Optimality Conditions and Duality in Nondifferentiable Minimax Fractional Programming with Generalized Convexity[1]

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Abstract. We establish sufficient optimality conditions for a class of nondifferentiable minimax fractional programming problems involving (F, α, ρ, d)-convexity. Subsequently, we apply the optimality conditions to formulate two types of dual problems and prove appropriate duality theorems.

Key Words. Minimax fractional programming, sublinear functionals, optimality conditions, duality, generalized convexity.

1. Introduction

Schmitendorf (Ref. 1) established necessary and sufficient optimality conditions for a minimax problem. Tanimoto (Ref. 2) applied these optimality conditions to define a dual problem and derived duality theorems, which were extended for the fractional analogue of a generalized minimax problem by Yadav and Mukherjee (Ref. 3). In Ref. 3, Yadav and Mukherjee employed also the optimality conditions of Ref. 1 to construct two dual problems and derived duality theorems for differentiable fractional minimax programming. Chandra and Kumar (Ref. 4) pointed out certain omissions and inconsistencies in the formulation of Yadav and Mukherjee (Ref. 3); they constructed two modified dual problems and proved duality theorems for differentiable fractional minimax programming. Bector and Bhatia (Ref. 5) and Weir (Ref. 6) relaxed the convexity assumptions in the sufficient optimality conditions of Ref. 1, employed the optimality conditions to construct several dual problems which involve pseudoconvex and quasiconvex functions, and discussed weak and strong duality theorems. In Ref. 7, Zalmai

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used an infinite-dimensional version of the Gordan theorem of the alternative to derive first-order and second-order necessary optimality conditions for a class of minimax programming problems in a Banach space; he established several sufficient optimality conditions and duality formulations under generalized invexity assumptions. Liu and Wu (Ref. 8) and Ahmad (Ref. 9) recently derived sufficient optimality conditions and duality theorems for minimax fractional programming in the framework of (F, ρ)-convex functions and ρ-invex functions respectively.

Motivated by various concepts of generalized convexity, Liang et al. (Refs. 10, 11) introduced a unified formulation of generalized convexity, which was called (F, α, ρ, d)-convexity and obtained some corresponding optimality conditions and duality results for both single-objective fractional problems and multiobjective problems. Recently, Liang and Shi (Ref. 12) obtained sufficient conditions and duality theorems for minimax fractional problem under (F, α, ρ, d)-convexity. Lai et al. (Ref. 13) derived necessary and sufficient conditions for nondifferentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct a parametric dual model and also discussed duality theorems. Lai and Lee (Ref. 14) obtained duality theorems for two parameter-free dual models of nondifferentiable minimax fractional problem involving generalized convexity assumptions. Recently, in view of generalized univexity, Mishra et al. (Ref. 15) extended the results of Refs. 13–14.

In this paper, motivated by Liang et al. (Ref. 10), Lai et al. (Ref. 13), and Lai and Lee (Ref. 14), we establish sufficient conditions for a nondifferentiable minimax fractional programming problem with (F, α, ρ, d)-convexity. When the sufficient conditions are utilized, two dual models can be formulated and the usual duality results are presented. In view of the generalized convexity, we extend the results of Refs. 8, 9 and 12–14.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we establish sufficient optimality conditions. Duality results are presented in Sections 4 and 5.

2. Notations and Preliminary Results

Let R^n be the n-dimensional Euclidean space and let X be an open set in R^n.

Definition 2.1.   A functional $F : X \times X \times R^n \to R$ is said to be sublinear if, $\forall x, \bar{x} \in X$,

(i) $F(x, \bar{x}; a_1 + a_2) \leq F(x, \bar{x}; a_1) + F(x, \bar{x}; a_2), \quad \forall a_1, a_2 \in R^n$,
(ii) $F(x, \bar{x}; \alpha a) = \alpha F(x, \bar{x}; a), \quad \forall \alpha \in R_+, a \in R^n$.

The concept of sublinear functional was given in Ref. 16. By (ii), it is clear that $F(x, \bar{x}; 0) = 0$.

Based upon the concept of sublinear functional, we recall a unified formulation of generalized convexity [i.e., (F, α, ρ, d)-convexity], which was introduced by Liang et al. (Ref. 10) as follows.

Definition 2.2. Let $F : X \times X \times R^n \to R$ be a sublinear functional; let the function $\zeta : X \to R$ be differentiable at $\bar{x} \in X, \alpha : X \times X \to R_+ \backslash \{0\}, \rho \in R$, and $d(., .) : X \times X \to R$. The function ζ is said to be (F, α, ρ, d)-convex at $\bar{x}$ if

$$\zeta(x) - \zeta(\bar{x}) \geq F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) + \rho d^2(x, \bar{x}), \quad \forall x \in X.$$

The function ζ is said to be (F, α, ρ, d)-convex over X if it is (F, α, ρ, d)-convex at $\bar{x}, \forall x \in X; \zeta$ is said to be strongly (F, α, ρ, d)-convex or (F, ρ)-convex if $\rho > 0$ or $\rho = 0$, respectively.

Special Cases. From Definition 2.2, there are the following special cases.

(I)   If $\alpha(x, \bar{x}) = 1$, for all $x, \bar{x} \in X$, then the (F, α, ρ, d)-convexity is the (F, ρ)-convexity defined in Ref. 16.

(II) If $F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) = \nabla \zeta(\bar{x})^t \eta(x, \bar{x})$ for a certain map $\eta : X \times X \to R^n$, then the (F, α, ρ, d)-convexity is the ρ-invexity of Ref. 17.

(III) If $\rho = 0$ or $d(x, \bar{x}) = 0$ for all $x, \bar{x} \in X$ and if $F(x, \bar{x}; \alpha(x, \bar{x}) \nabla \zeta(\bar{x})) = \nabla \zeta(\bar{x})^t \eta(x, \bar{x})$ for a certain map $\eta : X \times X \to R^n$, then the (F, α, ρ, d)-convexity reduces to the invexity introduced in Ref. 18.

We consider now the following nondifferentiable minimax fractional programming problem:

$$(P) \quad \min_{x \in R^n} \quad \sup_{y \in Y} \frac{f(x, y) + (x^t B x)^{1/2}}{h(x, y) - (x^t D x)^{1/2}},$$

$$\text{s.t.} \quad g(x) \leq 0, \quad x \in X,$$

where Y is a compact subset of R^m, $f, h : R^n \times R^m \to R$ are C^1 functions on $R^n \times R^m$, and $g : R^n \to R^p$ is a C^1 function on R^n; B and D are $n \times n$ positive-semidefinite matrices.

