Substitution

You can use **substitution** to convert a complicated integral into a simpler one. In these problems, I'll let u equal some convenient x-stuff — say $u = f(x)$. To complete the substitution, I must also substitute for dx. To do this, compute $\dfrac{du}{dx} = f'(x)$, so $du = f'(x)\,dx$. Then $dx = \dfrac{du}{f'(x)}$.

Example. Compute $\displaystyle\int (2x+3)^{100}\,dx$.

$$\int (2x+3)^{100}\,dx = \int u^{100}\cdot\frac{du}{2} = \frac{1}{2}\int u^{100}\,du = \frac{1}{202}u^{101} + C = \frac{1}{202}(2x+3)^{101} + C.$$

$$\left[u = 2x+3, \quad du = 2\,dx, \quad dx = \frac{du}{2}\right] \quad \square$$

Example. Compute $\displaystyle\int \frac{dx}{\sqrt{4-7x}}$.

$$\int \frac{dx}{\sqrt{4-7x}} = \int \frac{1}{\sqrt{u}}\cdot\left(-\frac{du}{7}\right) = -\frac{1}{7}\int u^{-1/2}\,du = -\frac{2}{7}u^{1/2} + C = -\frac{2}{7}(4-7x)^{1/2} + C.$$

$$\left[u = 4-7x, \quad du = -7\,dx, \quad dx = -\frac{du}{7}\right] \quad \square$$

Example. Later on, I'll derive the integration formula

$$\int \frac{dx}{x} = \ln|x| + C.$$

Use this formula to compute $\displaystyle\int \frac{1}{3x+1}\,dx$.

$$\int \frac{1}{3x+1}\,dx = \int \frac{1}{u}\cdot\frac{du}{3} = \frac{1}{3}\int\frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x+1| + C.$$

$$\left[u = 3x+1, \quad du = 3\,dx, \quad dx = \frac{du}{3}\right] \quad \square$$

Example. Compute $\displaystyle\int x(x^2+5)^{50}\,dx$.

$$\int x(x^2+5)^{50}\,dx = \int xu^{50}\cdot\frac{du}{2x} = \frac{1}{2}\int u^{50}\,du = \frac{1}{102}u^{51} + C = \frac{1}{102}(x^2+5)^{51} + C.$$

$$\left[u = x^2+5, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}\right] \quad \square$$

Notice that in the second step in the last example, the x's cancelled out, leaving only u's. If the x's had failed to cancel, I wouldn't have been able to complete the substitution.

But what made the x's cancel? It was the fact that I got an x from the derivative of $u = x^2 + 5$. This leads to the following rule of thumb.

Substitute for something whose derivative is also there.

Example. Compute $\displaystyle\int \sin(3x+1)\,dx$.

$$\int \sin(3x+1)\,dx = \int \sin u \cdot \frac{du}{3} = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(3x+1) + C.$$

$$\left[u = 3x+1, \quad du = 3\,dx, \quad dx = \frac{du}{3}\right] \quad \square$$

Example. Compute $\displaystyle\int (\sin 5x)^7 \cos 5x\,dx$.

$$\int (\sin 5x)^7 \cos 5x\,dx = \int u^7 \cos 5x \cdot \frac{du}{5\cos 5x} = \frac{1}{5}\int u^7\,du = \frac{1}{40}u^8 + C = \frac{1}{40}(\sin 5x)^8 + C.$$

$$\left[u = \sin 5x, \quad du = 5\cos 5x\,dx, \quad dx = \frac{du}{5\cos 5x}\right] \quad \square$$

Example. Compute $\displaystyle\int \frac{1}{\sqrt{x}(\sqrt{x}+9)^2}\,dx$.

$$\int \frac{1}{\sqrt{x}(\sqrt{x}+9)^2}\,dx = \int \frac{1}{\sqrt{x}u^2} \cdot 2\sqrt{x}\,du = 2\int u^{-2}\,du = -\frac{2}{u} + C = -\frac{2}{\sqrt{x}+9} + C.$$

$$\left[u = \sqrt{x}+9, \quad du = \frac{dx}{2\sqrt{x}}, \quad dx = 2\sqrt{x}\,du\right] \quad \square$$

Example. Compute $\displaystyle\int \frac{\sin \frac{1}{x}}{x^2}\,dx$.

$$\int \frac{\sin \frac{1}{x}}{x^2}\,dx = \int \frac{\sin u}{x^2} \cdot (-x^2\,du) = -\int \sin u\,du = \cos u + C = \cos \frac{1}{x} + C.$$

$$\left[u = \frac{1}{x}, \quad du = -\frac{dx}{x^2}, \quad dx = -x^2\,du\right] \quad \square$$

Example. Compute $\displaystyle\int (2x-1)(2x+3)^{40}\,dx$.

$$\int (2x-1)(2x+3)^{40}\,dx = \int (u-4)u^{40}\,du = \int (u^{41} - 4u^{40})\,du =$$

$$\left[u = 2x+3, \quad du = 2\,dx, \quad dx = \frac{du}{2}; \quad x = \frac{1}{2}(u-3)\right]$$

$$\frac{1}{42}u^{42} - \frac{4}{41}u^{41} + C = \frac{1}{42}(2x+3)^{42} - \frac{4}{41}(2x+3)^{41} + C. \quad \square$$

Math 162
1-24-1998

Integration by Parts

If u and v are functions of x, the Product Rule says that

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrate both sides:

$$\int \frac{d(uv)}{dx}\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx,$$

$$uv = \int u\,dv + \int v\,du,$$

$$\int u\,dv = uv - \int v\,du.$$

This is the **integration by parts** formula. The integral on the left corresponds to the integral you're trying to do. Parts replaces it with some junk (uv) and another integral ($\int v\,du$). You'll make progress if the new integral is easier to do than the old one.

