2.8. *(Intermediate Value Theorem) Suppose that f is* ***continuous*** *on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ (at least one) such that $f(c) = N$.*





Example 2.17. (Application of Intermediate Value Theorem) Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2. (see the book for the complete solution, page 126)

Example 2.18. Use the Intermediate Value Theorem to Show that there is a root of the equation $\cos x = x$ between 0 and 1.

Example 2.19. For what value of the constant c is the function $f(x) = \begin{cases} cx^2 + 2x & \text{for} \quad x < 2 \\ x^3 - cx & \text{for} \quad x \geq 2 \end{cases}$ continuous on $(-\infty, \infty)$?

*************************** Go over the examples in the book ***************************

2.6 Limits at Infinity; Horizontal Asymptotes

Definition 2.8.

i) Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

ii) Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Definition 2.9. The line $y = L$ is called a **horizontal asymptote** if either:

i) $\lim\limits_{x \to \infty} f(x) = L$ or

ii) $\lim\limits_{x \to -\infty} f(x) = L$

Example 2.20. Find the equation(s) of the horizontal asymptote(s) of:

i) $y = \tan^{-1} x$

ii) $y = \dfrac{3x^4 + 2}{5x^4 + 1}$

iii) $y = \dfrac{3x^2 + 2}{5x^4 + 1}$

iv) $y = \dfrac{3x^4 + 2}{5x^2 + 1}$

Note 9. For a rational function $f(x)$ of the form $\dfrac{q(x)}{h(x)}$ where both $q(x)$ and $h(x)$ are polynomials:

i) If the degree of $q(x)$ equals to the degree of $h(x)$, $\lim\limits_{x\to\infty} f(x) = \dfrac{\text{leading coefficient of } q(x)}{\text{leading coefficient of } h(x)} = \dfrac{A}{B}$. Then $y = \frac{A}{B}$ is a horizontal asymptote.

ii) If the degree of $q(x)$ is less than the degree of $h(x)$, $\lim\limits_{x\to\infty} f(x) = 0$. Then $y = 0$ is a horizontal asymptote.

iii) If the degree of $q(x)$ is greater than the degree of $h(x)$, $\lim\limits_{x\to\infty} f(x) = \pm\infty$. Then there is no horizontal asymptote.

We can repeat the same for the case as $x \to -\infty$.

Example 2.21. Find the horizontal and vertical asymptotes of the graph of the function:

i) $y = \dfrac{\sqrt{25x^2 + 3}}{2x - 1}$

ii) $y = \sqrt{\dfrac{2x^3 - 3x + 5}{3x + 18x^3}}$

iii) $y = \cot^{-1}\left(\dfrac{1}{x-2}\right)$

Example 2.22. Evaluate:

i) $\lim\limits_{x\to 0^-} e^{\frac{1}{x}}$

ii) $\lim\limits_{x\to\infty} \cos x$

Infinite Limits at Infinity

The notation

$$\lim_{x\to\infty} f(x) = \infty\,(-\infty)$$

indicates that the values of $f(x)$ become large (large negative) as x becomes large and same as $x \to -\infty$.

Example 2.23. Find the limit as x approaches $\pm\infty$?

i) $y = x^3$ (sketch)

ii) $y = e^x$ (sketch)

iii) $y = \sqrt{x^2+1} - x$

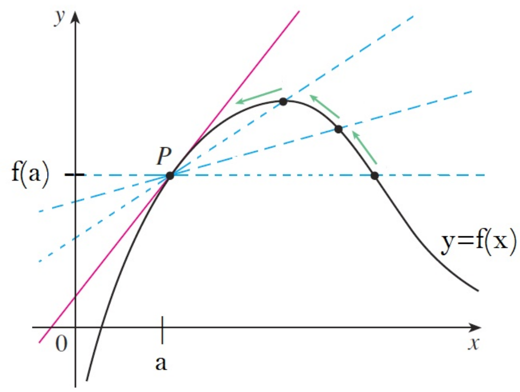
*************************** Go over the examples in the book ***************************

2.7 Derivatives and Rates of Change

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P

with slope: $$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.