Let

$$S = \{x \in X : g(x) \leq 0\}$$

denote the set of all feasible solutions of (P). Any point $x \in S$ is called the feasible point of (P). For each $(x, y) \in R^n \times R^m$, we define

$$\phi(x, y) = \frac{f(x, y) + (x^t B x)^{1/2}}{h(x, y) - (x^t D x)^{1/2}},$$

such that, for each $(x, y) \in S \times Y$,

$$f(x, y) + (x^t B x)^{1/2} \geq 0 \quad \text{and} \quad h(x, y) - (x^t D x)^{1/2} > 0.$$

For each $x \in S$, we define

$$J(x) = \{j \in J : g_j(x) = 0\},$$

where

$$J = \{1, 2, \ldots, p\},$$

$$Y(x) = \left\{ y \in Y : \frac{f(x, y) + (x^t B x)^{1/2}}{h(x, y) - (x^t D x)^{1/2}} = \sup_{z \in Y} \frac{f(x, z) + (x^t B x)^{1/2}}{h(x, z) - (x^t D x)^{1/2}} \right\}.$$

$$K(x) = \left\{ (s, t, \tilde{y}) \in N \times R_+^s \times R^{ms} : 1 \leq s \leq n+1, t = (t_1, t_2, \ldots, t_s) \in R_+^s, \right.$$

$$\left. \text{with } \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s), \text{ with } \bar{y}_i \in Y(x), i = 1, 2, \ldots, s \right\}.$$

Since f and h are continuously differentiable and Y is compact in R^m, it follows that, for each $x^* \in S$, $Y(x^*) \neq \emptyset$ and for any $\bar{y}_i \in Y(x^*)$, we have a positive constant

$$k_\circ = \phi(x^*, \bar{y}_i) = \frac{f(x^*, \bar{y}_i) + (x^{*t} B x^*)^{1/2}}{h(x^*, \bar{y}_i) - (x^{*t} D x^*)^{1/2}}.$$

We shall need the following generalized Schwartz inequality.

Let B be a positive-semidefinite matrix of order n. Then, for all $x, w \in R^n$,

$$x^t B w \leq (x^t B x)^{1/2} (w^t B w)^{1/2}. \tag{1}$$

We observe that equality holds if $Bx = \lambda B w$ for some $\lambda \geq 0$. Evidently, if $(w^t B w)^{1/2} \leq 1$, we have

$$x^t B w \leq (x^t B x)^{1/2}.$$

If the functions f, g, h in problem (P) are continuously differentiable with respect to $x \in R^n$, then Lai et al. (Ref. 13) derived the following necessary conditions for optimality of (P). In what follows, ∇ stands for the gradient vector with respect to x.

Theorem 2.1. Necessary Conditions. If x^* is a solution of problem (P) satisfying $x^{*t} B x^* > 0$, $x^{*t} D x^* > 0$ and if $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent, then there exist $(s, t^*, \bar{y}) \in K(x^*)$, $k_\circ \in R_+$, $w, v \in R^n$, and $\mu^* \in R_+^p$ such that

$$\sum_{i=1}^{s} t_i^* \{\nabla f(x^*, \bar{y}_i) + Bw - k_\circ(\nabla h(x^*, \bar{y}_i) - Dv)\} + \nabla \sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \tag{2}$$

$$f(x^*, \bar{y}_i) + (x^{*t} B x^*)^{1/2} - k_\circ(h(x^*, \bar{y}_i) - (x^{*t} D x^*)^{1/2}) = 0, \quad i = 1, 2, \ldots, s, \tag{3}$$

$$\sum_{j=1}^{p} \mu_j^* g_j(x^*) = 0, \tag{4}$$

$$t_i^* \geq 0, \quad i = 1, 2, \ldots, s, \quad \sum_{i=1}^{s} t_i^* = 1, \tag{5}$$

$$w^t B w \leq 1, \quad v^t D v \leq 1, \tag{6a}$$

$$(x^{*t} B x^*)^{1/2} = x^{*t} B w, \tag{6b}$$

$$(x^{*t} D x^*)^{1/2} = x^{*t} D v. \tag{6c}$$

Remark 2.1. In the above theorem, both matrices B and D are positive semidefinite. If either $x^{*t} B x^*$ or $x^{*t} D x^*$ is zero, or if both B and D are singular, then for $(s, t^*, \bar{y}) \in K(x^*)$, we can take a set $Z_{\bar{y}}(x^*)$ as defined in Ref. 13 by

$$Z_{\bar{y}}(x^*) = \{ z \in R^n : z^t \nabla g_j(x^*) \leq 0, j \in J(x^*)$$

satisfying one of the following conditions (i), (ii), (iii)}:

(i) $\quad x^{*t} B x^* > 0, \; x^{*t} D x^* = 0$

$$\Rightarrow z^t \left(\sum_{i=1}^{s} t_i^* \{ \nabla f(x^*, \bar{y}_i) + B x^* / (x^{*t} B x^*)^{1/2} \right.$$

$$\left. - k_\circ \nabla h(x^*, \bar{y}_i) \} \right) + \left(z^t (k_\circ^2 D) z \right)^{1/2} < 0,$$

(ii) $\quad x^{*t} B x^* = 0, \; x^{*t} D x^* > 0$

$$\Rightarrow z^t \left(\sum_{i=1}^{s} t_i^* \{ \nabla f(x^*, \bar{y}_i) - k_\circ (\nabla h(x^*, \bar{y}_i) \right.$$

$$\left. - D x^* / (x^{*t} D x^*)^{1/2}) \} \right) + (z^t B z)^{1/2} < 0,$$

(iii) $\quad x^{*t} B x^* = 0, \quad x^{*t} D x^* = 0$

$$\Rightarrow z^t \left(\sum_{i=1}^{s} t_i^* \{ \nabla f(x^*, \bar{y}_i) - k_\circ \nabla h(x^*, \bar{y}_i) \} \right)$$

$$+ \left(z^t (k_\circ^2 D) z \right)^{1/2} + (z^t B z)^{1/2} < 0.$$

If we take $Z_{\bar{y}}(x^*) = \emptyset$ in Theorem 2.1, then the result of Theorem 2.1 still holds.

3. Sufficient Conditions

We establish now the sufficient conditions for optimality of (P) under the assumptions of (F, α, ρ, d)-convexity.

Theorem 3.1. Sufficient Conditions. Let x^* be a feasible solution of (P) and let there exist a positive integer s, $1 \le s \le n + 1$, $t^* \in R^s_+$, $\bar{y}_i \in Y(x^*)$, $i = 1, 2, \ldots s$, $k_\circ \in R_+$, $w, v \in R^n$, and $\mu^* \in R^p_+$ satisfying relations (2)–(6). If $f(., \bar{y}_i) + (.)^t Bw$ is (F, α, ρ_i, d_i)-convex, if $-h(., \bar{y}_i) + (.)^t Dv$ is $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convex, and if $g_j(.)$, for $j = 1, 2, \ldots, p$ is (F, β_j, ν_j, c_j)-convex at x^* and

$$\sum_{i=1}^{s} t_i^* \left\{ \frac{\rho_i d_i^2(x, x^*)}{\alpha(x, x^*)} + k_\circ \frac{\bar{\rho}_i \bar{d}_i^2(x, x^*)}{\alpha(x, x^*)} \right\} + \sum_{j=1}^{p} \mu_j^* \nu_j \frac{c_j^2(x, x^*)}{\beta_j(x, x^*)} \ge 0, \qquad (7)$$

then x^* is a global optimal solution of (P).