I'm going to set up parts computations using tables; it is *much* easier to do repeated parts computations this way than to use the standard u-dv-v-du approach. To see where the table comes from, start with the parts equation:

$$\int u\,dv = uv - \int v\,du.$$

Apply parts to the integral on the right, differentiating $\dfrac{du}{dx}$ and integrating v. This gives

$$\int u\,dv = uv - \left[\left(\frac{du}{dx}\right)\left(\int v\,dx\right) - \int \left(\int v\,dx\right)\left(\frac{d^2u}{dx^2}\right)\,dx\right] = uv - \left(\frac{du}{dx}\right)\left(\int v\,dx\right) + \int \left(\int v\,dx\right)\left(\frac{d^2u}{dx^2}\right)\,dx.$$

If I apply parts yet again to the new integral on the right, I would get

$$\int u\,dv = uv - \left(\frac{du}{dx}\right)\left(\int v\,dx\right) + \left(\frac{d^2u}{dx^2}\right)\left(\int\left(\int v\,dx\right)\,dx\right) - \int\left(\int\left(\int v\,dx\right)\,dx\right)\left(\frac{d^3u}{dx^3}\right)\,dx.$$

There's a pattern here, and it's captured by the following table:

$$\begin{array}{cccc}
 & \dfrac{d}{dx} & & \displaystyle\int dx \\
+ & u & & dv \\
 & & \searrow & \\
- & \dfrac{du}{dx} & & v \\
 & & \searrow & \\
+ & \dfrac{d^2u}{dx^2} & & \displaystyle\int v\,dx \\
 & & \searrow & \\
- & \dfrac{d^3u}{dx^3} & & \displaystyle\int\left(\int v\,dx\right)\,dx \\
\vdots & \vdots & & \vdots
\end{array}$$

To make the table, put alternating $+$'s and $-$'s in the left-hand column. Take the original integral and break it into a u (second column) and a dv (third column). (I'll discuss how you choose u and dv later.)

Differentiate repeatedly down the u-column, and integrate repeatedly down the dv-column. (You don't write down the dx; it's kind of implicitly there in the third column, since you're integrating.)

How do you get from the table to the messy equation above? Consider the first term on the right: uv. You get that from the table by taking the $+$ sign, taking the u next to it, and then moving "southeast" to grab the v.

If you compare the table with the equation, you'll see that you get the rest of the terms on the right side by multiplying terms in the table according to the same pattern:

$$\begin{array}{ccccc} ☺ & \rightarrow & ☺ & & \\ & & & \searrow & \\ & & & & ☺ \end{array}$$

The table continues downward indefinitely, so how do you stop? If you look at the last messy equation above and compare it to the table, you can see how to stop: *Just integrate all the terms in a row of the table.*

You'll see that in many examples, *the process will stop naturally when the derivative column entries become* 0.

Example. Compute $\displaystyle\int x^3 e^{2x}\,dx$.

Parts is often useful when you have *different kinds of functions in the same integral.* Here I have a power (x^3) and and exponential (e^{2x}), and this suggests using parts.

I have to "allocate" $x^3e^{2x}\,dx$ between u and dv — remember that dx implicitly goes into dv. I will use $u = x^3$ and $dv = e^{2x}\,dx$. Here's the parts table:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & x^3 & & e^{2x} \\ & & \searrow & \\ - & 3x^2 & & \dfrac{1}{2}e^{2x} \\ & & \searrow & \\ + & 6x & & \dfrac{1}{4}e^{2x} \\ & & \searrow & \\ - & 6 & & \dfrac{1}{8}e^{2x} \\ & & \searrow & \\ + & 0 & \rightarrow & \dfrac{1}{16}e^{2x} \end{array}$$

You can see the derivatives of x^3 in one column and the integrals of e^{2x} in another. Notice that when I get a 0, I cut off the computation.

Therefore,

$$\int x^3e^{2x}\,dx = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{6}{8}xe^{2x} - \frac{6}{16}e^{2x} + \int 0\,dx.$$

But $\displaystyle\int 0\,dx$ is just 0 (up to an arbitrary constant), so I can write

$$\int x^3e^{2x}\,dx = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{6}{8}xe^{2x} - \frac{6}{16}e^{2x} + C.$$

Before leaving this problem, it's worth thinking about why the x^3 went into the derivative column and the e^{2x} went into the integral column. Here's what would happen if the two were reversed:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & e^{2x} & & x^3 \\ & & \searrow & \\ - & 2e^{2x} & & \dfrac{1}{4}x^4 \\ & & \searrow & \\ + & 4e^{2x} & & \dfrac{1}{20}x^5 \\ \vdots & \vdots & & \vdots \end{array}$$

This is bad for two reasons. First, I'm not getting that nice 0 I got by repeatedly differentiating x^3. Worse, the powers in the last column are getting bigger! This means that the problem is getting *more* complicated, rather than less.

Here's another attempt which doesn't work:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & 1 & & x^3 e^{2x} \\ & & \searrow & \\ - & 0 & & \displaystyle\int x^3 e^{2x}\, dx \end{array}$$

I got a 0 this time, but how can I find the integral in the second row? — it's the same as the original integral! *Putting the entire integrand into the integration column never works.* $\square$

Here's a rule of thumb which reflects the preceding discussion. When you're trying to decide which part of an integral to put into the derivative column, the order of preference is roughly

Logs **I**nverse trigs **P**owers **T**rig **E**xponentials

L-I-P-T-E.

According to this rule, in

$$\int x(\ln x)^2\, dx,$$

you'd try the log $(\ln x)^2$ in the derivative column ahead of the power x.

And in

$$\int e^{2x} \sin 5x\, dx,$$

you'd try the trig function $\sin 5x$ in the derivative column, because it has precedence over the exponential e^{2x}.

Example. Compute $\displaystyle\int \ln x\, dx$.

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & \ln x & & 1 \\ & & \searrow & \\ - & \dfrac{1}{x} & \rightarrow & x \end{array}$$

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C. \quad \square$$

Example. Compute $\displaystyle\int (\ln x)^2\,dx$.

