King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 202 Exam I
Semester II, 2006- (052)
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Question #		Points
1		30
2		30
3		30
4		30
5		20
6		10
7		15
8		15
9		15
10		15
11		10
Total:		220

1. Solve: $xy' = 2xe^x - y + 6x^2$ (#13/page 73)
(Show all your work)

~~divid by x~~

$\cancel{y' = 2e^x - \frac{y}{x} + 6x}$

$$x \frac{dy}{dx} = 2xe^x - y + 6x^2$$

$$x\,dy = (2xe^x - y + 6x^2)\,dx$$

$$(2xe^x - y + 6x^2)\,dx - x\,dy = 0 \quad \text{———} (1)$$

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Equation (1) is $\boxed{\text{exact}}$ with

$$M(x,y) = 2xe^x - y + 6x^2, \quad N(x,y) = -x$$

$$\frac{\partial M}{\partial y} = -1 \quad \text{and} \quad \frac{\partial N}{\partial x} = -1 \Rightarrow (1) \text{ is Exact}$$

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$$f(x,y) = \int N(x,y)\,dy + g(x) = \int -x\,dy + g(x)$$

$$f(x,y) = -xy + g(x) \Rightarrow \frac{\partial f}{\partial x} = -y + g'(x)$$

Now, $\frac{\partial f}{\partial x} = M(x,y) \Rightarrow -y + g'(x) = 2xe^x - y + 6x^2$

$$\Rightarrow g'(x) = 2xe^x + 6x^2 \Rightarrow g(x) = (2x-2)e^x + 2x^3$$ 5

5

Hence, $f(x,y) = -xy + 2(x-1)e^x + 2x^3$ 5

Now, $\boxed{-xy + 2(x-1)e^x + 2x^3 = C}$ is the general Solution

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Another: $f = \int M\,dx + g(y) = 2(x-1)e^x - yx + 2x^3 + g(y)$

$\frac{\partial f}{\partial y} = -x + g'(y) = N \Rightarrow g'(y) = 0 \Rightarrow \boxed{g(y) = 0}$

$\boxed{f(x,y) = -xy + 2(x-1)e^x + 2x^3} \Rightarrow \boxed{-xy + 2(x-1)e^x + 2x^3 = c}$

2. Solve: $(3x+y)dy = (x+3y)dx$ (#7/page 78) —— (1)
(Show all your work. Hint: it is homog of degree 1)

Let $y = ux$ —— (2)

$dy = u\,dx + x\,du$ —— (3)

use (2) & (3) in (1)

$(3x+ux)(u\,dx + x\,du) = (x+3ux)\,dx$

$x(3+u)(u\,dx + x\,du) = x(1+3u)\,dx$

divid by x

$(3+u)(u\,dx + x\,du) = (1+3u)\,dx$

$\cancel{3u\,dx} + \underline{\underline{3x\,du}} + u^2\,dx + \underline{\underline{xu\,du}} = dx + \cancel{3u\,dx}$

$(3x + xu)\,du = (1-u^2)\,dx$

divid by $x(1-u^2)$

$$\frac{x(3+u)\,du}{x(1-u^2)} = \frac{(1-u^2)}{x(1-u^2)}\,dx$$

$$\left(\frac{3+u}{1-u^2}\right)du = \frac{1}{x}\,dx$$

$$\left[\frac{2}{1-u} + \frac{1}{1+u}\right]du = \frac{dx}{x}$$

integrate

$2\ln|1-u| + \ln|1+u| = \ln|x| + c$

$2\ln\left|1-\frac{y}{x}\right| + \ln\left|1+\frac{y}{x}\right| = \ln|x| + c$

3. Solve: $y^{\frac{1}{2}}\, y' + y^{\frac{3}{2}} - x^2 = 0$ ——— (1)
(Show all your work)

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multiply (1) by $y^{-1/2}$

$$y' + y - x^2 y^{-1/2} = 0 \Rightarrow \boxed{y' + y = x^2 y^{-1/2}} \text{ ——— } (2)$$

