

Dr. Y. AL-Dweik

King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101 - Calculus I
Exam I
Term (121)

Tuesday, October 2, 2012

Time Allowed: 2 hours

Name: Solutions

ID Number: KEY

Section Number: _____ Serial Number: _____

Class Time: _____ Instructor's Name: _____

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 10 Problems)



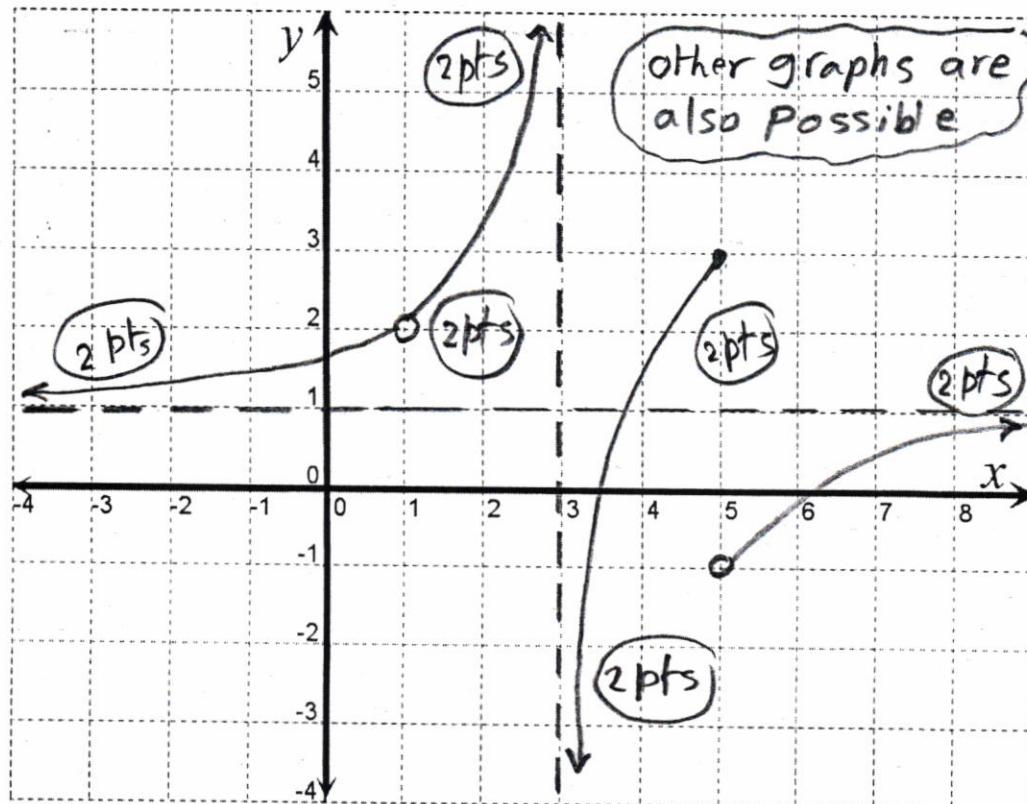
| | Points | Maximum Points |
|--------|--------|----------------|
| Page 1 | | 20 |
| Page 2 | | 13 |
| Page 3 | | 12 |
| Page 4 | | 17 |
| Page 5 | | 22 |
| Page 6 | | 10 |
| Page 7 | | 6 |
| Total | | 100 |

1. (2 points) Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[-1, 1]$.

$$\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1 \quad (2 \text{ pts})$$

2. (12 points) Sketch the graph of a function f that satisfies the following conditions:

- (i) $\lim_{x \rightarrow \pm\infty} f(x) = 1$, (ii) $\lim_{x \rightarrow 3^-} f(x) = \infty$, (iii) $\lim_{x \rightarrow 3^+} f(x) = -\infty$,
 (iv) f has a removable discontinuity at 1, (v) f has a jump discontinuity at 5.



3. (6 points) Using the Sandwich Theorem, show that if $\lim_{x \rightarrow 1} |f(x)| = 0$, then $\lim_{x \rightarrow 1} f(x) = 0$.

since $-|f(x)| \leq f(x) \leq |f(x)| \quad (2 \text{ pts})$

and $\lim_{x \rightarrow 1} |f(x)| = 0 \Rightarrow \lim_{x \rightarrow 1} -|f(x)| = 0, \quad (2 \text{ pts})$

it follows from the Sandwich Theorem

that $\lim_{x \rightarrow 1} f(x) = 0 \quad (2 \text{ pts})$

4. Evaluate the limit or show that it does not exist.

i) (4 points) $\lim_{x \rightarrow 2^-} \frac{|x-2|(x-4)}{(x-2)}$

$$= \lim_{x \rightarrow 2^-} \frac{-(x-2)(x-4)}{(x-2)}$$

(2 pts)

$$= \lim_{x \rightarrow 2^-} - (x-4) = -(-2) = 2$$

(2 pts)

ii) (6 points) $\lim_{\theta \rightarrow 0} \tan 3\theta \cdot \csc \theta$

$$= \lim_{\theta \rightarrow 0} \left(\frac{\tan 3\theta}{3\theta} \cdot 3 \cdot \frac{\theta}{\sin \theta} \right)$$

(3 pts)

$$= 3 \lim_{\theta \rightarrow 0} \frac{\tan 3\theta}{3\theta} \cdot \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta}$$

{ (3 pts)

$$= 3 \cdot 1 \cdot 1 = 3$$

iii) (3 points) $\lim_{x \rightarrow 1} \left[\frac{1}{x} \right]$, where $[y]$ is the greatest integer less than or equal to y .