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & (\ln x)^2 & & 1 \\ & & \searrow & \\ - & \dfrac{2\ln x}{x} & \rightarrow & x \end{array}$$

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx = x(\ln x)^2 - 2(x\ln x - x) + C. \quad \square$$

Example. Compute $\displaystyle\int x(x+4)^{50}\,dx$.

First,

$$\int (x+4)^{50}\,dx = \int u^{50}\,du = \frac{1}{51}u^{51} + C = \frac{1}{51}(x+4)^{51} + C.$$

$$\left[u = x+4, \quad du = dx\right]$$

The same substitution shows that

$$\int (x+4)^{51}\,dx = \frac{1}{52}(x+4)^{52} + C.$$

Now do the original integral by parts:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & x & & (x+4)^{50} \\ & & \searrow & \\ - & 1 & & \dfrac{1}{51}(x+4)^{51} \\ & & \searrow & \\ + & 0 & & \dfrac{1}{2652}(x+4)^{52} \end{array}$$

$$\int x(x+4)^{50}\,dx = \frac{1}{51}x(x+4)^{51} - \frac{1}{2652}(x+4)^{52} + C. \quad \square$$

Example. Compute $\displaystyle\int \arctan x\,dx$.

Parts is also useful when the integrand is a single, unsimplifiable lump. You can't do anything interesting with $\arctan x$, so use parts.

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & \arctan x & & 1 \\ & & \searrow & \\ - & \dfrac{1}{x^2+1} & \rightarrow & x \end{array}$$

Therefore,

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{x^2+1}\,dx.$$

I can do the new integral by substituting $u = x^2+1$. Then $du = 2x\,dx$, so $dx = \dfrac{du}{2x}$:

$$x\arctan x - \int \frac{x}{x^2+1}\,dx = x\arctan x - \int \frac{x}{u}\cdot\frac{du}{2x} = x\arctan x - \frac{1}{2}\int \frac{du}{u} = x\arctan x - \frac{1}{2}\ln|u| + C =$$

$$x\arctan x - \frac{1}{2}\ln|x^2+1| + C.\quad \square$$

Example. Compute $\displaystyle\int_0^{\pi/2} x\sin x\,dx$.

If you do a definite integral using parts, compute the antiderivative using parts as usual, then slap on the limits of integration at the end.

$$\begin{array}{ccccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & x & & \sin x \\ & & \searrow & \\ - & 1 & & -\cos x \\ & & \searrow & \\ + & 0 & & -\sin x \end{array}$$

Thus,

$$\int_0^{\pi/2} x\sin x\,dx = \left[-x\cos x + \sin x\right]_0^{\pi/2} = 1.\quad \square$$

Example. Compute $\displaystyle\int e^x \sin 2x\,dx$.

$$\begin{array}{ccccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & e^x & & \sin 2x \\ & & \searrow & \\ - & e^x & & -\dfrac{1}{2}\cos 2x \\ & & \searrow & \\ + & e^x & & -\dfrac{1}{4}\sin 2x \end{array}$$

$$\int e^x\sin 2x\,dx = -\frac{1}{2}e^x\cos 2x + \frac{1}{4}e^x\sin 2x - \frac{1}{4}\int e^x\sin 2x\,dx.$$

What's this? All that work and you get the original integral again!

Actually, you're almost done. Jog your brain and get out of "parts mode". Instead, look at the equation as *an equation to be solved for the original integral.* It looks like this:

$$\text{original integral} = \text{some junk} - \text{original integral}.$$

Just move the copy of the original integral on the right back to the left!

$$\int e^x\sin 2x\,dx = -\frac{1}{2}e^x\cos 2x + \frac{1}{4}e^x\sin 2x - \frac{1}{4}\int e^x\sin 2x\,dx,$$

$$\frac{5}{4}\int e^x\sin 2x\,dx = -\frac{1}{2}e^x\cos 2x + \frac{1}{4}e^x\sin 2x,$$

$$\int e^x\sin 2x\,dx = -\frac{2}{5}e^x\cos 2x + \frac{1}{5}e^x\sin 2x + C.\quad \square$$

Partial Fractions

Partial fractions is the opposite of adding fractions over a common denominator. It applies to integrals of the form

$$\int \frac{P(x)}{Q(x)}\,dx, \quad \text{where} P(x) \quad \text{and} \quad Q(x) \quad \text{are polynomials.}$$

The idea is to break $\dfrac{P(x)}{Q(x)}$ into a sum of smaller terms which are easier to integrate.

(A function of the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, is called a **rational function**.)

I'll by doing an example to give you a feel for the procedure. Then I'll go back and explain the steps in the method. The procedure is a bit long, and requires a substantial amount of algebra. Therefore, before using partial fractions, you should be sure that there isn't an easier way to do the integral.

First, I want to mention a formula that often comes up in these problems:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C, \quad a \neq 0.$$

(Do you see how to work it out? Substitute $u = ax + b$, so $du = a\,dx$.) For example,

$$\int \frac{1}{x-7}\,dx = \ln|x-7| + C,$$

$$\int \frac{1}{7x+5}\,dx = \frac{1}{7}\ln|7x+5| + C,$$

$$\int \frac{1}{3-2x}\,dx = -\frac{1}{2}\ln|3-2x| + C.$$

Example. Compute $\displaystyle\int \frac{17-3x}{x^2-2x-3}\,dx$.