Note 10. i) We sometimes refer to the slope of the tangent line to a curve at a point as the slope of the curve at the point.

ii) If we let $h = x - a$ in the above definition, we get another representation for the slope of the tangent line:
$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 2.24. Find an equation of the tangent line to the graph of $y = \dfrac{3}{5-x}$ at the point $(2, 1)$.

Velocities

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where is s the displacement (directed distance) of the object from the origin at time t. The function $f(t)$ that describes the motion is called the position function of the object. In the time interval from

$t = a$ to $t = a + h$,

$$\text{average velocity} = v_{\text{ave.}} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

and

$$\text{instantaneous velocity at time a} = v_{\text{ins.}} = v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 2.25. Let $s = f(t) = \dfrac{1}{5-t}$ be the position function in meter of an object. What is the instantaneous velocity when time equals 2 sec?

See EXAMPLE 3 on page 145.

Derivatives

The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Equivalent way of stating the definition of the derivative is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example 2.26. If $f(x) = \dfrac{1}{5-x}$, find $f'(2)$?

Rates of Change

Let $y = f(x)$. If x changes from x_1 to x_2, then the change (or increment) in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1).$$

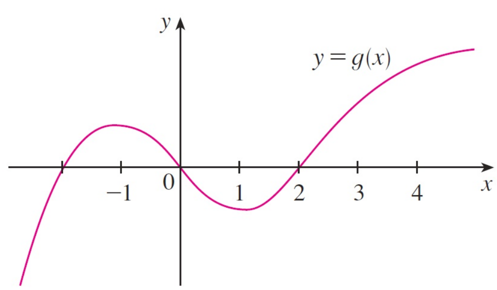
So, the average rate of change of y with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and the instantaneous rate of change is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Example 2.27. For the function t whose graph is given, arrange the following numbers in increasing order and explain your reasoning:



0

$g'(-2)$

$g'(0)$

$g'(2)$

$g'(4)$

*************************** Go over the examples in the book ***************************

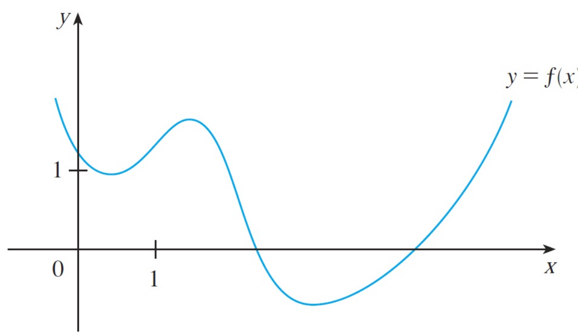
2.8 The Derivative as a Function

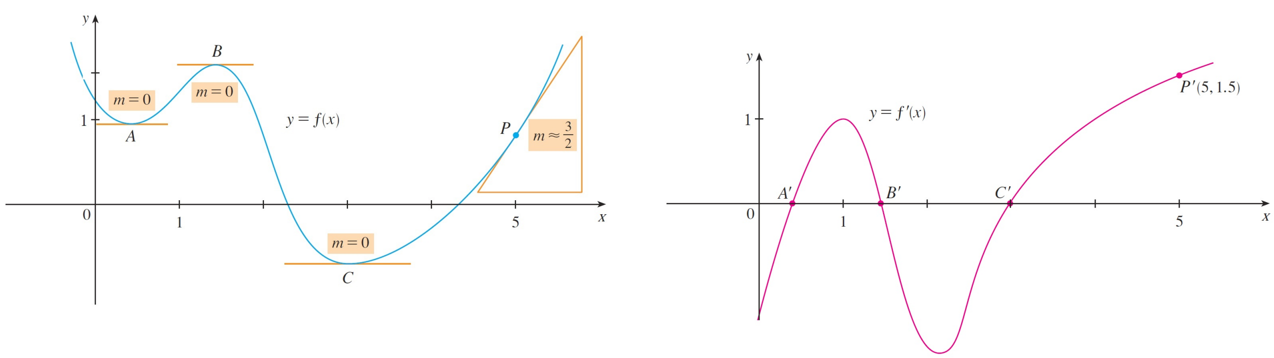
If we replace a, in the definition of the derivative at a point, by any number x we obtained

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists.