Since $\lim_{x \rightarrow 1^-} \left[\frac{1}{x} \right] = 1$ (1 pt)

and $\lim_{x \rightarrow 1^+} \left[\frac{1}{x} \right] = 0$, (1 pt)

it follows that $\lim_{x \rightarrow 1} \left[\frac{1}{x} \right]$ DNE (1 pt)

iv) (6 points) $\lim_{x \rightarrow 3} \frac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

$$= \lim_{x \rightarrow 3} \frac{(x^2 - 9)(\sqrt{x^2 + 7} + 4)}{(\sqrt{x^2 + 7} - 4)(\sqrt{x^2 + 7} + 4)}$$

(2 pts)

$$= \lim_{x \rightarrow 3} \frac{(x^2 - 9)(\sqrt{x^2 + 7} + 4)}{(x^2 + 7) - 16} = \lim_{x \rightarrow 3} \frac{(x^2 - 9)(\sqrt{x^2 + 7} + 4)}{x^2 - 9}$$

(2 pts)

$$= \lim_{x \rightarrow 3} (\sqrt{x^2 + 7} + 4) = \sqrt{16} + 4 = 8$$

(2 pts)

v) (6 points) $\lim_{x \rightarrow 0} \frac{x - x \cos x}{\sin^2 x}$

$$= \lim_{x \rightarrow 0} \frac{x(1 - \cos x)}{\sin^2 x}$$

(1 pt)

$$= \lim_{x \rightarrow 0} \frac{x \frac{(1 - \cos x)}{x^2}}{\frac{\sin^2 x}{x^2}}$$

(2 pts)

$$= \lim_{x \rightarrow 0} \frac{\frac{1 - \cos x}{x}}{\frac{(\sin x)^2}{x^2}}$$

(1 pt)

$$= \frac{0}{1} = 0$$

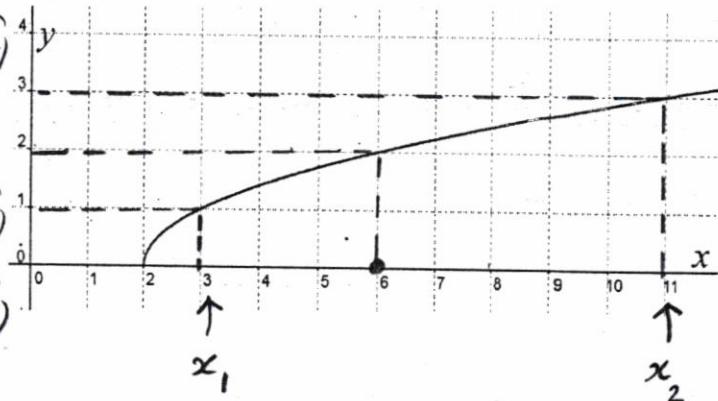
(2 pts)

5. (9 points) Use the graph of $f(x) = \sqrt{x-2}$ to find $\delta > 0$ such that if $0 < |x-6| < \delta$, then $|f(x) - 2| < 1$

$$|f(x) - 2| < 1 \Rightarrow \varepsilon = 1 \quad (1 \text{ pt})$$

From the graph:

- $f(x_1) = 1 \Rightarrow x_1 = 3 \quad (2 \text{ pts})$
 - $f(x_2) = 3 \Rightarrow x_2 = 11 \quad (2 \text{ pts})$
 - $6 - x_1 = 6 - 3 = 3 \text{ and } x_2 - 6 = 11 - 6 = 5 \quad (2 \text{ pts})$
- $\Rightarrow \delta = 3 \quad (\text{or } 0 < \delta \leq 3) \quad (2 \text{ pts})$



6. (8 points) Use the Intermediate Value Theorem to prove that the equation $x^3 - 3x - 1 = 0$ has a solution.