$x^2 - 2x - 3 = (x-3)(x+1)$. Write

$$\frac{17-3x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}.$$

Multiply both sides by $(x-3)(x+1)$ to clear denominators:

$$17 - 3x = A(x+1) + B(x-3).$$

Let $x = 3$. I get

$$17 - 9 = 4A + 0, \quad \text{so} \quad 8 = 4A, \quad \text{or} \quad A = 2.$$

Let $x = -1$. I get

$$17 + 3 = 0 - 4B, \quad \text{so} \quad 20 = -4B, \quad \text{or} \quad B = -5.$$

Therefore,

$$\frac{17-3x}{(x-3)(x+1)} = \frac{2}{x-3} - \frac{5}{x+1}.$$

So

$$\int \frac{17-3x}{x^2-2x-3}\,dx = \int \left(\frac{2}{x-3} - \frac{5}{x+1}\right)dx = 2\ln|x-3| - 5\ln|x+1| + C. \quad \square$$

Consider an integral of the form

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ and $Q(x)$ are polynomials.

Recall that the **degree** of a polynomial is the highest power of the variable that occurs in it. Nonzero constants have degree 0; by convention, 0 has degree $-\infty$.

Step 1. If the degree of the top is greater than or equal to the degree of the bottom, divide the bottom into the top.

Example. Compute $\displaystyle\int \frac{2x^3 - 5x^2 - x + 5}{x^2 - 1}\,dx$.

The top has degree 3 while the bottom has degree 2. Divide the bottom into the top:

$$\frac{2x^3 - 5x^2 - x + 5}{x^2 - 1} = 2x - 5 + \frac{x}{x^2 - 1}.$$

So

$$\int \frac{2x^3 - 5x^2 - x + 5}{x^2 - 1}\,dx = \int \left(2x - 5 + \frac{x}{x^2 - 1}\right)\,dx.$$

Integrate $\dfrac{x}{x^2 - 1}$ by substitution:

$$\int \frac{x}{x^2 - 1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2 - 1| + C.$$

$$\left[u = x^2 - 1, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}\right]$$

Hence,

$$\int \left(2x - 5 + \frac{x}{x^2 - 1}\right)\,dx = x^2 - 5x + \frac{1}{2}\ln|x^2 - 1| + C.$$

In this problem, division simplified the integral enough that it wasn't necessary to go any further in the partial fractions procedure. □

Example. Consider the integral $\displaystyle\int \frac{x^2}{x^2 - 1}\,dx$. The top and the bottom have the same degree. Divide the bottom into the top:

$$\frac{x^2}{x^2 - 1} = 1 + \frac{1}{x^2 - 1}.$$

Therefore,

$$\int \frac{x^2}{x^2 - 1}\,dx = \int \left(1 + \frac{1}{x^2 - 1}\right)\,dx.$$

There's more to do: The integral is passed along to the next step in the partial fractions procedure. □

Example. Consider the integral $\displaystyle\int \frac{7x^2 - 25x + 20}{x(x - 2)^2}\,dx$.

The top has degree 2 while the bottom has degree 3, so you do not need to divide. The integral is passed along to the next step in the partial fractions procedure. □

Step 1 is a preliminary operation which puts the integral into a good form for the rest of the procedure. Before going on, check to see whether you can use a simple technique (like substitution) to do the integrals you've obtained. Sometimes (as in the example above) you can complete the integration immediately. If you don't see a simple way to finish, then proceed to Step 2.

Step 2. You will now have an integral that looks like

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ and $Q(x)$ are polynomials, and the top is smaller in degree than the bottom. (The division process may spew out some other junk, but that can be integrated immediately. You only need to worry about the remaining fractional part.)

Factor the bottom of the fraction into a product of **linear** terms and **irreducible quadratic** terms.

A **linear** term is a term where the variable occurs only to the first power. Here are some linear terms:

$$x-1, \quad 2x-3, \quad x.$$

An **irreducible quadratic** term is a quadratic term with only imaginary (complex) roots. That is, it is a quadratic which "doesn't factor". Here are some irreducible quadratic terms:

$$x^2+1, \quad x^2-2x+5.$$

You can check that a quadratic is irreducible by using the general quadratic formula to find its roots. If the roots are complex numbers, the quadratic does not factor.

Warning: x^2-2 is *not* irreducible:

$$x^2-2=(x-\sqrt{2})(x+\sqrt{2}).$$

("Ugly" factors are allowed.) And x^2 is not considered an irreducible quadratic: It is $(x-0)(x-0)$, the square of a linear term. This distinction will become important in Step 3.

Example. In the second example above, I obtained

$$\int \left(1+\frac{1}{x^2-1}\right)\,dx.$$

The "1" integrates immediately to x. So consider $\displaystyle\int \frac{1}{x^2-1}\,dx$. Factor the denominator of the fraction:

$$\int \frac{1}{x^2-1}\,dx=\int \frac{1}{(x-1)(x+1)}\,dx.$$

Now pass the integral along to the next step. □

Example. In an earlier example, I obtained

$$\int \frac{7x^2-25x+20}{x(x-2)^2}\,dx.$$

The bottom is already factored. Note that $(x-2)^2$ is *not* considered a quadratic factor; it's considered to be a linear factor raised to a power. $\square$

Example. Consider the integral $\displaystyle\int \frac{1}{x^3-1}\,dx$.

Factor the denominator of the fraction:

$$\int \frac{1}{x^3-1}\,dx = \int \frac{1}{(x-1)(x^2+x+1)}\,dx.$$

Note that x^2+x+1 has only imaginary roots. $\square$

Remark. It is known that any polynomial can be factored into linear terms and irreducible quadratic terms. However, the "can be" in the last sentence does not mean that you can always do it in practice! In fact, another branch of mathematics called **Galois theory** says that there is no general formula for finding exactly the roots of polynomials of degree 5 or higher.

To understand *how partial fractions works*, it's not necessary to consider huge or intractable examples. For that reason, the polynomials in the problems will usually be easy to factor. Moreover, programs like *Mathematica* can approximate roots, and that is often good enough for applications. However, you should keep in mind that factoring polynomials is, in general, not a simple thing.

Step 3. Obtain the **partial fractions decomposition** for the fraction.

