Hyperimmunity in $2^{\mathbb{N}}$.

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Abstract

We investigate the notion of hyperimmunity with respect to how it can be applied to Π^0_1 classes and their Muchnik degrees. We show that hyperimmunity is a strong enough concept to prove the existence of Π^0_1 classes with intermediate Muchnik degree - in contrast to Post's attempts to construct intermediate c.e. degrees.

1 Introduction

1.1 Motivation

This work is an attempt to develop and explore a computability theory on Π^0_1 classes of $2^{\mathbb{N}}$ in direct analogy to the study of c.e. Turing degrees. The two primary concepts of that study are c.e. subsets of $\mathbb{N}$ and Turing reducibility — both of which we assume the reader is very familiar with.

The analogous concepts in $2^{\mathbb{N}}$ that we deal with are Π^0_1 subclasses of of $2^{\mathbb{N}}$ and *Muchnik* reducibility. We ask ourselves how concepts developed in the study of c.e. Turing degrees can be profitably applied to our developing understanding of Π^0_1 Muchnik degrees. This paper is meant to be read as much as a suggestion of a course of study as a record of results.

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This work owes a lot to discussions with Stephen G. Simpson.

It could of course be argued that c.e. subsets of $\mathbb{N}$ are more properly analogous to Σ^0_1 rather than Π^0_1 subclasses of $2^{\mathbb{N}}$. One response to this is that it is really an historical artefact that c.e. (i.e. Σ^0_1) subsets of $\mathbb{N}$ rather than co-c.e. (i.e. Π^0_1) were studied. Indeed most of the properties of c.e. sets that we are concerned with are usually defined explicitly in terms of their complements. But really the analogy that we draw here is not meant to be exact but rather a guide to research, and often it is where the analogy fails that the real research interest lies.

Π^0_1 subclasses of $2^{\mathbb{N}}$ are already an established and ongoing area of research in computability theory (see for example [5]). One fruitful way to conceive of a Π^0_1 is of the set of paths through some computable binary tree. Muchnik reducibility is less-studied but a completely natural concept. Just as Turing reducibility is an idea that applies to arbitrary subsets of $\mathbb{N}$, Muchnik reducibility can be applied to arbitrary subsets of $\mathbb{N}^{\mathbb{N}}$. $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Muchnik reducible to $B \subseteq \mathbb{N}^{\mathbb{N}}$ (written $A \leqslant_w B$) if for all $f \in B$ there is a $g \in A$ such that $f \geqslant_T g$. The idea is that A and B are the respective sets of solutions to two mathematical (*mass*) problems and every solution to the problem represented by B computes a solution to the problem represented by A. In our case the problems involved will simply be those of finding paths through given computable trees. Two problems are Muchnik equivalent if any solution to either computes a solution to the other. The resulting structure of Π^0_1 classes modulo Muchnik equivalence is called the *Muchnik lattice* and is denoted $\mathcal{P}_w$. A Muchnik *degree* is the equivalence class of some subset of $2^{\mathbb{N}}$.

Our basic program is to study properties of Π^0_1 classes and see how this influences their Muchnik degrees. However we will not be concerned with arbitrary properties of Π^0_1 classes but only those properties that have a strong computability theoretic character. Namely those properties that are preserved by computable permutations of $2^{\mathbb{N}}$ (any such property we refer to as being a *computably topological property* because any computable permutation automatically respects the topology on $2^{\mathbb{N}}$). This is the same criterion we use when we define a computability theoretic property of subsets of $\mathbb{N}$. According to Rogers charaterisation of Klein's program in [15] Chapter 4, this specification of the class of objects studied and the characterisation of the type of properties studied specifies a mathematical subject.

In this paper we look at the analogy of Post's problem in $\mathcal{P}_w$. Post was the first to ask if there existed a c.e. set of intermediate Turing degree. He tried to create such a set by describing various properties that he hoped would guarantee incompleteness while not requiring computability. Such properties as *immunity* or *hyperimmunity* were tried. None of these properties succeeded in describing an intermediate Turing degree and Post's problem was solved later by other methods.

Here we revive Post's method in another context — that of the Muchnik lattice, and here his ideas are a lot more fruitful. We use Post's idea of hyperimmunity and use it to define computable topological properties of Π^0_1 classes. This we do this in 5 different ways to get 5 distinct properties. Each the property determines the nature of the set of *branching nodes* of P. That is the set of binary strings σ with the property that $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ have extensions in P. The nature of this set (and other similar sets) has implications for the Muchnik degree of a Π^0_1 class P. The most straightforward result is that if P has no computable element and the set of branching nodes of P is hyperimmune then P is of intermediate Muchnik degree.

Of the five properties defined, three imply Muchnik incompleteness and the other two imply (at least) another type of incompleteness - *Medvedev* incompleteness.

We also apply the stronger property of *dense immunity* to Π^0_1 classes and try to show where measure and the well-known property of *thinness* fit into the scheme of things.

These ideas create a panoply of open questions - some of which should be reasonably easy to answer and some of which will probably require significantly different methods to those used here. We end with a section on some directions for further research.

1.2 Basics

Most of the notation we use is standard. Novel notation specially for this paper is introduced in this section. The other material in this section can be found in more detail in [5], [4], [2] or [18].