(2) is Bernoulli with $n = -1/2$, $p(x) = 1$, $f(x) = x^2$

Let $u = y^{1-n} \Rightarrow u = y^{1+\frac{1}{2}} \Rightarrow \boxed{u = y^{3/2}}$ (***)

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$$\underbrace{y = u^{2/3}}_{(*)} \Rightarrow \underbrace{y' = \frac{2}{3} u^{-1/3} \cdot u'}_{(**)}$$

use (*) and (**) in (2):

$$\frac{2}{3} u^{-1/3} \cdot u' + u^{2/3} = x^2 \cdot u^{-1/3} \text{ ——— } (3)$$

multiply (3) by $\frac{3}{2} u^{1/3}$

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$$\boxed{u' + \frac{3}{2} u = x^2} \text{ ——— } (4)$$ is linear in u.

integrating facter $= u_1 = e^{\frac{3}{2}\int dx} = e^{\frac{3}{2}x}$

multiply (4) by $e^{3/2 x}$

$$e^{3/2 x} u' + \frac{3}{2} e^{3/2 x} u = x^2 e^{3/2 x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{\frac{3}{2}x} \cdot u \right] = x^2 e^{\frac{3}{2}x}$$ 5

integrate

$$e^{\frac{3}{2}x} \cdot u = \int x^2 e^{\frac{3}{2}x} . dx$$

$$e^{\frac{3}{2}x} \cdot u = \left(\frac{2}{3}x^2 - \frac{8}{9}x + \frac{16}{27}\right) e^{\frac{3}{2}x} + c$$

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$$\Rightarrow u = \frac{2}{3}x^2 - \frac{8}{9}x + \frac{16}{27} + c e^{-\frac{3}{2}x}$$ use $u = y^{3/2}$ give

$$\Rightarrow y^{3/2} = \frac{2}{3}x^2 - \frac{8}{9}x + \frac{16}{27} + c e^{-3/2 x}$$ 5

Aside:

$$\begin{array}{lcl} & & e^{3/2 x} \\ x^2 & \xrightarrow{+} & \frac{2}{3} e^{3/2 x} \\ 2x & \xrightarrow{-} & 4/9\, e^{3/2 x} \\ 2 & \xrightarrow{+} & \frac{8}{27} e^{3/2 x} \\ 0 & & \end{array}$$

$\int x^2 e^{\frac{3}{2}x} dx =$

$\left(\frac{2}{3}x^2 - \frac{8}{9}x + \frac{16}{27}\right) e^{\frac{3}{2}x} + c$

4. Solve: $(2x + y + 1)y' = 1$ (#14/page 85)
(Show all your work)

$$y' = \frac{1}{2x+y+1} \quad \text{is in the form } y' = f(Ax+By+c) \longrightarrow (1)$$

Let $u = 2x + y + 1$ 5

$$\Rightarrow y = u - 2x - 1 \quad \text{——— } (2)$$

$$\Rightarrow y' = u' - 2 \quad \text{——— } (3)$$

use (2) & (3) in (1) :

$$u' - 2 = \frac{1}{u}$$

$$\Rightarrow u' = \frac{1}{u} + 2 \quad \Rightarrow u' = \frac{1+2u}{u}$$

$$\Rightarrow du = \left(\frac{1+2u}{u}\right) dx$$

$$\Rightarrow \left(\frac{u}{1+2u}\right) du = dx$$ 10 separable

Aside:

$$\begin{array}{r} \tfrac{1}{2} \\ 1+2u \overline{)\, u \qquad} \\ \mp u \mp \tfrac{1}{2} \\ \hline -\tfrac{1}{2} \end{array}$$