This is the heart of the partial fractions method. It is basically a lot of algebra, but it's sufficiently complicated that the best way to describe it is by doing some examples.

Example. Compute $\displaystyle\int \frac{1}{(x-1)(x+1)}\,dx$.

I want to find numbers A and B such that

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

is an **algebraic identity**. This means that the equation should be true for all values of x for which both sides are defined.

First, multiply through to clear denominators.

$$1 = A(x+1) + B(x-1).$$

Since this equation is to be true for *all* x, it must true for $x=1$. Plug in $x=1$. I obtain $1=2A$, so $A=1/2$.

I chose $x=1$ because it *killed* the $B(x-1)$-term, and this allowed me to solve for A.

I see that setting $x=-1$ will kill the $A(x+1)$ term. Doing so, I get $1=2B$, so $B=1/2$.

Therefore,

$$\frac{1}{(x-1)(x+1)} = \frac{1}{2}\,\frac{1}{x-1} + \frac{1}{2}\,\frac{1}{x+1}.$$

(Check: If you add the fractions on the right, you'll get the fraction on the left.)

Now it's easy to do the integral:

$$\int \frac{1}{(x-1)(x+1)}\,dx = \int \left(\frac{1}{2}\,\frac{1}{x-1} + \frac{1}{2}\,\frac{1}{x+1}\right) = \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C. \quad \square$$

Example. Compute $\displaystyle\int \frac{2-x^2}{x(x-1)^2}\,dx$.

This example will show how to handle **repeated factors** — in this case, $(x-1)^2$. Here's what you do:

$$\frac{2-x^2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

For a **repeated factor**, you have one term for each power up to the power the factor is raised to. In this case, you have a term for $x-1$ and a term for $(x-1)^2$.

Multiply to clear denominators:

$$2-x^2 = A(x-1)^2 + Bx(x-1) + Cx.$$

Let $x=0$. I get $2=A$.
Let $x=1$. I get $1=C$.
Plug the A and C values back in:

$$2-x^2 = 2(x-1)^2 + Bx(x-1) + x. \qquad (*)$$

With only B left, I can plug in any number and solve for B. I'll let $x=2$:

$$2-4 = 2+2B+2, \quad -6 = 2B, \quad B = -3.$$

Therefore,

$$\frac{2-x^2}{x(x-1)^2} = \frac{2}{x} - \frac{3}{x-1} + \frac{1}{(x-1)^2}.$$

Hence,

$$\int \frac{2-x^2}{x(x-1)^2}\,dx = \int \left(\frac{2}{x} - \frac{3}{x-1} + \frac{1}{(x-1)^2}\right) dx = 2\ln|x| - 3\ln|x-1| - \frac{1}{x-1} + C.$$

Alternatively, take equation (*). Multiply out the B-term:

$$2-x^2 = 2(x-1)^2 + B(x^2-x) + x.$$

Differentiate!

$$-2x = 4(x-1) + B(2x-1) + 1.$$

Differentiate again!

$$-2 = 4 + 2B, \quad -6 = 2B, \quad B = -3.$$

At any point, you can plug in any number for x, or you can differentiate both sides of the equation. □

Example. Compute $\displaystyle\int \frac{7x^2-25x+20}{x(x-2)^2}\,dx$.

In this case, the **repeated factor** is the "$(x-2)^2$". I want A, B, and C so that

$$\frac{7x^2-25x+20}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Clear denominators:

$$7x^2-25x+20 = A(x-2)^2 + Bx(x-2) + Cx.$$

Plugging in $x = 0$ and $x = 2$ will kill lots of terms. First, set $x = 0$. I obtain $20 = 4A + 0 + 0$, so $A = 5$. Next, set $x = 2$. Then $28 - 50 + 20 = 0 + 0 + 2C$ so $C = -1$.
Put $A = 5$ and $C = -1$ back into the equation:

$$7x^2 - 25x + 20 = 5(x-2)^2 + Bx(x-2) - x.$$

Set $x = 1$. I get $2 = 5 - B - 1$, so $B = 2$.
Substitute the A, B, and C values into the original decomposition:

$$\frac{7x^2 - 25x + 20}{x(x-2)^2} = \frac{5}{x} + \frac{2}{x-2} - \frac{1}{(x-2)^2}.$$

Finally, do the integral:

$$\int \frac{7x^2 - 25x + 20}{x(x-2)^2}\,dx = \int \left(\frac{5}{x} + \frac{2}{x-2} - \frac{1}{(x-2)^2}\right) dx = 5\ln|x| + 2\ln|x-2| + \frac{1}{x-2} + C. \quad \square$$

Example. How would you try to decompose

$$\frac{5x^4 - 3x + 1}{(x-3)^4(x+2)^2}$$

using partial fractions? That is, what is the initial partial fractions equation?

The linear factor $x - 3$ is repeated 4 times, and the linear factor $x + 2$ is repeated 2 times. So you use

$$\frac{5x^4 - 3x + 1}{(x-3)^4(x+2)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{(x-3)^4} + \frac{E}{x+2} + \frac{F}{(x+2)^2}.$$

You could do the $x+2$ terms first instead. Notice that the numerator $5x^4 - 3x + 1$ has no effect on the decomposition. $\square$

Example. How would you try to decompose

$$\frac{3x^3 + 4x - 17}{x^3(2x-1)^2}$$

using partial fractions? That is, what is the initial partial fractions equation?

The linear factor x is repeated 3 times and the linear factor $2x - 1$ is repeated twice. Therefore, you should try to solve

$$\frac{3x^3 + 4x - 17}{x^3(2x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{2x-1} + \frac{E}{(2x-1)^2}.$$

Notice that *the top of the fraction is irrelevant in deciding how to set up the decomposition.* It only comes in during the solution process.

Notice also that "x^3" is considered a linear term (x) raised to the third power. You get one term on the right for x, one for x^2, and one for x^3 — no "skipping"! $\square$

Example. Compute $\displaystyle\int \frac{1}{(x-1)(x^2+x+1)}\,dx$.