Proof. Suppose to the contrary that x^* is not an optimal solution of (P). Then, there exists an $x \in S$ such that

$$\sup_{y \in Y} \frac{f(x, y) + (x^t Bx)^{1/2}}{h(x, y) - (x^t Dx)^{1/2}} < \sup_{y \in Y} \frac{f(x^*, y) + (x^{*t} Bx^*)^{1/2}}{h(x^*, y) - (x^{*t} Dx^*)^{1/2}}.$$

We note that

$$\sup_{y \in Y} \frac{f(x^*, y) + (x^{*t} Bx^*)^{1/2}}{h(x^*, y) - (x^{*t} Dx^*)^{1/2}} = \frac{f(x^*, \bar{y}_i) + (x^{*t} Bx^*)^{1/2}}{h(x^*, \bar{y}_i) - (x^{*t} Dx^*)^{1/2}} = k_\circ,$$

for $\bar{y}_i \in Y(x^*)$, $i = 1, 2, \ldots, s$, and

$$\frac{f(x, \bar{y}_i) + (x^t Bx)^{1/2}}{h(x, \bar{y}_i) - (x^t Dx^*)^{1/2}} \le \sup_{y \in Y} \frac{f(x, y) + (x^t Bx)^{1/2}}{h(x, y) - (x^t Dx)^{1/2}}.$$

Therefore, we have

$$\frac{f(x, \bar{y}_i) + (x^t Bx)^{1/2}}{h(x, \bar{y}_i) - (x^t Dx)^{1/2}} < k_\circ, \quad \text{for } i = 1, 2, \ldots, s.$$

It follows that

$$f(x, \bar{y}_i) + (x^t Bx)^{1/2} - k_\circ[h(x, \bar{y}_i) - (x^t Dx)^{1/2}] < 0, \quad \text{for } i = 1, 2, \ldots, s. \tag{8}$$

From relations (1), (3), (5), (6), (8), we obtain

$$\phi_\circ(x) = \sum_{i=1}^{s} t_i^* \{ f(x, \bar{y}_i) + x^t Bw - k_\circ(h(x, \bar{y}_i) - x^t Dv) \}$$

$$\le \sum_{i=1}^{s} t_i^* \{ f(x, \bar{y}_i) + (x^t Bx)^{1/2} - k_\circ(h(x, \bar{y}_i) - (x^t Dx)^{1/2}) \}$$

$$< 0 = \sum_{i=1}^{s} t_i^* \{ f(x^*, \bar{y}_i) + (x^{*t} Bx^*)^{1/2} - k_\circ(h(x^*, \bar{y}_i) - (x^{*t} Dx^*)^{1/2}) \}$$

$$= \sum_{i=1}^{s} t_i^* \{f(x^*, \bar{y}_i) + x^{*t} B w - k_\circ(h(x^*, \bar{y}_i) - x^{*t} Dv)\} = \phi_\circ(x^*).$$

It follows that

$$\phi_\circ(x) < \phi_\circ(x^*). \tag{9}$$

We use the (F, α, ρ_i, d_i)-convexity of $f(., \bar{y}_i) + (.)^t B w$ and the $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convexity of $-h(., \bar{y}_i) + (.)^t Dv$ at x^* for $i = 1, 2, \ldots, s$, i.e.,

$$f(x, \bar{y}_i) + x^t B w - f(x^*, \bar{y}_i) - x^{*t} B w$$
$$\geq F(x, x^*; \alpha(x, x^*)(\nabla f(x^*, \bar{y}_i) + B w)) + \rho_i d_i^2(x, x^*), \quad i = 1, 2, \ldots, s, \tag{10}$$

and

$$-h(x, \bar{y}_i) + x^t Dv + h(x^*, \bar{y}_i) - x^{*t} Dv$$
$$\geq F(x, x^*; \alpha(x, x^*)(-\nabla h(x^*, \bar{y}_i) + Dv)) + \bar{\rho}_i \bar{d}_i^2(x, x^*), \quad i = 1, 2, \ldots, s. \tag{11}$$

Multiplying (10) by t_i^*, (11) by $t_i^* k_\circ$, and then summing up these inequalities, using the sublinearity of F, we have [by (9)]

$$F\left(x, x^*; \alpha(x, x^*) \sum_{i=1}^{s} t_i^* \{\nabla f(x^*, \bar{y}_i) + B w - k_\circ(\nabla h(x^*, \bar{y}_i) - Dv)\}\right)$$
$$+ \sum_{i=1}^{s} t_i^* \{\rho_i d_i^2(x, x^*) + k_\circ \bar{\rho}_i \bar{d}_i^2(x, x^*)\} \leq \phi_\circ(x) - \phi_\circ(x^*) < 0.$$

Since $\alpha(x, x^*) > 0$, by the sublinearity of F, we obtain

$$F\left(x, x^*; \sum_{i=1}^{s} t_i^* \{\nabla f(x^*, \bar{y}_i) + B w - k_\circ(\nabla h(x^*, \bar{y}_i) - Dv)\}\right)$$
$$+ \sum_{i=1}^{s} t_i^* \left\{ \frac{\rho_i d_i^2(x, x^*)}{\alpha(x, x^*)} + k_\circ \frac{\bar{\rho}_i \bar{d}_i^2(x, x^*)}{\alpha(x, x^*)} \right\} < 0. \tag{12}$$

On the other hand, by the (F, β_j, ν_j, c_j)-convexity of $g_j(.)$ for $j = 1, 2, \ldots, p$, we have

$$g_j(x) - g_j(x^*) \geq F(x, x^*; \beta_j(x, x^*)\nabla g_j(x^*)) + \nu_j c_j^2(x, x^*),$$
$$j = 1, 2, \ldots, p.$$

By $\mu^* \geq 0$, $\beta(x, x^*) > 0$, and the sublinearity of F,

$$\sum_{j=1}^{p} \mu_j^* \frac{g_j(x) - g_j(x^*)}{\beta_j(x, x^*)}$$

$$\geq F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right) + \sum_{j=1}^{p} \mu_j^* v_j \frac{c_j^2(x, x^*)}{\beta_j(x, x^*)}. \tag{13}$$

Since the feasibility of x, $\beta(x, x^*) > 0$, and (4) imply that

$$\sum_{j=1}^{p} \mu_j^* \frac{g_j(x) - g_j(x^*)}{\beta_j(x, x^*)} \leq 0,$$

then (13) leads to

$$F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right) + \sum_{j=1}^{p} \mu_j^* v_j \frac{c_j^2(x, x^*)}{\beta_j(x, x^*)} \leq 0. \tag{14}$$

From (2), (12), (14) and the sublinearity of F, we have [by (7)]

$$0 = F\left(x, x^*; \sum_{i=1}^{s} t_i^* \{\nabla f(x^*, \bar{y}_i) + Bw - k_\circ(\nabla h(x^*, \bar{y}_i) - Dv)\}\right.$$

$$\left. + \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right)$$

$$\leq F\left(x, x^*; \sum_{i=1}^{s} t_i^* \{\nabla f(x^*, \bar{y}_i) + Bw - k_\circ(\nabla h(x^*, \bar{y}_i) - Dv)\}\right)$$

$$+ F\left(x, x^*; \sum_{j=1}^{p} \mu_j^* \nabla g_j(x^*)\right)$$

$$< -\sum_{i=1}^{s} t_i^* \left\{\frac{\rho_i d_i^2(x, x^*)}{\alpha(x, x^*)} + k_\circ \frac{\bar{\rho}_i \bar{d}_i^2(x, x^*)}{\alpha(x, x^*)}\right\} - \sum_{j=1}^{p} \mu_j^* v_j \frac{c_j^2(x, x^*)}{\beta_j(x, x^*)}$$

$$\leq 0.$$

Thus, we have a contradiction. Hence, the proof is complete. $\square$

Corollary 3.1. Let x^* be a feasible solution of (P) and let there exist a positive integer s, $1 \leq s \leq n + 1$, $t^* \in R_+^s$, $\bar{y}_i \in Y(x^*)$, $i = 1, 2, \ldots, s$, $k_\circ \in R_+$, $w, v \in R^n$, and $\mu^* \in R_+^p$ satisfying the relations (2)–(6). If $f(., \bar{y}_i) + (.)^t Bw$ is strongly (F, α, ρ_i, d_i)-convex or (F, α)-convex, if $-h(., \bar{y}_i) + (.)^t Dv$ is strongly $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convex or (F, α)-convex, and if $g_j(.)$, for $j = 1, 2, \ldots, p$, is strongly

(F, β_j, ν_j, c_j)-convex or (F, β_j)-convex at x^*, then x^* is a global optimal solution of (P).