$$\frac{u}{1+2u} = \frac{1}{2} - \frac{1}{2}\frac{1}{1+2u}$$

$$\int \frac{u}{1+2u} du = \frac{1}{2}u - \frac{1}{4}\ln|1+2u| +$$

integrate

$$\int \frac{u}{1+2u} du = \int dx$$

$$\frac{1}{2}u - \frac{1}{4}\ln|1+2u| + C_1 = x + C_2$$

$$\Rightarrow 2u - \ln|1+2u| + 4C_1 = 4x + 4C_2$$

$$2u - \ln|1+2u| - 4x = C_3$$ 10

Hence, $\boxed{2(2x+y+1) - \ln|4x+2y+3| - 4x = C}$ 5

is one-parameter family of solutions

5. Use an appropriate substitution to reduce the DE

$$y' = -x^4 + \frac{2}{x}y + y^2 \tag{1}$$

into a linear DE. Write the new DE in the following form

$$u' + p(x)u = f(x) \tag{2}$$

where $y_1 = x^2$ is a known solution of the DE. [Note: Just reduce it to linear DONOT SOLVE]

(1) is Riccati equation.

Let $y = y_1 + \frac{1}{u}$

$y = x^2 + \frac{1}{u}$ ——— (3) 5

$\Rightarrow y' = 2x - \frac{1}{u^2}u'$ ——— (4) 5

use (3) & (4) in (1):

$$2x - \frac{1}{u^2}u' = -x^4 + \frac{2}{x}\left(x^2 + \frac{1}{u}\right) + \left(x^2 + \frac{1}{u}\right)^2$$

$$2x - u^{-2}u' = -x^4 + 2x + \frac{2}{x}u^{-1} + x^4 + 2x^2u^{-1} + u^{-2}$$

$$-u^{-2}\cdot u' = \left(\frac{2}{x} + 2x^2\right)u^{-1} + u^{-2}$$

multiply by $-u^2$

$$u' = -\left(\frac{2}{x} + 2x^2\right)u - 1$$ 10

$$\Rightarrow \boxed{u' + \left(\frac{2}{x} + 2x^2\right)u = -1} \text{ ——— (5)}$$

(5) is linear DE with

$p(x) = \frac{2}{x} + 2x^2$ and $f(x) = -1$

6. $y = 2\frac{1+c\ e^{4x}}{1-c\ e^{4x}}$ is a one-parameter family of solutions of the first-order DE $y' = y^2 - 4$. Which one of the following statements is TRUE.

(a) $y = 2$ is a singular solution

(b) $y = 2$ is a trivial solution

(c) $y = 0$ is a trivial solution

$\boxed{\text{(d)}}$ $y = -2$ is a singular solution

(e) $y = 0$ is a particular solution

$y = -2$ is a solution.

Check: $LHS = y' = \frac{d}{dx}[-2] = 0$

$RHS = y^2 - 4 = (-2)^2 - 4 = 4 - 4 = 0$

but no value for c so that $-2 = 2\frac{1+ce^{4x}}{1-ce^{4x}}$

Hence, $y = -2$ is not a member of the famil

$\Rightarrow y = -2$ is singular solution.

7. The DE

$$y^2\ x^{\frac{3}{2}}dx + y^2x^{\frac{3}{2}}dy = xy^{\frac{9}{2}}dy \tag{3}$$

is classified as

(a) separable

(b) linear in y

(c) linear in x

(d) exact

(e) made exact

(f) homog. of degree α

(g) Bernoulli in y

(h) Bernoulli in x

(i) $y' = f(Ax + By + C)$

(j) Riccati in y

(k) Riccati in x

None

8. Find an appropriate integrating factor which make the non-exact DE

$$6xydx + (4y + 9x^2) = 0 \tag{4}$$

an exact DE.

