1. Use the graph of $f(x)$ below to answer the following:

(a) (5 points) Evaluate the limit if it exists. If it does not exist, explain why. Use the symbols ∞ or $-\infty$ as appropriate.

1 pt. i. $\lim_{x\to 3^+} f(x) = \infty$

1 pt. ii. $\lim_{x\to\infty} f(x) = 1$

1 pt. iii. $\lim_{x\to -3} f(x) = -4$

1 pt. iv. $\lim_{x\to 0} f(x) =$ D.N.E $\left(\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)\right)$

1 pt. v. $\lim_{x\to 5} f(x) = 0$

(b) (4 points) At which values of x, if any, is f discontinuous? Why?

1 pt ① $x=0$ (Jump)

1 pt ② $x = 3$ ($f(3)$ is undefined)

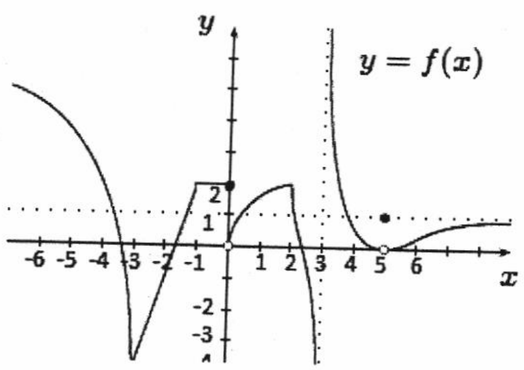
2 pt. ③ $x = 5$ ($\lim_{x\to 5} f(x) \neq f(5)$).

1 pt (c) (1 points) State the equations of any horizontal asymptotes.

$y = 1$

1 pt. (d) (1 points) State the equations of any vertical asymptotes.

$x = 3$



2. Evaluate the limit if it exists. If it does not exist, explain why. Use the symbols ∞ or $-\infty$ as appropriate.

(a) (2 points) $\lim_{x \to e} \left(\pi^{x-e} + 2\ln x\right)$

$$= \underbrace{\pi^{e-e} + 2\ln e}_{1\text{ pt}} = \underbrace{\boxed{3}}_{1\text{ pt.}}$$

(b) (4 points) $\lim_{x \to -2} \dfrac{x^3 - 4x}{x^2 - x - 6}$

$$= \lim_{x \to -2} \frac{x(x-2)(x+2)}{(x+2)(x-3)} \Big\} \text{ 2 pts.}$$

$$= \lim_{x \to -2} \frac{x(x-2)}{x-3} = \frac{-2(-4)}{-5} = \boxed{-\frac{8}{5}} \Big\} \text{ 2 pts.}$$

(c) (3 points) $\lim_{x \to \infty} \dfrac{2x^3 - 2x^2 + 4x - 1}{8 - 5x + 3x^2 - 3x^3}$

$$= \boxed{-\frac{2}{3}} \Big\} \text{ 3 pts.}$$

(d) (3 points) $\lim_{x \to 0^-} \dfrac{4x + |x|}{x}$

$$= \lim_{x \to 0} \underbrace{\frac{4x - x}{x}}_{2\text{ pts}} = \lim_{x \to 0} \frac{3x}{x} = \underbrace{\boxed{3}}_{1\text{ pt.}}$$

(e) (3 points) $\lim_{x \to 3^-} \dfrac{2x - [[x]]}{x + [[x]]}$, where $[[x]]$ is the greatest integer less than or equal to x

$$= \underbrace{\frac{2(3) - 2}{3 + 2}}_{2\text{ pts.}} = \underbrace{\boxed{\frac{4}{5}}}_{1\text{ pt}}$$

3. Evaluate the limit if it exists. If it does not exist, explain why. Use the symbols ∞ or $-\infty$ as appropriate.

(a) (3 points) $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$. $= 0$

$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$

$-x^2 \le x^2\cos\left(\frac{1}{x^2}\right) \le x^2$ } 1 pt.

1 pt. { by the squeeze Theorem

$\to 0 \leftarrow$ 1 pt.

(b) (5 points) $\lim_{x\to -\infty} \frac{\sqrt{4x^2+9}}{2x+1}$.