Proof. Under the assumptions of this corollary, we know that inequality, holds [by (7)]. Therefore, x^* is a global optimal solution of (P). □

4. Duality Model I

In this section, we consider the following dual problem to (P):

(DI) $\quad \max\limits_{(s,t,\bar{y}) \in K(z)} \sup\limits_{(z,\mu,k,v,w) \in H_1(s,t,\bar{y})} k,$

where $H_1(s, t, \bar{y})$ denotes the set of all $(z, \mu, k, v, w) \in R^n \times R_+^p \times R_+ \times R^n \times R^n$ satisfying

$$\sum_{i=1}^{s} t_i \{\nabla f(z, \bar{y}_i) + Bw - k(\nabla h(z, \bar{y}_i) - Dv)\} + \nabla \sum_{j=1}^{p} \mu_j g_j(z) = 0, \quad (15)$$

$$\sum_{i=1}^{s} t_i \{f(z, \bar{y}_i) + z^t Bw - k(h(z, \bar{y}_i) - z^t Dv)\} \geq 0, \quad (16)$$

$$\sum_{j=1}^{p} \mu_j g_j(z) \geq 0, \quad (17)$$

$$(s, t, \bar{y}) \in K(z), \quad (18)$$

$$w^t Bw \leq 1, \quad v^t Dv \leq 1. \quad (19)$$

For a triplet $(s, t, \bar{y}) \in K(z)$, if the set $H_1(s, t, \bar{y}) = \emptyset$, then we define the supremum over it to be $-\infty$. Throughout this section, we denote

$$\phi_1(.) = \sum_{i=1}^{s} t_i \{f(., \bar{y}_i) + (.)^t Bw - k(h(., \bar{y}_i) - (.)^t Dv)\}.$$

Now, we derive the following weak, strong, and strict converse duality theorems.

Theorem 4.1. Weak Duality. Let x and $(z, \mu, k, v, w, s, t, \bar{y})$ be the feasible solutions of (P) and (DI) respectively. Suppose that $f(., \bar{y}_i) + (.)^t Bw$ and $-h(., \bar{y}_i) + (.)^t Dv$, for $i = 1, 2, \ldots, s$, are respectively (F, α, ρ_i, d_i)-convex and $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convex at z. Also, let $g_j(.)$, for $j = 1, 2, \ldots, p$, be (F, β_j, ν_j, c_j)-convex at z and let the inequality

$$\sum_{i=1}^{s} t_i \left\{ \frac{\rho_i d_i^2(x, z)}{\alpha(x, z)} + k \frac{\bar{\rho}_i \bar{d}_i^2(x, z)}{\alpha(x, z)} \right\} + \sum_{j=1}^{p} \mu_j \nu_j \frac{c_j^2(x, z)}{\beta_j(x, z)} \geq 0 \quad (20)$$

hold. Then,

$$\sup_{y \in Y} \frac{f(x, y) + (x^t B x)^{1/2}}{h(x, y) - (x^t D x)^{1/2}} \geq k.$$

Proof.   Suppose to the contrary that

$$\sup_{y \in Y} \frac{f(x, y) + (x^t B x)^{1/2}}{h(x, y) - (x^t D x)^{1/2}} < k.$$

Therefore, we get the following relation:

$$f(x, \bar{y}_i) + (x^t B x)^{1/2} - k[h(x, \bar{y}_i) - (x^t D x)^{1/2}] < 0, \quad \text{for all } \bar{y}_i \in Y.$$

It follows from $t_i \geq 0$, $i = 1, 2, \ldots, s$, with $\sum_{i=1}^{s} t_i = 1$, that

$$t_i \{ f(x, \bar{y}_i) + (x^t B x)^{1/2} - k(h(x, \bar{y}_i) - (x^t D x)^{1/2}) \} \leq 0, \qquad (21)$$

with at least one strict inequality because $t = (t_1, t_2, \ldots, t_s) \neq 0$.
    From (1), (16), (19), (21), we have

$$\phi_1(x) = \sum_{i=1}^{s} t_i \{ f(x, \bar{y}_i) + x^t B w - k(h(x, \bar{y}_i) - x^t D v) \}$$

$$\leq \sum_{i=1}^{s} t_i \{ f(x, \bar{y}_i) + (x^t B x)^{1/2} - k(h(x, \bar{y}_i) - (x^t D x)^{1/2}) \}$$

$$< 0 \leq \sum_{i=1}^{s} t_i \{ f(z, \bar{y}_i) + z^t B w - k(h(z, \bar{y}_i) - z^t D v) \}$$

$$= \phi_1(z).$$

Hence,

$$\phi_1(x) < \phi_1(z).$$

Similar to the proof of Theorem 3.1, we have

$$F\left(x, z; \sum_{i=1}^{s} t_i \{ \nabla f(z, \bar{y}_i) + B w - k(\nabla h(z, \bar{y}_i) - D v) \} \right)$$

$$+ \sum_{i=1}^{s} t_i \left\{ \frac{\rho_i d_i^2(x, z)}{\alpha(x, z)} + k \frac{\bar{\rho}_i \bar{d}_i^2(x, z)}{\alpha(x, z)} \right\} < 0, \qquad (22)$$

$$\sum_{j=1}^{p} \mu_j \frac{g_j(x) - g_j(z)}{\beta_j(x, z)} \geq F\left(x, z; \sum_{j=1}^{p} \mu_j \nabla g_j(z) \right) + \sum_{j=1}^{p} \mu_j v_j \frac{c_j^2(x, z)}{\beta_j(x, z)}. \qquad (23)$$

Utilizing the feasibility of x for (P) and (17), we get

$$\sum_{j=1}^{p} \mu_j g_j(x) \leq 0 \leq \sum_{j=1}^{p} \mu_j g_j(z). \tag{24}$$

Therefore, from (23), (24), and $\beta(x, z) > 0$, we obtain

$$F\left(x, z; \sum_{j=1}^{p} \mu_j \nabla g_j(z)\right) + \sum_{j=1}^{p} \mu_j v_j \frac{c_j^2(x, z)}{\beta_j(x, z)} \leq 0. \tag{25}$$