In this example, there's an irreducible quadratic factor x^2+x+1. In this case, I try

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

Thus, a quadratic factor (or a quadratic factor to a power) will produce terms on the right with "two letters" on top.

The rationale is the same as the one I gave for repeated factors. I don't know what kind of fraction to expect, so I have to take the most general case.

(You might ask: "Well, why not try $\dfrac{Bx^2+Cx+D}{x^2+x+1}$ — or higher powers?" The answer is that if I had a quadratic or something bigger on top, I could divide first to reduce to a $Bx+C$ form.)

The solution procedure is similar to those used above. Clear denominators:

$$1 = A(x^2+x+1) + (Bx+C)(x-1).$$

Set $x=1$. This gives $1=3A$, so $A=1/3$. Plug it back in:

$$1 = \frac{1}{3}(x^2+x+1) + (Bx+C)(x-1).$$

$x=0$ will kill B, leaving C to be solved for. Setting $x=0$, I get $1=1/3-C$, so $C=-2/3$. Plug it back in:

$$1 = \frac{1}{3}(x^2+x+1) + (Bx-\frac{2}{3})(x-1).$$

Now I can either plug in a value for x at random, or differentiate. I'll differentiate:

$$0 = \frac{1}{3}(2x+1) + (Bx-\frac{2}{3}) + B(x-1).$$

Differentiate again:

$$0 = \frac{2}{3} + B + B.$$

I get $B=-1/3$.

Plug the values back into the original fractional decomposition:

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{1}{3}\,\frac{1}{x-1} - \frac{1}{3}\,\frac{x+2}{x^2+x+1}.$$

The integral is

$$\int \frac{1}{(x-1)(x^2+x+1)}\,dx = \int \left(\frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{3}\cdot\frac{x+2}{x^2+x+1}\right)\,dx.$$

I'll do the integrals separately. First,

$$\int \frac{1}{3}\cdot\frac{1}{x-1} = \frac{1}{3}\ln|x-1| + C.$$

Next,

$$\frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx = \frac{1}{6}\int \frac{2x+4}{x^2+x+1}\,dx = \frac{1}{6}\int \frac{2x+1+3}{x^2+x+1}\,dx =$$
$$\frac{1}{6}\left(\int \frac{2x+1}{x^2+x+1}\,dx + 3\int \frac{1}{x^2+x+1}\,dx\right).$$

The first integral succumbs to a substitution:

$$\int \frac{2x+1}{x^2+x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2+x+1| + C.$$

$$\left[u = x^2+x+1, \quad du = (2x+1)\,dx, \quad dx = \frac{du}{2x+1}\right]$$

The second requires completing the square:

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \frac{\sqrt{3}}{2}\int \frac{1}{\frac{3}{4}u^2+\frac{3}{4}}\,du =$$

$$\left[x+\frac{1}{2} = \frac{\sqrt{3}}{2}u, \quad dx = \frac{\sqrt{3}}{2}\,du\right]$$

$$\frac{2}{\sqrt{3}}\int \frac{1}{u^2+1}\,du = \frac{2}{\sqrt{3}}\arctan u + C = \frac{2}{\sqrt{3}}\arctan \frac{2(x+\frac{1}{2})}{\sqrt{3}} + C.$$

(Whew!) Putting the two together,

$$\frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx = \frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan \frac{2(x+\frac{1}{2})}{\sqrt{3}} + C.$$

Finally, the original problem is

$$\int \frac{1}{(x-1)(x^2+x+1)}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\arctan \frac{2(x+\frac{1}{2})}{\sqrt{3}} + C.$$

What a mess! This is why you should consider other methods before you turn to partial fractions! □

Trigonometric Integrals

For trig integrals involving powers of sines and cosines, there are two important cases:

1. The integral contains an odd power of sine or cosine.
2. The integral contains only even powers of sines and cosines.

I will look at the odd power case first. It turns out that the same idea can be used to integrate some powers of secants and tangents, so I'll digress to do some examples of those as well.

Example.

$$\int (\sin 5x)^3 \left(1 + 4(\cos 5x)^2\right)\, dx = \int (\sin 5x)^2 \left(1 + 4(\cos 5x)^2\right) \sin 5x\, dx =$$

$$\int \left(1 - (\cos 5x)^2\right) \left(1 + 4(\cos 5x)^2\right) \sin 5x\, dx =$$

$$\left[u = \cos 5x, \quad du = -5 \sin 5x\, dx, \quad dx = -\frac{du}{5 \sin 5x}\right]$$

$$-\frac{1}{5} \int (1 - u^2)(1 + 4u^2)\, du = -\frac{1}{5} \int (1 + 3u^2 - 4u^4)\, du = -\frac{1}{5}(u + u^3 - \frac{4}{5}u^5) + C =$$

$$-\frac{1}{5}\left(\cos 5x + (\cos 5x)^3 - \frac{4}{5}(\cos 5x)^5\right) + C = -\frac{1}{5}\cos 5x - \frac{1}{5}(\cos 5x)^3 + +\frac{4}{25}(\cos 5x)^5 + C.$$

In this example, the key point was in the second line. I obtained an integral with lots of $\cos 5x$'s and a single $\sin 5x$. This allowed me to make the substitution $u = \cos 5x$, because the $\sin 5x$ was available to make du.