3 pts { $= \lim_{x\to -\infty} \frac{\sqrt{x^2(4+\frac{9}{x^2})}}{2x+1}$

$= \lim_{x\to -\infty} \frac{|x|\sqrt{4+9/x^2}}{2x+1} = \lim_{x\to -\infty} \frac{-x\sqrt{4+9/x^2}}{2x+1} =$

$= -\frac{1}{2}\cdot 2 = \boxed{-1}$ 2 pts

(c) (5 points) $\lim_{\theta\to 0} \frac{\sqrt{2\theta+3}-\sqrt{3}}{\sin\theta}$

$= \lim_{\theta\to 0} \frac{\sqrt{2\theta+3}-\sqrt{3}}{\sin\theta}\cdot\frac{\sqrt{2\theta+3}+\sqrt{3}}{\sqrt{2\theta+3}+\sqrt{3}}$ } 3 pts

$= \boxed{\lim_{\theta\to 0} \frac{2\theta}{\sin\theta\left[\sqrt{2\theta+3}+\sqrt{3}\right]}}$ 2 pts $= \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{2\theta}} \cdot \lim_{\theta\to 0} \frac{1}{\sqrt{2\theta+3}+\sqrt{3}}$

$= \frac{1}{\frac{1}{2}} \cdot \frac{1}{2\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}}$

1 pt

(d) (5 points) $\lim_{x\to 0} \frac{x^2-1+\cos^2 x}{x-\sin x}$.

$= \lim_{x\to 0} \frac{x^2-1+1-\sin^2 x}{x-\sin x}$ } 1 pt.

$= \lim_{x\to 0} \frac{x^2-\sin^2 x}{x-\sin x} = \lim_{x\to 0} \frac{\overbrace{(x-\sin x)(x+\sin x)}^{2\text{ pts}}}{x-\sin x}$

$= \lim_{x\to 0} (x+\sin x) = 0$ } 2 pts.

4. (6 points) Let

$$f(x)=\begin{cases} ax+10 & \text{if } x\le -2 \\ b\,|x| & \text{if } -2<x\le 2 \\ a\,(x-2)^2+6 & \text{if } x>2 \end{cases}$$

Find the values of the constants a and b that make $f(x)$ continuous on $(-\infty,\infty)$.

① $f(x)$ is cont. at $x=-2 \Rightarrow$

1 pt $\quad \lim_{x\to -2^+} f(x) = \lim_{x\to -2^-} f(x) \Rightarrow$

$b|-2| = a(-2)+10$

1 pt. $\quad 2b = -2a+10 \quad$ ---- I

② $f(x)$ is cont. at $x=2 \Rightarrow$

1 pt. $\quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$

1 pt. $\quad 6 = b|-2| \quad$ --- II.

$\Rightarrow \quad 6 = 2b \Rightarrow \boxed{b=3}$ } 1 pt.

$2(3) = -2a+10$

$6 = -2a+10$

$\frac{6-10}{-2} = a \Rightarrow$

$\boxed{a=2}$. } 1 pt.

5. (8 points) Use the ***definition*** of the derivative to find $f'(a)$ for $f(x) = \sqrt{5-3x}$.

2 pts.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{5-3(a+h)} - \sqrt{5-3a}}{h}$$

2 pts.

$$= \lim_{h \to 0} \frac{\sqrt{5-3(a+h)} - \sqrt{5-3a}}{h} \cdot \frac{\sqrt{5-3(a+h)} + \sqrt{5-3a}}{\sqrt{5-3(a+h)} + \sqrt{5-3a}}$$

$$= \lim_{h \to 0} \frac{5-3(a+h) - (5-3a)}{h\left[\sqrt{5-3(a+h)} + \sqrt{5-3a}\right]}$$

2 pts.

$$= \lim_{h \to 0} \frac{5 - 3a - 3h - 5 + 3a}{h\left[\sqrt{5-3(a+h)} + \sqrt{5-3a}\right]}$$

$$= \lim_{h \to 0} \frac{-3h}{h\left[\sqrt{5-3(a+h)} + \sqrt{5-3a}\right]}$$

$$= \lim_{h \to 0} \frac{-3}{\sqrt{5-3(a+h)} + \sqrt{5-3a}}$$

$$= \frac{-3}{\sqrt{5-3a} + \sqrt{5-3a}} = \boxed{\frac{-3}{2\sqrt{5-3a}}}$$

2 pts.