Thus, from (15), (22), (25), we get [by (20)]

$$0 = F\left(x, z; \sum_{i=1}^{s} t_i \{\nabla f(z, \bar{y}_i) + Bw - k(\nabla h(z, \bar{y}_i) - Dv)\} + \sum_{j=1}^{p} \mu_j \nabla g_j(z)\right)$$

$$\leq F\left(x, z; \sum_{i=1}^{s} t_i \{\nabla f(z, \bar{y}_i) + Bw - k(\nabla h(z, \bar{y}_i) - Dv)\}\right)$$

$$+ F\left(x, z; \sum_{j=1}^{p} \mu_j \nabla g_j(z)\right)$$

$$< -\sum_{i=1}^{s} t_i \left\{\frac{\rho_i d_i^2(x, z)}{\alpha(x, z)} + k \frac{\bar{\rho}_i \bar{d}_i^2(x, z)}{\alpha(x, z)}\right\} - \sum_{j=1}^{p} \mu_j v_j \frac{c_j^2(x, z)}{\beta_j(x, z)}$$

$$\leq 0.$$

We have a contradiction. Hence, the proof is complete.   □

Corollary 4.1.   Let x and $(z, \mu, k, v, w, s, t, \bar{y})$ be feasible solutions of (P) and (DI) respectively. Suppose that $f(., \bar{y}_i) + (.)^t Bw$ and $-h(., \bar{y}_i) + (.)^t Dv$, for $i = 1, 2, \ldots, s$, are respectively strongly (F, α, ρ_i, d_i)-convex or (F, α)-convex and strongly $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convex or (F, α)-convex at z. Also let $g_j(.)$, for $j = 1, 2, \ldots, p$, be strongly (F, β_j, v_j, c_j)-convex or (F, β_j)-convex at z. Then,

$$\sup_{y \in Y} \frac{f(x, y) + (x^t Bx)^{1/2}}{h(x, y) - (x^t Dx)^{1/2}} \geq k.$$

Proof.   Under the assumptions of this corollary, we know that the inequality (20) holds. So, we get the corollary from Theorem 4.1.   □

Theorem 4.2.   Strong Duality. Assume that x^* is an optimal solution for (P) and let $\nabla g_j(x^*)$, $j \in J(x^*)$, be linearly independent. Then, there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}) \in H_1(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is feasible for (DI). Further, if the weak duality holds for all feasible

$(z, \mu, k, v, w, s, t, \bar{y})$ of (DI), then $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is optimal for (P) and the two objectives have the same optimal values.

Proof. By Theorem 2.1, there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}) \in H_1(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is feasible for (DI) and

$$\bar{k} = \frac{f(x^*, \bar{y}_i^*) + (x^{*t} B x^*)^{1/2}}{h(x^*, \bar{y}_i^*) - (x^{*t} D x^*)^{1/2}}.$$

Since (P) and (DI) have the same objective values, the optimality of this feasible solution follows from Theorem 4.1.   □

Theorem 4.3. Strict Converse Duality. Let x^* and $(\bar{z}, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ be the optimal solutions for (P) and (DI), respectively. Suppose that $f(., \bar{y}_i^*) + (.)^t B \bar{w}$ and $-h(., \bar{y}_i^*) + (.)^t D \bar{v}$, for $i = 1, 2, \ldots, \bar{s}$, are respectively (F, α, ρ_i, d_i)-convex and $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convex at $\bar{z}$ for all $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(\bar{z}, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}) \in H_1(\bar{s}, \bar{t}, \bar{y}^*)$. Let $g_j(.)$, for $j = 1, 2, \ldots, p$, be (F, β_j, ν_j, c_j)-convex at $\bar{z}$ and let the inequality

$$\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{ \frac{\rho_i d_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} + k \frac{\bar{\rho}_i \bar{d}_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} \right\} + \sum_{j=1}^{p} \bar{\mu}_j \nu_j \frac{c_j^2(x^*, \bar{z})}{\beta_j(x^*, \bar{z})} > 0 \tag{26}$$

hold; let $\nabla g_j(x^*)$, $j \in J(x^*)$, be linearly independent. Then, $x^* = \bar{z}$; that is, $\bar{z}$ is optimal for (P) and

$$\sup_{y \in Y} \frac{f(\bar{z}, \bar{y}^*) + (\bar{z}^t B \bar{z})^{1/2}}{h(\bar{z}, \bar{y}^*) - (\bar{z}^t D \bar{z})^{1/2}} = \bar{k}.$$

Proof. We assume that $x^* \neq \bar{z}$ and reach a contradiction. From Theorem 4.2, we know that

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + (x^{*t} B x^*)^{1/2}}{h(x^*, \bar{y}^*) - (x^{*t} D x^*)^{1/2}} = \bar{k}.$$

Similar to the proof of Theorem 4.1, we have

$$\frac{\phi_1(x^*) - \phi_1(\bar{z})}{\alpha(x^*, \bar{z})} \geq F\left(x^*, \bar{z}; \sum_{i=1}^{\bar{s}} \bar{t}_i \{\nabla f(\bar{z}, \bar{y}_i^*) + B \bar{w} - \bar{k}(\nabla h(\bar{z}, \bar{y}_i^*) - D \bar{v})\}\right)$$

$$+ \sum_{i=1}^{\bar{s}} \bar{t}_i \left\{ \frac{\rho_i d_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} + \bar{k} \frac{\bar{\rho}_i \bar{d}_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} \right\}, \tag{27}$$

$$\sum_{j=1}^{p} \bar{\mu}_j \frac{g_j(x^*) - g_j(\bar{z})}{\beta_j(x^*, \bar{z})} \geq F\left(x^*, \bar{z}; \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(\bar{z})\right) + \sum_{j=1}^{p} \bar{\mu}_j v_j \frac{c_j^2(x^*, \bar{z})}{\beta_j(x^*, \bar{z})}.$$

(28)

By both the feasibility of x^* and (17), we have

$$\sum_{j=1}^{p} \bar{\mu}_j g_j(x^*) \leq 0 \leq \sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{z}).$$

Thus, from (28) and $\beta(x^*, \bar{z}) > 0$, we obtain

$$F\left(x^*, \bar{z}; \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(\bar{z})\right) + \sum_{j=1}^{p} \bar{\mu}_j v_j \frac{c_j^2(x^*, \bar{z})}{\beta_j(x^*, \bar{z})} \leq 0.$$

(29)

Relations (15), (26), (29) along with the sublinearity of F yield

$$F\left(x^*, \bar{z}; \sum_{i=1}^{\bar{s}} \bar{t}_i \{\nabla f(\bar{z}, \bar{y}_i^*) + B\bar{w} - \bar{k}(\nabla h(\bar{z}, \bar{y}_i^*) - D\bar{v})\}\right)$$

$$\geq -F\left(x^*, \bar{z}; \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(\bar{z})\right)$$

$$\geq \sum_{j=1}^{p} \bar{\mu}_j v_j \frac{c_j^2(x^*, \bar{z})}{\beta_j(x^*, \bar{z})}$$

$$> -\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{\frac{\rho_i d_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} + \bar{k} \frac{\bar{\rho}_i \bar{d}_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})}\right\}.$$

(30)

Hence, from (27), (30), and $\alpha(x^*, \bar{z}) > 0$, we get

$$\phi_1(x^*) - \phi_1(\bar{z}) > 0.$$

Now, we get the following relation [by (16)]:

$$\sum_{i=1}^{\bar{s}} \bar{t}_i \{f(x^*, \bar{y}_i^*) + (x^{*t} B x^*)^{1/2} - \bar{k}(h(x^*, \bar{y}_i^*) - (x^{*t} D x^*)^{1/2})\}$$

$$> \sum_{i=1}^{\bar{s}} \bar{t}_i \{f(\bar{z}, \bar{y}_i^*) + (\bar{z}^t B \bar{z})^{1/2} - \bar{k}(h(\bar{z}, \bar{y}_i^*) - (\bar{z}^t D \bar{z})^{1/2})\} \geq 0.$$

Therefore, there exists a certain $i_\circ$, such that

$$f(x^*, \bar{y}_{i_\circ}^*) + (x^{*t} B x^*)^{1/2} - \bar{k}[h(x^*, \bar{y}_{i_\circ}^*) - (x^{*t} D x^*)^{1/2}] > 0.$$