I got the $\sin 5x$ by "pulling it off" the odd power of $\sin 5x$. Then I converted the rest of the stuff to $\cos 5x$'s using the identity $(\sin\theta)^2 + (\cos\theta)^2 = 1$. This is the generic procedure when you have at least one odd power of sine or cosine. □

Example.

$$\int \left(5(\sin x)^{2/3} + 1\right) (\cos x)^3\, dx = \int \left(5(\sin x)^{2/3} + 1\right) \left(1 - (\sin x)^2\right) (\cos x)\, dx =$$

$$\left[u = \sin x, \quad du = \cos x\, dx, \quad dx = \frac{du}{\cos x}\right]$$

$$\int \left(5u^{2/3} + 1\right) \left(1 - u^2\right)\, du = \int \left(5u^{2/3} + 1 - 5u^{8/3} - u^2\right)\, du = 3u^{5/3} + u - \frac{15}{11}u^{11/3} - \frac{1}{3}u^3 + C =$$

$$3(\sin x)^{5/3} + \sin x - \frac{15}{11}(\sin x)^{11/3} - \frac{1}{3}(\sin x)^3 + C. \quad □$$

Example. You can use a similar idea to integrate some powers of secants and tangents.

$$\int (\sec 3x)^4\, dx = \int (\sec 3x)^2 (\sec 3x)^2\, dx = \int \left(1 + (\tan 3x)^2\right) (\sec 3x)^2\, dx =$$

$$\left[u = \tan 3x, \quad du = 3(\sec 3x)^2\, dx, \quad dx = \frac{du}{3(\sec 3x)^2}\right]$$

$$\frac{1}{3}\int (1+u^2)\,du = \frac{1}{3}(u+\frac{1}{3}u^3) + C = \frac{1}{3}\tan 3x + \frac{1}{9}(\tan 3x)^3 + C.$$

In this example, I grabbed a $(\sec 3x)^2$, then converted the rest of the stuff to $\tan 3x$'s using $1+(\tan\theta)^2 = (\sec\theta)^2$. The $(\sec 3x)^2$ was exactly what I needed to make du for the substitution $u = \tan 3x$.

Notice that the argument $3x$ did not play an important role in the problem. □

Example.

$$\int (\tan\theta)^3\,d\theta = \int (\tan\theta)^2 \tan\theta\,d\theta = \int \left((\sec\theta)^2 - 1\right)\tan\theta\,d\theta =$$

$$\int (\sec\theta)^2\tan\theta\,d\theta - \int \tan\theta\,d\theta = \int \sec\theta(\sec\theta\tan\theta\,d\theta) - \int \frac{\sin\theta}{\cos\theta}\,d\theta.$$

I can do the first integral using $u = \sec\theta$, so $du = \sec\theta\tan\theta\,d\theta$ and $d\theta = \dfrac{du}{\sec\theta\tan\theta}$:

$$\int \sec\theta(\sec\theta\tan\theta\,d\theta) = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sec\theta)^2 + C.$$

I can do the second integral using $w = \cos\theta$, so $dw = -\sin\theta\,d\theta$ and $d\theta = \dfrac{dw}{-\sin\theta}$:

$$\int \frac{\sin\theta}{\cos\theta}\,d\theta = \int \frac{1}{w}\,dw = \ln|w| + C = \ln|\cos\theta| + C.$$

Therefore,

$$\int (\tan\theta)^3\,d\theta = \frac{1}{2}(\sec\theta)^2 + \ln|\cos\theta| + C. \quad \square$$

The examples show that certain patterns that arise in trig integrals are good, in the sense that they allow you to do a substitution which makes the integral easy. Here are some of the "good patterns":

- Lots of $\cos x$'s and a single $\sin x$.
- Lots of $\sin x$'s and a single $\cos x$.
- Lots of $\tan x$'s and a single $(\sec x)^2$.
- Lots of $\sec x$'s and a single $\sec x\tan x$.

You should aim for these patterns whenever possible.

If you have an integral involving sines and cosine in which all the powers are *even*, the method I just described usually won't work. Instead, it is better to apply the following **double angle formulas**:

$$(\sin\theta)^2 = \frac{1}{2}(1-\cos 2\theta)$$
$$(\cos\theta)^2 = \frac{1}{2}(1+\cos 2\theta)$$

Any even power of $\sin x$ or $\cos x$ can be expressed as a power of $(\sin x)^2$ or $(\cos x)^2$. Use the identities above to substitute for $(\sin x)^2$ or $(\cos x)^2$, and multiply out the result. The net effect is to **reduce the powers** that occur in the integral, while at the same time increasing the arguments ($x \to 2x$).

Example.

$$\int (\cos 5x)^2 (\sin 5x)^2\, dx = \int \left(\frac{1}{2}(1+\cos 10x)\right)\left(\frac{1}{2}(1-\cos 10x)\right)\, dx = \frac{1}{4}\int \left(1-(\cos 10x)^2\right)\, dx =$$

$$\frac{1}{4}\int (\sin 10x)^2\, dx = \frac{1}{4}\int \frac{1}{2}(1-\cos 20x)\, dx = \frac{1}{8}(x - \frac{1}{20}\sin 20x) + C. \quad \square$$

Example. It is usually *not* a good idea to use the double angle formulas with odd powers. Consider the following computation:

$$\int (\cos x)^3\, dx = \int (\cos x)^2 \cos x\, dx = \int \frac{1}{2}(1+\cos 2x)\cos x\, dx.$$

The integral can be done in this form, but you either need to apply one of the angle addition formulas to $\cos 2x \cos x$ or use integration by parts. The problem is that *having trig functions with different arguments in the same integral makes the integral a bit harder to do.*

It would have been better to do the integral by using the "odd power" technique:

$$\int (\cos x)^3\, dx = \int (\cos x)^2 \cos x\, dx = \int \left(1-(\sin x)^2\right)\cos x\, dx =$$

$$\left[u = \sin x, \quad du = \cos x\, dx, \quad dx = \frac{du}{\cos x}\right]$$

$$\int (1-u^2)\, du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}(\sin x)^3 + C. \quad \square$$

Trig Substitution

Trig substitution reduces certain integrals to integrals of trig functions. The idea is to match the given integral against one of the following trig identities:

$$1-(\sin\theta)^2=(\cos\theta)^2$$
$$1+(\tan\theta)^2=(\sec\theta)^2$$
$$(\sec\theta)^2-1=(\tan\theta)^2$$

- If the integral contains an expression of the form a^2-x^2, try a substitution based on the first identity: $x=a\sin\theta$.
- If the integral contains an expression of the form a^2+x^2, try a substitution based on the second identity: $x=a\tan\theta$.
- If the integral contains an expression of the form x^2-a^2, try a substitution based on the third identity: $x=a\sec\theta$.