Example 2.28. Using the graph of f, sketch a possible graph for f'.





Example 2.29. Compare the graphs of f and f' for:

a) $f(x) = x^3 - x^2$

b) $f(x) = |x|$

c) $f(x) = \sqrt{x}$

Derivative Notations

Let $y = f(x)$, so x is the independent variable and y is the dependent variable. Some common notations for the derivative are:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and $\frac{d}{dx}$ are called differentiation operators and $\frac{dy}{dx}$ Leibniz' notation. To find a value for the derivative at some point $f'(a)$ we use this notation: $\left.\frac{dy}{dx}\right|_{x=a}$

Definition 2.10. A function f is **differentiable** at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 2.30. Where is the function $f(x) = |x|$ differentiable?

Theorem 2.9. *If f is differentiable at a, then it f is continuous at a.*

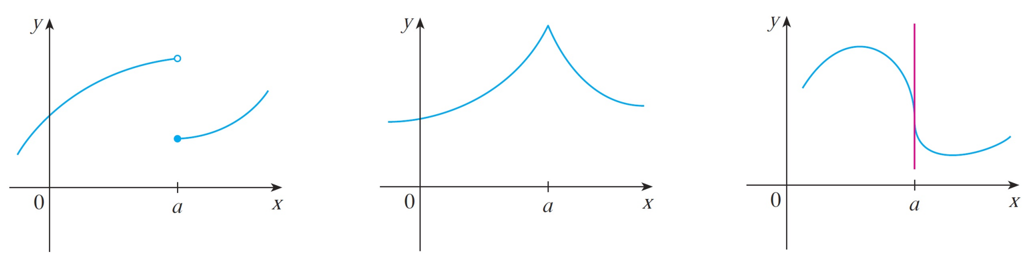
Question: How about the converse? Is the continuity implies the differentiable. Give a counter example?

Cases of Non-differentiability of a function f at a:

i) f is **discontinuous** at a

ii) f has a **corner** at a,i.e. f is continuous at a but $\lim_{x\to a^-} f'(x) \neq \lim_{x\to a^+} f'(x)$. For example $f(x) = |x|$.

iii) f has a **vertical tangent** at a. f is continuous at a but $\lim_{x\to a} f'(x) = \infty$ or $-\infty$. For example, $f(x) = \sqrt[3]{x}$ (try to sketch)



A discontinuity A corner A vertical tangent

Higher Derivatives

Let $y = f(x)$ and assume it is differentiable. So, the 1st derivative has this notation

$$f' = \frac{dy}{dx}.$$

The 2nd derivative has this notation

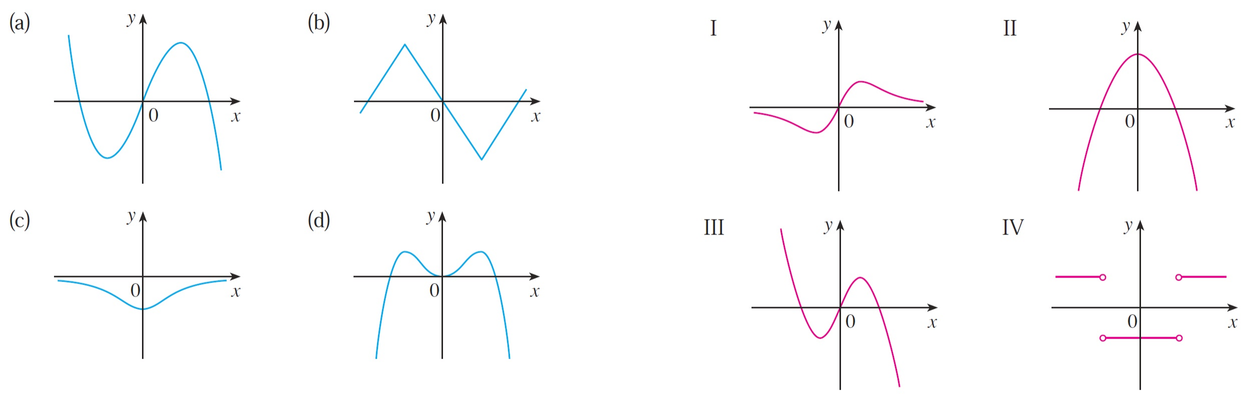
$$(f')' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''.$$