It follows that

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + (x^{*t} B x^*)^{1/2}}{f(x^*, \bar{y}^*) - (x^{*t} D x^*)^{1/2}} \geq \frac{f(x^*, \bar{y}_{t_\circ}^*) + (x^{*t} B x^*)^{1/2}}{h(x^*, \bar{y}_{i_\circ}^*) - (x^{*t} D x^*)^{1/2}} > \bar{k}. \qquad (31)$$

Finally, we have a contradiction and the proof is complete.   □

Remark 4.1.   If we take

$$F(x, \bar{x}; \alpha(x, \bar{x}) \nabla f(\bar{x})) = (x - \bar{x})^t \nabla f(\bar{x}), \quad \text{for } x, \bar{x} \in X,$$

in Theorems 4.1–4.3, we get Theorems 4.1–4.3 in Ref. 13. If we remove the quadratic terms from the numerator and denominator of the objective function and if take $\alpha(x, \bar{x}) = 1$, for all $x \in X$, we obtain Theorems 4.1–4.3 in Ref. 8. Also, if we take

$$F(x, \bar{x}; \alpha(x, \bar{x}) \nabla f(\bar{x})) = \eta(x, \bar{x})^t \nabla f(\bar{x})$$

for a certain map $\eta : X \times X \to R^n$ and remove the quadratic terms from the numerator and denominator of the objective function, we get Theorems 4.1–4.3 in Ref. 9.

5. Duality Model II

In this section, we introduce a Wolfe type dual model to problem (P). In order to discuss the following duality model, we state first another version of Theorem 2.1, by replacing the parameter $k_\circ$ with $\frac{f(\bar{x}, \bar{y}_i) + (\bar{x}^t B \bar{x})^{1/2}}{h(\bar{x}, \bar{y}_i) - (\bar{x}^t D \bar{x})^{1/2}}$ and by rewriting the multiplier functions associated with the inequality constraints.

Theorem 5.1.   Let $\bar{x}$ be a solution for (P) and let $\nabla g_j(\bar{x}), j \in J(\bar{x})$ be linearly independent. Then, there exist $(\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$ and $\bar{\mu} \in R_+^p$ such that

$$\sum_{i=1}^{\bar{s}} \bar{t}_i \{ (h(\bar{x}, \bar{y}_i) - (\bar{x}^t D \bar{x})^{1/2})(\nabla f(\bar{x}, \bar{y}_i) + Bw)$$

$$- (f(\bar{x}, \bar{y}_i) + (\bar{x}^t B \bar{x})^{1/2})(\nabla h(\bar{x}, \bar{y}_i) - Dv) \} + \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{p} \bar{\mu}_j g_j(\bar{x}) \geq 0,$$

$$\bar{\mu} \in R_+^p, \quad \bar{t}_i \geq 0, \quad \sum_{i=1}^{\bar{s}} \bar{t}_i = 1, \quad \bar{y}_i \in Y(\bar{x}), \quad i = 1, 2, \ldots, \bar{s}.$$

(DII) $\max\limits_{(x, t, \bar{y}) \in K(z)} \; \sup\limits_{(z, \mu, v, w) \in H_2(s, t, \bar{y})} F(z),$

where

$$F(z) = \sup_{y \in Y} \frac{f(z, y) + (z^t Bx)^{1/2}}{h(z, y) - (z^t Dx)^{1/2}}$$

and $H_2(s, t, \bar{y})$ denotes the set of all $(z, \mu, v, w) \in R^n \times R_+^p \times R^n \times R^n$ satisfying

$$\sum_{i=1}^{s} t_i \{(h(z, \bar{y}_i) - (z^t Dz)^{1/2})(\nabla f(z, \bar{y}_i) + Bw)$$

$$- (f(z, \bar{y}_i) + (z^t Bz)^{1/2})(\nabla h(z, \bar{y}_i) - Dv)\} + \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(z) = 0, \quad (32)$$

$$\sum_{j=1}^{p} \mu_j g_j(z) \geq 0, \tag{33}$$

$$(s, t, \bar{y}) \in K(z), \tag{34}$$

$$(z^t Bz)^{1/2} = z^t Bw, \quad (z^t Dz)^{1/2} = z^t Dv, \quad w^t Bw \leq 1, \quad v^t Dv \leq 1. \tag{35}$$

For a triplet $(s, t, \bar{y}) \in K(z)$, if the set $H_2(s, t, \bar{y})$ is empty, then we define the supremum over it to be $-\infty$. For convenience, we let

$$\phi_2(.) = \sum_{i=1}^{s} t_i \{(h(z, \bar{y}_i) - z^t Dv)(f(., \bar{y}_i) + (.)^t Bw)$$

$$- (f(z, \bar{y}_i) + z^t Bw)(h(., \bar{y}_i) - (.)^t Dv)\}.$$

Then, we can establish the following weak, strong, and strict converse duality theorems.

Theorem 5.2. Weak Duality. Let x and $(z, \mu, v, w, s, t, \bar{y})$ be the feasible solutions of (P) and (DII) respectively. Suppose that $f(., \bar{y}_i) + (.)^t Bw$ and $-h(., \bar{y}_i) + (.)^t Dv$, for $i = 1, 2, \ldots, s$, are respectively (F, α, ρ_i, d_i)-convex and $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convex at z. Also, let $g_j(.)$, for $j = 1, 2, \ldots, p$, be (F, β_j, ν_j, c_j)-convex at z and let the inequality

$$\sum_{i=1}^{s} t_i \left\{ \frac{(h(z, \bar{y}_i) - (z^t Dz)^{1/2})\rho_i d_i^2(x, z)}{\alpha(x, z)} + \frac{(f(z, \bar{y}_i) + (z^t Bz)^{1/2})\bar{\rho}_i \bar{d}_i^2(x, z)}{\alpha(x, z)} \right\}$$

$$+ \sum_{j=1}^{p} \mu_j \nu_j \frac{c_j^2(x, z)}{\beta_j(x, z)} \geq 0 \tag{36}$$

hold. Then,

$$\sup_{y \in Y} \frac{f(x, y) + (x^t Bx)^{1/2}}{h(x, y) - (x^t Dx)^{1/2}} \geq F(z).$$

Proof.   Suppose to the contrary that

$$\sup_{y \in Y} \frac{f(x, y) + (x^t Bx)^{1/2}}{h(x, y) - (x^t Dx)^{1/2}} < F(z). \tag{37}$$

Since $\bar{y}_i \in Y(z), i = 1, 2, \ldots, s$, we get

$$F(z) = \frac{f(z, \bar{y}_i) + (z^t Bz)^{1/2}}{h(z, \bar{y}_i) - (z^t Dz)^{1/2}}. \tag{38}$$

By (37) and (38), we have

$$[h(z, \bar{y}_i) - (z^t Dz)^{1/2}][f(x, \bar{y}_i) - (x^t Bx)^{1/2}]$$
$$-[f(z, \bar{y}_i) + (z^t Bz)^{1/2}][h(x, \bar{y}_i) - (x^t Dx)^{1/2}] < 0,$$

for all $i = 1, 2, \ldots, s$ and $\bar{y}_i \in Y$. From $\bar{y}_i \in Y(z) \subset Y$ and $t \in R_+^s$, with $\sum_{i=1}^s t_i = 1$, we obtain

$$\sum_{i=1}^s t_i \{[h(z, \bar{y}_i) - (z^t Dz)^{1/2}][f(x, \bar{y}_i) + (x^t Bx)^{1/2}]$$
$$-[f(z, \bar{y}_i) + (z^t Bz)^{1/2}][h(x, \bar{y}_i) - (x^t Dx)^{1/2}]\} < 0. \tag{39}$$