(a) y^6

(b) x^2

(c) y^{-2}

(d) $12x$

$\boxed{\text{(e)}}$ y^2

$M = 6xy$, $N = 4y + 9x^2$

$M_y = 6x$, $N_x = 18x$

observe that: $\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{12x}{6xy} = \frac{2}{y}$

$=$ function of y alone.

integrating factor $= \mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$

$= e^{\int \frac{2}{y} dy} = e^{\ln y^2} = y^2$

9. If $y(x)$ is the solution of the IVP

$$x^2y' = y(1-x), \qquad y(-1) = -1 \tag{5}$$

Then $y(2) =$ [Note: equation (5) is separable]

$\boxed{\text{(a)}}$ $\frac{1}{2}e^{-3/2}$
(b) $-\frac{1}{2}e^{-1/2}$
(c) $\frac{1}{2}e^{3/2}$
(d) 0
(e) $\frac{1}{2}e^{-1/2}$

10. Determine a region of the xy plane for which the differential equation

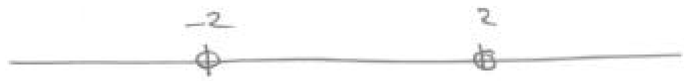
$$y' = \frac{y^2+4}{x^2-4} \tag{6}$$

would have a unique solution.

$$f(x,y) = \frac{y^2+4}{x^2-4} = \left(\frac{1}{x^2-4}\right)y^2 + \frac{4}{x^2-4}$$

$$\frac{\partial f}{\partial y} = \frac{2}{x^2-4}\,y = \frac{2y}{x^2-4}$$

(a) $(-4,4)$
(b) $(0,+\infty)$
(c) $(-\infty,0)$
$\boxed{\text{(d)}}$ $(4,+\infty)$
(e) $(-\infty,4)$



$f(x,y)$ and $\frac{\partial f}{\partial y}$ ~~continuous~~ discont at $x = \pm 2$

The interval ~~contains~~ $(4,+\infty)$ does not contain $x = \pm 2$.

(5) Can be written as $\frac{dy}{y} = \frac{1-x}{x^2}\,dx$ separable

integrate $\Rightarrow c_1 + \ln|y| = -\frac{1}{x} - \ln|x| + c_2$

$\Rightarrow \ln|y| + \ln|x| = c_3 - \frac{1}{x} \Rightarrow \ln|xy| = c_3 - \frac{1}{x}$

$\Rightarrow |xy| = e^{c_3 - \frac{1}{x}} \Rightarrow |xy| = e^{c_3}\cdot e^{-1/x} \Rightarrow xy = \pm e^{c_3}\cdot e^{-1/x}$

$\Rightarrow xy = Ce^{-1/x} \Rightarrow y = \frac{Ce^{-1/x}}{x}$ ——— (*)

Now, $y(-1) = -1 \Rightarrow -1 = \frac{Ce^{-1/-1}}{-1} \Rightarrow -1 = \frac{Ce}{-1} \Rightarrow C = \frac{1}{e} = e^{-1}$

(*) and $C = e^{-1} \Rightarrow y = \frac{e^{-1}\cdot e^{-1/x}}{x} \Rightarrow \boxed{y = \frac{e^{-(1+\frac{1}{x})}}{x}}$ is the sol of the IVP

Now, $y(2) = \frac{e^{-(1+\frac{1}{2})}}{2} = \frac{e^{-\frac{3}{2}}}{2} = \frac{1}{2}e^{-3/2}$

Aside:

$\frac{1-x}{x^2} = \frac{1}{x^2} - \frac{1}{x}$

$\int \frac{1-x}{x^2}\,dx = -\frac{1}{x} - \ln|x| + c_2$

11. Give an example of an exact linear first-order DE where $y = 2x$ is a particular solution.

<u>one example</u>:

$$y = 2x \Rightarrow y' = 2$$

Consider the DE: $\boxed{y' = 2}$

* $y = 2x$ is a solution
* it is first-order
* $y' = 2$ is linear.
  with $p(x) = 0$, $f(x) = 2$
  $y' + 0 \cdot y = 2$
* $y' = 2$ is exact
  $y' = 2 \Rightarrow \frac{dy}{dx} = 2 \Rightarrow dy = 2dx$
  $\Rightarrow 2dx - dy = 0$
  $M = 2$, $N = -1$
  $M_y = 0$, $N_x = 0$.
  $\Rightarrow M_y = N_x$.