The process can be continued:

$$\frac{d^3y}{dx^3} = f''', \ \frac{d^4y}{dx^4} = f^{(4)}, \ \ldots, \frac{d^ny}{dx^n} = f^{(n)}.$$

Example 2.31. Double check that if $f(x) = x^3 - x^2$, then $f'(x) = 3x^2 - 2x$, $f''(x) = 6x - 2$, $f'''(x) = 6$ and $f^{(4)} = 0$.

Example 2.32. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.



************************** Go over the examples in the book **************************

Old Exam - Exam 1 - Term 162

1) $\lim_{x\to\infty} \dfrac{\sqrt{9x^2-9}}{2x-6} =$

a) $\dfrac{1}{3}$
b) 1
c) -2
d) $\dfrac{3}{2}$
e) $\dfrac{2}{9}$

2) The function $f(x) = \dfrac{\ln(2+\cos e^x)}{x^2-4}$ is continuous for all x in the interval

a) $[-2, 2]$
b) $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
c) $(-\infty, 0)$
d) $(-\infty, 0) \cup (0, \infty)$
e) $[0, \infty)$

3) The equations of the vertical asymptote(s) of $h(x) = \dfrac{x^2-1}{1-x-2x^2}$ is (are)

a) $y = -\dfrac{1}{2}$
b) $x = -1\,,\ x = \dfrac{1}{2}$
c) $x = -\dfrac{1}{4}$
d) $x = -\dfrac{1}{2},\ x = \dfrac{1}{2}$
e) $x = \dfrac{1}{2}$

4) Let $f(x) = \dfrac{x^3+3x^2-9x-27}{x^3-9x}$. If R is the number of **removable** discontinuities of f and I is the number of **infinite** discontinuities of f, then

a) $R = 3$ and $I = 3$
b) $R = 0$ and $I = 3$
c) $R = 3$ and $I = 0$
d) $R = 1$ and $I = 2$
e) $R = 2$ and $I = 1$

5) If $\lim_{x\to 2} f(x) = 7$ and $\lim_{x\to 2} g(x) = 3$, then $\lim_{x\to 2} \dfrac{\sqrt{x+f(x)}}{|x-2|-(g(x))^2} =$

a) 0

b) $-\dfrac{2}{3}$

c) -1

d) 1

e) $-\dfrac{1}{3}$

6) If $f(x) = \begin{cases} ax+2b & \text{for} \quad x \le 0 \\ x^2+3a-b & \text{for} \quad 0 < x \le 2 \\ 3x-5 & \text{for} \quad x > 2 \end{cases}$ is a continuous function everywhere, then $f(1) =$

a) 3

b) -2

c) 0

d) 1

e) -5

7) If $\lim_{x\to 1} \dfrac{f(x)-4}{x-1} = 8$, then $\lim_{x\to 1} \dfrac{f(x)}{x+1}$

a) equals 6

b) equals 2

c) Does not exist

d) equals 8

e) equals 4

8) If $f(x) = \begin{cases} x^3 \sin\dfrac{1}{x} & \text{for} \quad x \ne 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$, then

a) $f'(0) = 1$

b) $f'(0) = -1$

c) $f'(0) = 0$

d) $f(x)$ is not differentiable at $x = 0$

e) $f'(0) = \dfrac{1}{2}$

9) $\lim_{x \to \infty} \left[\tan^{-1} \left(\frac{1}{e^{-x} - 1} \right) \right] =$

a) $\frac{\pi}{4}$
b) $-\frac{\pi}{4}$
c) 0
d) $-\frac{\pi}{2}$
e) $\frac{\pi}{2}$