By (1), (35), and (39), we have

$$\phi_2(x) = \sum_{i=1}^s t_i \{(h(z, \bar{y}_i) - z^t Dv)(f(x, \bar{y}_i) + x^t Bw)$$
$$- (f(z, \bar{y}_i) + z^t Bw)(h(x, \bar{y}_i) - x^t Dv)\}$$
$$\leq \sum_{i=1}^s t_i \{(h(z, \bar{y}_i) - (z^t Dz)^{1/2})(f(x, \bar{y}_i) + (x^t Bx)^{1/2})$$
$$- (f(z, \bar{y}_i) + z^t Bz)^{1/2})(h(x, \bar{y}_i) - (x^t Dx)^{1/2})\}$$
$$< 0 = \phi_2(z).$$

Hence,

$$\phi_2(x) < \phi_2(z). \tag{40}$$

Using the (F, α, ρ_i, d_i)-convexity of $f(., \bar{y}_i) + (.)^t Bw$ and the $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convexity of $-h(., \bar{y}_i) + (.)^t Dv$ at z for $i = 1, 2, \ldots, s$, i.e.,

$$f(x, \bar{y}_i) + x^t Bw - f(z, \bar{y}_i) - z^t Bw$$
$$\geq F(x, z; \alpha(x, z)(\nabla f(z, \bar{y}_i) + Bw)) + \rho_i d_i^2(x, z), \quad i = 1, 2, \ldots, s, \tag{41}$$
$$- h(x, \bar{y}_i) + x^t Dv + h(z, \bar{y}_i) - z^t Dv$$
$$\geq F(x, z; \alpha(x, z)(-\Delta h(z, \bar{y}_i) + Dv)) + \bar{\rho}_i \bar{d}_i^2(x, z), \quad i = 1, 2, \ldots, s. \tag{42}$$

Multiplying (41) by $t_i[h(z, \bar{y}_i) - (z^t D z)^{1/2}]$, (42) by $t_i[f(z, \bar{y}_i) + (z^t B z)^{1/2}]$, and then summing up these inequalities with the sublinearity of F, we get

$$
\frac{\phi_2(x) - \phi_2(z)}{\alpha(z, x)} \geq F\left(x, z; \sum_{i=1}^{s} t_i\{(h(z, \bar{y}_i) - (z^t D z)^{1/2})(\nabla f(z, \bar{y}_i) + B w)\right.
$$

$$
\left. - (f(z, \bar{y}_i) + (z^t B z)^{1/2})(\nabla h(z, \bar{y}_i) - D v)\}\right)
$$

$$
+ \sum_{i=1}^{s} t_i \left\{ \frac{(h(z, \bar{y}_i) - (z^t D z)^{1/2})\rho_i d_i^2(x, z)}{\alpha(x, z)} \right.
$$

$$
\left. + \frac{(f(z, \bar{y}_i) + (z^t B z)^{1/2})\bar{\rho}_i \bar{d}_i^2(x, z)}{\alpha(x, z)} \right\},
$$

which by using the (F, β_j, ν_j, c_j)-convexity of $g_j(.)$, (32), and (36), along with the sublinearity of F, yields

$$
\frac{\phi_2(x) - \phi_2(z)}{\alpha(z, x)} \geq F\left(x, z; -\sum_{j=1}^{p} \mu_j \nabla g_j(z)\right) - \sum_{j=1}^{p} \mu_j \nu_j \frac{c_j^2(x, z)}{\beta_j(x, z)}
$$

$$
\geq \frac{-\sum_{j=1}^{p} \mu_j g_j(x) - \sum_{j=1}^{p} \mu_j g_j(z)}{\beta_j(x, z)}
$$

$$
\geq 0,
$$

by the feasibility of x, (33), and $\beta_j(x, z) > 0$. Since $\alpha(x, z) > 0$, we get $\phi_2(x) \geq \phi_2(z)$, which contradicts (40). Hence, the proof is complete. $\qquad\square$

Corollary 5.1. Let x and $(z, \mu, v, w, s, t, \bar{y})$ be feasible solutions of (P) and (DII) respectively. Let $f(., \bar{y}_i) + (.)^t B w$ and $-h(., \bar{y}_i) + (.)^t D v$, for $i = 1, 2, \ldots, s$, be respectively strongly (F, α, ρ_i, d_i)-convex or (F, α)-convex and strongly $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convex or (F, α)-convex at z. Also, let $g_j(.)$, for $j = 1, 2, \ldots, p$, be strongly (F, β_j, ν_j, c_j)-convex or (F, β_j)-convex at z. Then,

$$
\sup_{y \in Y} \frac{f(x, y) + (x^t B x)^{1/2}}{h(x, y) - (x^t D x)^{1/2}} \geq F(z).
$$

Proof. Under the assumptions of this corollary, we know that the inequality (36) holds. So, we can get the corollary from Theorem 5.2. $\qquad\square$

Theorem 5.3. Strong Duality. Let x^* be an optimal solution for (P) and let $\nabla g_j(x^*)$, $j \in J(x^*)$, be linearly independent. Then, there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{v}, \bar{w}) \in H_2(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is feasible for (DII). Further, let the weak duality holds for all feasible $(z, \mu, v, w, s, t, \bar{y})$ of

(DII). Then, $(x^*, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is optimal for (DII) and the two objectives have the same optimal values.

Proof. If x^* is an optimal solution for (P), then by Theorem 2.1, there exist $\bar{v}, \bar{w} \in R^n$, and $\bar{\mu} \in R_+^p$ to satisfy the expression (2), that is, the expression (32) by substituting

$$k_\circ = \frac{f(\bar{x}, \bar{y}_i^*) + (\bar{x}^t B \bar{x})^{1/2}}{h(\bar{x}, \bar{y}_i^*) - (\bar{x}^t D \bar{x})^{1/2}}$$

in (2). It follows that there are $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{v}, \bar{w}) \in H_2(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ is feasible for (DII) and such that problems (P) and (DII) have the same objective values. The optimality of this feasible solution for (DII) thus follows from Theorem 5.2. $\qquad\square$

Theorem 5.4. Strict Converse Duality. Let x^* and $(\bar{z}, \bar{\mu}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*)$ be the optimal solutions for (P) and (DII), respectively. Suppose that $f(., \bar{y}_i^*) + (.)^t B \bar{w}$ and $-h(., \bar{y}_i^*) + (.)^t D \bar{v}$, for $i = 1, 2, \ldots, \bar{s}$, are respectively (F, α, ρ_i, d_i)-convex and $(F, \alpha, \bar{\rho}_i, \bar{d}_i)$-convex at $\bar{z}$ for all $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(\bar{z}, \bar{\mu}, \bar{v}, \bar{w}) \in H_2(\bar{s}, \bar{t}, \bar{y}^*)$; let $g_j(.)$, for $j = 1, 2, \ldots, p$, be (F, β_j, ν_j, c_j)-convex at $\bar{z}$, let the inequality

$$\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{ \frac{(h(\bar{z}, \bar{y}_i^*) - (\bar{z}^t D \bar{z})^{1/2}) \rho_i d_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} + \frac{(f(\bar{z}, \bar{y}_i^*) + (\bar{z}^t B \bar{z})^{1/2}) \bar{\rho}_i \bar{d}_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} \right\}$$

$$+ \sum_{j=1}^{p} \bar{\mu}_j \nu_j \frac{c_j^2(x^*, \bar{z})}{\beta_j(x^*, \bar{z})} > 0 \tag{43}$$

hold, and let $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent. Then, $x^* = \bar{z}$; that is, $\bar{z}$ is optimal for (P) and

$$\sup_{y \in Y} \frac{f(\bar{z}, \bar{y}^*) + (\bar{z}^t B \bar{z})^{1/2}}{h(\bar{z}, \bar{y}^*) - (\bar{z}^t D \bar{z})^{1/2}} = F(\bar{z}).$$