6. The displacement (in meters) of a particle moving in a straight line is given by $s(t) = 3\,t^2+1$, where t is measured in seconds and $0 \le t \le 5$.

(a) (4 points) Find the average velocity over the interval $[0,5]$.

$$V_{ave} = \underbrace{\frac{S(5) - S(0)}{5 - 0}}_{2\text{ pts}}$$

$$= \frac{76 - 1}{5} = \underbrace{15}_{2\text{ pts.}} \text{ m/s}$$

(b) (4 points) Use limits to find the instantaneous velocity v when $t = 4$.

$$v_{ins}(4) = \underbrace{\lim_{t \to 4} \frac{s(t) - s(4)}{t - 4}}_{2\text{ pts.}}$$

$$= \lim_{t \to 4} \frac{3t^2 + 1 - 49}{t - 4}$$

$$= \lim_{t \to 4} \frac{3t^2 - 48}{t - 4}$$

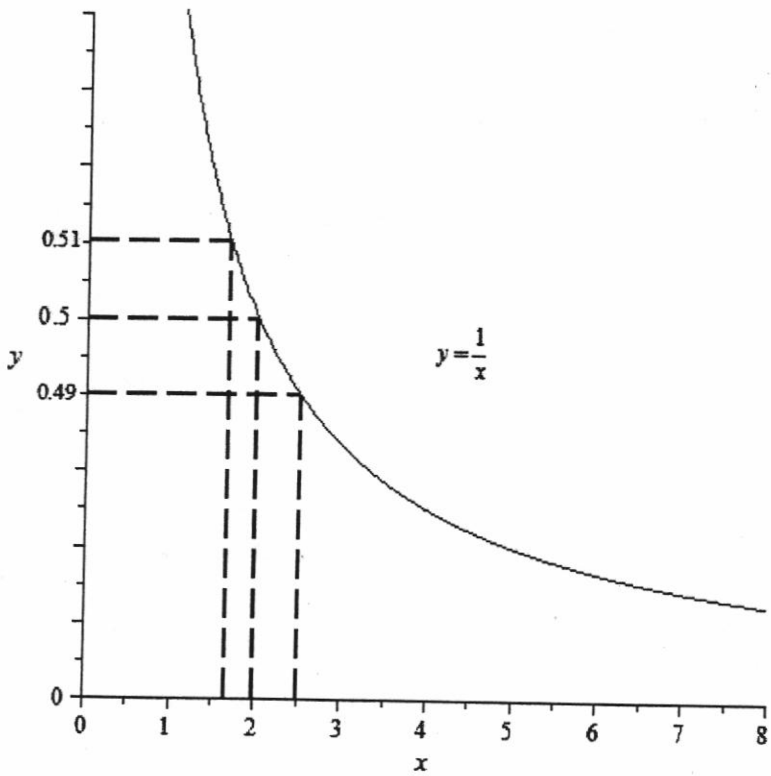
$$= \lim_{t \to 4} \underbrace{\frac{3(t-4)(t+4)}{(t-4)}}_{1\text{ pt}}$$

$$= \lim_{t \to 4} 3(t+4) = \underbrace{24}_{1\text{ pt.}}$$

7. (6 points) Use the given graph of $f(x) = \dfrac{1}{x}$ to find a number δ such that

$$|f(x) - 0.5| < 0.01 \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

Write your answer in the form $\delta = \frac{p}{q}$ where p and q are integers.



$f(x) = \frac{1}{x} \Rightarrow$

$y_1 = 0.51 \Rightarrow \frac{1}{x_1} = 0.51 \Rightarrow x_1 = \frac{100}{51} \xrightarrow{1 \text{ pt.}}$

$y_2 = 0.49 \Rightarrow \frac{1}{x_2} = 0.49 \Rightarrow x_2 = \frac{100}{49} \xrightarrow{1 \text{ pt.}}$

$\delta = \min.\left\{ 2 - \frac{100}{51}, \frac{100}{49} - 2 \right\} \xrightarrow{3 \text{ pts.}}$

$= \min\left\{ \frac{2}{51}, \frac{2}{49} \right\}$

$= \frac{2}{51} \xrightarrow{1 \text{ pt.}}$

8. Consider $f(x) = \frac{x}{2x^2 + x}$.

(a) (2 points) Find all values of x where f is ***not*** continuous.