Let $f(x) = x^3 - 3x - 1 \quad (2 \text{ pts})$

$\Rightarrow f$ is continuous everywhere because it is a polynomial. (2 pts)

If $x = -1$, then $f(-1) = 1$, (1 pt)

If $x = 0$, then $f(0) = -1 \quad (1 \text{ pt})$

Other values of x are also possible

Then $x^3 - 3x - 1 = 0$ for some x between -1 and 0 according to the Intermediate Value Theorem! (2 pts)

7. (12 points) Use limits to find the values of a and b for which the function

$$g(x) = \begin{cases} ax + 5b, & x \leq 0 \\ x^2 + a - 3b, & 0 < x \leq 2 \\ 5x - 3, & x > 2 \end{cases}$$

is continuous at every x .

g is continuous everywhere except possibly at 0 and 2 1 pt

g is continuous at 0 and 2 if :

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^+} g(x) = g(0) \quad \text{1 pt}$$

$$\Rightarrow \lim_{x \rightarrow 0^-} (ax + 5b) = \lim_{x \rightarrow 0^+} (x^2 + a - 3b) = 5b \quad \text{2 pts}$$

$$\Rightarrow 5b = a - 3b \Rightarrow a = 8b \quad \text{A} \quad \text{2 pts}$$

$$\text{And } \lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^+} g(x) = g(2) \quad \text{1 pt}$$

$$\Rightarrow 4 + a - 3b = 10 - 3 \Rightarrow a = 3b + 3 \quad \text{B} \quad \text{2 pts}$$

$$\text{A, B} \Rightarrow 8b = 3b + 3 \Rightarrow 5b = 3 \Rightarrow b = \frac{3}{5} \quad \text{2 pts}$$

$$\Rightarrow a = \frac{24}{5} \quad \text{1 pt}$$

8. (10 points) Use limits to find the horizontal asymptotes of the curve $y = \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

$$\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} = \lim_{x \rightarrow -\infty} \frac{x^3 \left(\frac{4}{x^3} - 3 \right)}{|x^3| \sqrt{1 + \frac{9}{x^6}}} = \lim_{x \rightarrow -\infty} \frac{-\left(\frac{4}{x^3} - 3\right)}{\sqrt{1 + \frac{9}{x^6}}}$$

$$= \frac{-(-3)}{1} = 3 \quad \text{1 pt} \quad \text{3 pts}$$

$$\lim_{x \rightarrow \infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} = \lim_{x \rightarrow \infty} \frac{x^3 \left(\frac{4}{x^3} - 3 \right)}{|x^3| \sqrt{1 + \frac{9}{x^6}}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{4}{x^3} - 3\right)}{\sqrt{1 + \frac{9}{x^6}}}$$

$$= \frac{-3}{1} = -3 \quad \text{1 pt} \quad \text{3 pts}$$

$\Rightarrow y = 3$ and $y = -3$ are the horizontal asymptotes 2 pts

9. (10 points) Use limits to find all asymptotes of the curve $y = \frac{x^2+1}{x-1}$.

- $\lim_{x \rightarrow 1^-} y = -\infty$ (or $\lim_{x \rightarrow 1^+} y = \infty$) 1 pt

$\Rightarrow x=1$ is a vertical asymptote 1 pt

- $y = \frac{x^2+1}{x-1} = \frac{x^2(1 + \frac{1}{x^2})}{x(1 - \frac{1}{x})} = \frac{x(1 + \frac{1}{x})}{(1 - \frac{1}{x})}$ 1 pt

$\Rightarrow \lim_{x \rightarrow \infty} y = \infty$ and $\lim_{x \rightarrow -\infty} y = -\infty$ 2 pts

\Rightarrow No horizontal asymptotes 1 pt

- $y = x+1 + \frac{2}{x-1}$ 1 pt

$\Rightarrow \lim_{x \rightarrow \pm\infty} (y - (x+1)) = \lim_{x \rightarrow \pm\infty} \frac{2}{x-1}$ 2 pts

$$\begin{array}{c} x+1 \\ \hline x-1 & \left[\begin{array}{r} x^2+1 \\ x^2-x \\ \hline x+1 \\ x-1 \\ \hline 2 \end{array} \right] \end{array}$$

$\Rightarrow y = x+1$ is an oblique (slant) asymptote 1 pt

10. To each of the following statements, give an example to show that the statement is False:

(a) (3 points) If $\lim_{x \rightarrow 1} (f(x) \cdot g(x))$ exists, then the limit must be $f(1) \cdot g(1)$

$$\text{Let } f(x) = x-1 \text{ and } g(x) = \begin{cases} 1 & , x \neq 1 \\ \infty & , x = 1 \end{cases}$$

$$\Rightarrow \lim_{x \rightarrow 1} (f(x) \cdot g(x)) = \lim_{x \rightarrow 1} 1 = 1 \neq f(1) \cdot g(1)$$

Other examples are also possible

(b) (3 points) If f has domain $(0, \infty)$ and has no horizontal asymptotes, then $\lim_{x \rightarrow \infty} f(x) = \infty$ or $\lim_{x \rightarrow \infty} f(x) = -\infty$

$$f(x) = \sin x, x \geq 0$$

has no horizontal asymptotes

$$\text{with } \lim_{x \rightarrow \infty} f(x) \neq \infty \text{ and } \lim_{x \rightarrow \infty} f(x) \neq -\infty$$

Other examples are also possible