10) The largest number $\delta > 0$, such that if $0 < |x - 2| < \delta$, then $|\sqrt{19 - x} - 3| < 1$, is

(You may use the graph of $y = \sqrt{19 - x}$)

a) $\delta = 1$
b) $\delta = 5$
c) $\delta = 3$
d) $\delta = 9$
e) $\delta = 7$

11) The sum of all values of k, for which $y = k$ is a horizontal asymptote to the graph of the function

$$f(x) = \begin{cases} \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}} & \text{for} \quad x > 4 \\ 1 & \text{for} \quad \sqrt[3]{\dfrac{3}{8}} \leq x \leq 4 \\ \left(\dfrac{x^3 + x - 3}{8x^3 - 3} \right)^{1/3} & \text{for} \quad x < \sqrt[3]{\dfrac{3}{8}} \end{cases}$$, equals to

a) 1
b) 0
c) $-\frac{1}{2}$
d) $\frac{3}{8}$
e) $-\frac{1}{3}$

12) Suppose $f(x)$ is a differentiable function that satisfies the following $f(x+y) = f(x) + f(y) + 2xy - 1$ for any real numbers x and y and $\lim_{x \to 0} \frac{f(x) - 1}{x} = -2$. Then $f'(x) =$

a) $2x$
b) $-2 + 2x$

c) $-2x$
d) $-2-x$
e) -2

13) To prove that $\lim_{x\to 2}(2x-1)=3$ by using the $\varepsilon-\delta$ definition of the limit, we find that for given $\varepsilon=0.002$, the largest possible value for δ that can be used is

a) 0.003
b) 0.002
c) 0.001
d) 0.05
e) 0.02

14) Which one of the following statements is **TRUE**?

a) $x^2=\cos x$ has no roots in $(-\pi,\pi)$
b) If $|f|$ is continuous at $x=a$, then f is continuous at $x=a$
c) If $f(x)=|x-6|$, then f is not differentiable at $x=0$
d) If f is continuous at $x=a$, then f is differentiable at $x=a$
e) $e^x=3-2x$ has one root in $(0,1)$

15) $\lim_{x\to 0^+}\sqrt{x}\,e^{\sin\left(\frac{\pi}{x}\right)}=$

a) -1
b) $\frac{1}{e}$
c) 0
d) $\sqrt{e}$
e) 1

16) If $f(x)=\begin{cases}\dfrac{1-\cos x}{\sin x} & \text{for } x<0\\[2mm] \dfrac{(1+x)^2-1}{x} & \text{for } x>0\end{cases}$, then

a) $\lim_{x\to 0}f(x)=0$
b) $\lim_{x\to 0}f(x)=\frac{1}{2}$
c) $\lim_{x\to 0}f(x)=\infty$
d) $\lim_{x\to 0}f(x)=1$
e) $\lim_{x\to 0}f(x)$ does not exist

17) Let a and b be real numbers. $\lim_{x\to-\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right) =$

a) $\frac{1}{\sqrt{2}}(a-b)$
b) 0
c) $\sqrt{a}-\sqrt{b}$
d) $\frac{1}{2}(b-a)$
e) $-\infty$

18) The equation of the tangent line to the curve $y = \frac{2}{1-3x}$ at the point with x-coordinate $x=0$ is

a) $y = 3x-3$
b) $y = 6x+2$
c) $y = 2x+2$
d) $y = \frac{-1}{6}x+2$
e) $y = \frac{1}{6}x+2$

19) $\lim_{t\to4}\frac{\frac{1}{2}-\frac{1}{\sqrt{t}}}{t-4} =$

a) $\frac{1}{16}$
b) $\frac{1}{8}$
c) $\frac{1}{2}$
d) $\frac{3}{14}$
e) $\frac{1}{4}$

20) $\lim_{x\to\frac{1}{3}^+}[\![3x]\!] =$

(where $[\![x]\!]$ is the greatest integer less than or equal to x)

a) 0
b) 3
c) 1
d) -1
e) 2