$f(x) = \frac{x}{x(2x+1)} \Rightarrow f(x)$ is NOT Cont.

at

① $x = 0$ $\xrightarrow{1 \text{ pt.}}$

② $x = -1/2$ $\xrightarrow{1 \text{ pt.}}$

(b) (4 points) Evaluate the limit of f at each of these values, and state the type of discontinuity at each point.

① $\lim_{x \to 0} \frac{x}{x(2x+1)} = \lim_{x \to 0} \frac{1}{2x+1} = 1$ $\xrightarrow{1 \text{ pt}}$

"Removable" $\xrightarrow{1 \text{ pt}}$

② $\lim_{x \to -1/2} \frac{x}{x(2x+1)}$ = does not exist. $\xrightarrow{1 \text{ pt.}}$

"Infinite" $\xrightarrow{1 \text{ pt}}$

9. (6 points) Find all the horizontal and vertical asymptotes of $f(x) = \frac{x}{\sqrt{x^2 + 1}}$.

① Vertical Asymptotes: None $\}$ $\xrightarrow{2 \text{ pts}}$

② Horizontal Asymptotes

(i) $\lim_{x \to \infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to \infty} \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}}$

$= 1 \longrightarrow 1$ pt.

$\Rightarrow \boxed{y = 1}$ $\xrightarrow{1}$ pt.

(ii) $\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to -\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}}$

$= -1$ $\xrightarrow{1 \text{ pt}}$

$\xleftarrow{1 \text{ pt}} \Rightarrow \{ \boxed{y = -1}$

2

10. (10 points) Suppose $f(x) = \frac{x}{x-3}$. If $f'(a) = \frac{-3}{(a-3)^2}$, Find the point (a, b) on the graph of $f(x)$ such that the tangent line to the graph at (a, b) passes through the point $(9, 1)$.

Solution:

The equation of the tangent line at (a,b) is

$$y - y_1 = m(x - x_1)$$

where $m = f'(a) = \frac{-3}{(a-3)^2}$ } 2 pts.

$x_1 = a$, $y_1 = b \Rightarrow$

$$y - b = \frac{-3}{(a-3)^2}(x-a)$$ } 2 pts.

Now: since (a,b) is on the graph of $f(x)$, then $b = f(a) = \frac{a}{a-3} \Rightarrow$

$$y - \frac{a}{a-3} = \frac{-3}{(a-3)^2}(x-a)$$ 2 pts.

Also, the line goes thrugh $(9,1)$ and so

$$1 - \frac{a}{a-3} = \frac{-3}{(a-3)^2}(9-a)$$ 2 pts

$\Rightarrow (a-3)^2 - a(a-3) = -3(9-a)$

$\Rightarrow a^2 - 6a + 9 - a^2 + 3a = -27 + 3a$

$\Rightarrow -6a = -36 \Rightarrow \boxed{a = 6}$ 1 pt.

and so $b = \frac{a}{a-3} = \frac{6}{6-3} = 3$

$\Rightarrow \boxed{b = 3}$ 1 pts

11. (6 points) Use the Intermediate Value Theorem to show that the functions $f(x) = \ln(x)$ and $g(x) = e^{-x}$ intersect over the interval $[1, e]$.

Solution:

Let $h(x) = f(x) - g(x)$

$= \ln x - e^{-x}$ } 2 pts

Then

1 pt ① $h(x)$ is cont. (both f & g are cont.)

1 pt. ② $h(1) = \ln 1 - e^{-1} = -\frac{1}{e} < 0$

1 pt. ③ $h(e) = \ln e - e^{-e}$

$= 1 - \frac{1}{e^e} > 0$

[note $e^e > 1 \Rightarrow \frac{1}{e^e} < 1$]

by the IMVT, there exists a number c such that

$1 < c < e$

and $h(c) = 0 \Rightarrow$

$\ln c - e^{-c} = 0$

$\Rightarrow \boxed{\ln c = e^{-c}}$

1 pt.