Proof. We shall assume that $x^* \neq \bar{z}$ and reach a contradiction. From conditions similar to the proof of Theorem 5.2, we have

$$\frac{\phi_2(x^*) - \phi_2(\bar{z})}{\alpha(x^*, \bar{z})} \geq F\left(x^*, \bar{z}; \sum_{i=1}^{\bar{s}} \bar{t}_i \{(h(\bar{z}, \bar{y}_i^*) - (\bar{z}^t D \bar{z})^{1/2})(\nabla f(\bar{z}, \bar{y}_i^*) + B \bar{w})\right.$$

$$\left. - (f(\bar{z}, \bar{y}_i^*) + (\bar{z}^t B \bar{z})^{1/2})(\nabla h(\bar{z}, \bar{y}_i^*) - D \bar{v})\}\right)$$

$$+ \sum_{i=1}^{\bar{s}} \bar{t}_i \left\{ \frac{(h(\bar{z}, \bar{y}_i^*) - (\bar{z}^t D \bar{z})^{1/2}) \rho_i d_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} + \frac{(f(\bar{z}, \bar{y}_i^*) + (\bar{z}^t B \bar{z})^{1/2}) \bar{\rho}_i \bar{d}_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} \right\},$$

$$\tag{44}$$

$$\sum_{j=1}^{p} \bar{\mu}_j \frac{g_j(x^*) - g_j(\bar{z})}{\beta_j(x^*, \bar{z})} \geq F\left(x^*, \bar{z}; \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(\bar{z})\right) + \sum_{j=1}^{p} \bar{\mu}_j v_j \frac{c_j^2(x^*, \bar{z})}{\beta_j(x^*, \bar{z})}. \tag{45}$$

By the feasibility of x^*, $\beta(x^*, \bar{z}) > 0$, and (33), inequality (45) gives

$$F\left(x^*, \bar{z}; \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(\bar{z})\right) \leq -\sum_{j=1}^{p} \bar{\mu}_j v_j \frac{c_j^2(x^*, \bar{z})}{\beta_j(x^*, \bar{z})}. \tag{46}$$

From (32), (43), and (46), along with the sublinearity of F, we obtain

$$F\left(x^*, \bar{z}; \sum_{i=1}^{\bar{s}} \bar{t}_i \{(h(\bar{z}, \bar{y}_i^*) - (\bar{z}^t D\bar{z})^{1/2})(\nabla f(\bar{z}, \bar{y}_i^*) + B\bar{w})\right.$$

$$\left. - (f(\bar{z}, \bar{y}_i^*) + (\bar{z}^t B\bar{z})^{1/2})(\nabla h(\bar{z}, \bar{y}_i^*) - D\bar{v})\}\right)$$

$$\geq -F\left(x^*, \bar{z}; \sum_{i=1}^{p} \bar{\mu}_j \nabla g_j(\bar{z})\right)$$

$$\geq \sum_{j=1}^{p} \bar{\mu}_j v_j \frac{c_j^2(x^*, \bar{z})}{\beta_j(x^*, \bar{z})}$$

$$> -\sum_{i=1}^{\bar{s}} \bar{t}_i \left\{ \frac{(h(\bar{z}, \bar{y}_i^*) - (\bar{z}^t D\bar{z})^{1/2})\rho_i d_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} \right.$$

$$\left. + \frac{(f(\bar{z}, \bar{y}_i^*) + (\bar{z}^t B\bar{z})^{1/2})\bar{\rho}_i \bar{d}_i^2(x^*, \bar{z})}{\alpha(x^*, \bar{z})} \right\}. \tag{47}$$

Using $\alpha(x^*, \bar{z}) > 0$ and (47), inequality (44) yields

$$\phi_2(x^*) - \phi_2(\bar{z}) > 0,$$

that is,

$$\sum_{i=1}^{\bar{s}} \bar{t}_i \{(h(\bar{z}, \bar{y}_i^*) - \bar{z}^t D\bar{v})(f(x^*, \bar{y}_i^*) + x^{*t} B\bar{w})$$

$$- (f(\bar{z}, \bar{y}_i^*) + \bar{z}^t B\bar{w})(h(x^*, \bar{y}_i^*) - x^{*t} D\bar{v})\}$$

$$> \sum_{i=1}^{\bar{s}} \bar{t}_i \{(h(\bar{z}, \bar{y}_i^*) - \bar{z}^t D\bar{v})(f(\bar{z}, \bar{y}_i^*) + \bar{z}^t B\bar{w})$$

$$- (f(\bar{z}, \bar{y}_i^*) + \bar{z}^t B\bar{w})(h(\bar{z}, \bar{y}_i^*) - \bar{z}^t D\bar{v})\} \geq 0.$$

Therefore, there exists a certain $i_\circ$ such that

$$\sum_{i=1}^{\bar{s}} \bar{t}_i \{(h(\bar{z}, \bar{y}_{i_\circ}^*) - \bar{z}^t D \bar{v})(f(x^*, \bar{y}_{i_\circ}^*) + x^{*t} B \bar{w})$$
$$- (f(\bar{z}, \bar{y}_{i_\circ}^*) + \bar{z}^t B \bar{w})(h(x^*, \bar{y}_{i_\circ}^*) - x^{*t} D \bar{v})\} > 0.$$

From the above inequality and (35), it follows that

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + (x^{*t} B x^*)^{1/2}}{h(x^*, \bar{y}^*) - (x^{*t} D x^*)^{1/2}} \geq \frac{f(x^*, \bar{y}_{i_\circ}^*) + (x^{*t} B x^*)^{1/2}}{h(x^*, \bar{y}_{i_\circ}^*) - (x^{*t} D x^*)^{1/2}} > F(\bar{z}). \qquad (48)$$

By strong duality theorem, we know that

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + (x^{*t} B x^*)^{1/2}}{h(x^*, \bar{y}^*) - (x^{*t} D x^*)^{1/2}} = F(\bar{z}). \qquad (49)$$

Inequality (48) contradicts (49); hence, the proof is complete. $\qquad \square$

Remark 5.1. If we replace $F(x, \bar{x}; \alpha(x, \bar{x}) \nabla f(\bar{x}))$ by $(x - \bar{x})^t \nabla f(\bar{x})$, for $x, \bar{x} \in X$, in Theorems 5.2–5.4, we get Theorems 1–3 in Ref. 14. Also, when $B = C = 0$ in the objective function, Theorems 4.5–4.7 of Ref. 12 can be obtained; in addition, if $\alpha(x, \bar{x}) = 1$, for all $x, \bar{x} \in X$, Theorems 4.5–4.7 of Ref. 8 can be found.

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