If you don't obtain one of the identities above after substituting, you've probably used the wrong substitution.

Example.

$$\int(4-x^2)^{3/2}\,dx=\int\left(4-4(\sin\theta)^2\right)^{3/2}(2\cos\theta)\,d\theta=$$

$$[x=2\sin\theta,\quad dx=2\cos\theta\,d\theta]$$

$$\int\left(4(\cos\theta)^2\right)^{3/2}(2\cos\theta)\,d\theta=16\int(\cos\theta)^4\,d\theta=16\int\left((\cos\theta)^2\right)^2\,d\theta=$$

$$16\int\left(\frac{1}{2}(1+\cos 2\theta)\right)^2\,d\theta=4\int\left(1+2\cos 2\theta+(\cos 2\theta)^2\right)\,d\theta=$$

$$4\int\left(1+2\cos 2\theta+\left(\frac{1}{2}(1+\cos 4\theta)\right)\right)\,d\theta=4\left(\theta+\sin 2\theta+\frac{1}{2}(\theta+\frac{1}{4}\sin 4\theta)\right)+C=$$

$$6\theta+4\sin 2\theta+\frac{1}{2}\sin 4\theta+C.$$

To "match" the "4" in "$4-x^2$", I had to use $x=2\sin\theta$ (since $2^2=4$). I used the **double angle formula** to reduce the **even powers of cosine**.

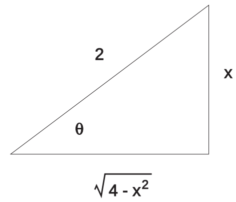
To put the x's back, I need to express everything in terms of trig functions of θ (as opposed to 2θ or 4θ). I use the **double angle formulas** for sine:

$$\sin 2\theta=2\sin\theta\cos\theta,\quad \sin 4\theta=2\sin 2\theta\cos 2\theta=2(2\sin\theta\cos\theta)\left(2(\cos\theta)^2-1\right)=4\sin\theta\cos\theta\left(2(\cos\theta)^2-1\right).$$

Therefore,

$$\int(4-x^2)^{3/2}\,dx=6\theta+8\sin\theta\cos\theta+2\sin\theta\cos\theta\left(2(\cos\theta)^2-1\right)=6\theta+6\sin\theta\cos\theta+4(\cos\theta)^3\sin\theta.$$

Now draw a right triangle which shows the substitution.



The triangle shows $\sin\theta = \dfrac{x}{2}$, and by Pythagoras the third side is $\sqrt{4-x^2}$. Therefore,

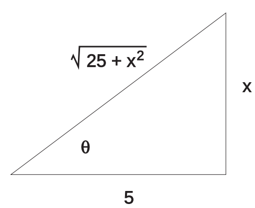
$$\int (1-x^2)^{3/2}\,dx = 6\arcsin\frac{x}{2} + \frac{3}{2}x\sqrt{4-x^2} + \frac{1}{4}x(4-x^2)^{3/2} + C. \quad \square$$

Example. Compute $\displaystyle\int \frac{dx}{\sqrt{25+x^2}}$.

$25+x^2$ looks like $1+(\tan\theta)^2$, so let $x = 5\tan\theta$. Then $dx = 5(\sec\theta)^2\,d\theta$, so

$$\int \frac{dx}{\sqrt{25+x^2}} = \int \frac{5(\sec\theta)^2\,d\theta}{\sqrt{25+25(\tan\theta)^2}} = \int \frac{5(\sec\theta)^2\,d\theta}{\sqrt{25(\sec\theta)^2}} = \int \frac{5(\sec\theta)^2\,d\theta}{5\sec\theta} = \int \sec\theta\,d\theta =$$

$$\ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C.$$



$\square$

Example. Compute $\displaystyle\int \frac{x\,dx}{\sqrt{25+x^2}}$.

This could be done using $x = 5\tan\theta$. But it's easier to do a u-substitution:

$$\int \frac{x\,dx}{\sqrt{25+x^2}} = \int \frac{x\cdot\dfrac{du}{2x}}{\sqrt{u}} = \frac{1}{2}\int\frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{25+x^2} + C.$$

$$\left[u = 25+x^2, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}\right] \quad \square$$

Example. Compute $\displaystyle\int \sqrt{x^2-4}\,dx$.

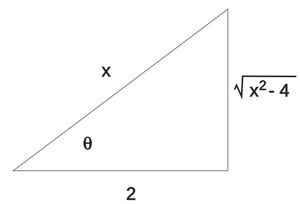
x^2-4 looks like $(\sec\theta)^2-1$, so let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$, and

$$\int \sqrt{x^2-4}\,dx = \int \sqrt{4(\sec\theta)^2-4}(2\sec\theta\tan\theta\,d\theta) = \int \sqrt{4(\tan\theta)^2}(2\sec\theta\tan\theta\,d\theta) =$$

$$\int (2\tan\theta)(2\sec\theta\tan\theta\,d\theta) = 4\int \sec\theta(\tan\theta)^2\,d\theta = 4\int \sec\theta\left((\sec\theta)^2 - 1\right)\,d\theta =$$

$$4\int (\sec\theta)^3\,d\theta - 4\int \sec\theta\,d\theta = 2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta| - 4\ln|\sec\theta + \tan\theta| + C =$$

$$2\sec\theta\tan\theta - 2\ln|\sec\theta + \tan\theta| = \frac{1}{2}x\sqrt{x^2 - 4} - 2\ln\left|\frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2}\right| + C.$$



□