$2^{\mathbb{N}}$ is the class of infinite binary sequences equipped with the natural product topology making it a totally disconnected Polish space. $2^{<\mathbb{N}}$ denotes the set of all finite binary strings. If $\sigma \in 2^{<\mathbb{N}}$, we denote by U_σ the set $\{f \in 2^{\mathbb{N}} : f \supset \sigma\}$. The collection $\{U_\sigma : \sigma \in 2^{<\mathbb{N}}\}$ forms a basis for the topology on $2^{\mathbb{N}}$. Any finite union of basis elements is clopen. Elements of $2^{<\mathbb{N}}$ will usually be denoted by σ, or τ and infinite binary sequences by f or g, or X or Y. Subsets of $\mathbb{N}$ will be identified with their characteristic function without further mention. $\sigma^\frown\tau$ and

$\sigma^\frown f$ will denote the concatenation of σ with τ or f.

All unexplained computability theory terminology and notation is standard and can be found in [19] or [15]. We review the concepts that will be particularly important here.

If $X = \{x_0 < x_1 < x_2 \dots\} \subseteq \mathbb{N}$ then the map $i \mapsto x_i$ is called the *principal function of* X and is denoted p_X.

If f and g are two functions from $\mathbb{N}$ to $\mathbb{N}$ and for all n, $f(n) \geqslant g(n)$ then f is said to *dominate* g. We say f dominates $X \subseteq \mathbb{N}$ if f dominates the principal function of X.

If $X \subseteq \mathbb{N}$ is infinite and p_X is not dominated by any computable function then X is called *hyperimmune*.

There is another useful characterisation of hyperimmunity. Every finite subset $F = \{x_0 < x_1 < x_2 < \cdots < x_n\}$ of $\mathbb{N}$ can be indexed canonically by $\prod_{i=0}^{n} p_i^{x_i}$, where p_i is the i^{th} prime number. D_n will denote the finite set canonically indexed by n. A *strong array* is a sequence of finite sets whose canonical indices are given by a computable function. A *disjoint strong array* is a strong array whose elements are pairwise disjoint. $\langle D_{f(n)} \rangle$ will denote a strong array will computable indexing function f.

A well-known theorem (Kuznecov, Medvedev, Uspenski [19] V.2.3) states that $X \subseteq \mathbb{N}$ is hyperimmune if and only if there is no disjoint strong array $\langle D_{f(n)} \rangle$ such that for all n $D_{f(n)} \cap X \neq \emptyset$. This is actually used as the definition of hyperimmunity and the equivalence to our text definition is the theorem.

A *tree* is a subset of $2^{<\mathbb{N}}$ that is closed under taking initial segments. The elements of a tree are called *nodes*. A tree is *computable* precisely when its set of nodes is. A *path* through a tree T is an element f of $2^{\mathbb{N}}$ such that for all n, $f|_n \in T$. A Π^0_1 *class* is the set of paths through some infinite computable tree. We will thus always assume that Π^0_1 classes are non-empty. If T is a computable tree the associated Π^0_1 class will be denoted $[T]$. If T is a tree and $\sigma \in T$ has the property that there exists $f \in [T]$ such that $f \supset \sigma$ then σ is called *extendible*. The set of extendible nodes of T is denoted $\mathrm{Ext}(T)$. Similarly, if P is a Π^0_1 class and T any tree such that $P = [T]$ then by $\mathrm{Ext}(P)$ we mean $\mathrm{Ext}(T)$ (it is not hard to check that this is well-defined).

In general, computable trees will have non-extendible nodes but in constructing a Π^0_1 class we can view it as a nested computable intersection of trees with no non-extendible nodes. In other words a Π^0_1 class is the set of paths through some co-c.e. tree that has no non-extendible nodes.

It is also very useful to view Π^0_1 classes syntactically. $P \subseteq 2^{\mathbb{N}}$ is a Π^0_1 class if and only if for some computable predicate $R \subseteq \mathbb{N} \times 2^{\mathbb{N}}$

$$P = \{f : \forall n R(n, f)\}.$$

The equivalence of these different ideas is set out in detail in [5].

We introduce some notation and definitions that will be useful. Throughout, P is a Π^0_1 class, σ an element of $2^{<\mathbb{N}}$, X an element of $2^{\mathbb{N}}$ and T a tree.

Notation:

- $P(\sigma) = \{f \in P : f \supset \sigma\}$.

- $f[n] = f|_n = \langle f(0), f(1), \ldots f(n-1)\rangle$ (with $f[0] = \emptyset$),
  $P[n] = \{f[n] : f \in P\} = \{\sigma \in \mathrm{Ext}(P) : |\sigma| = n\}$.

• $\{e\}^X[n]$ is the partial sequence $\langle x_i\rangle_{i=0}^{n-1}$ where $x_i = \{e\}^X(i)$ whenever it is defined and undefined otherwise. In particular, $\{e\}^X[n] \in T$ implies $\{e\}^X(m)\downarrow$ for all $m \leqslant n-1$. As above, $\{e\}^X[0] = \emptyset$.

• We will be particularly concerned with a subset of the extendible nodes of P - namely the *branching nodes of* P. σ is a branching node if $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ are both in $\mathrm{Ext}(P)$. The set of branching nodes of P is denoted $\mathrm{Br}(P)$. If $X \in P$ then by $\mathrm{Br}_X(P)$ we mean the set $\{n \in \mathbb{N} : X|_n \in \mathrm{Br}(P)\}$. The concept of a branching node can also be applied to any subset of $2^{\mathbb{N}}$.

• An important type of Π^0_1 class is a separating class. If $A, B \subseteq \mathbb{N}$ are disjoint c.e. sets, then the *separating class of* A *and* B, denoted $\mathcal{S}(A, B)$, is the set

$$\{f \in 2^{\mathbb{N}} : \forall n[(n \in A \Rightarrow f(n) = 1) \text{ and } (n \in B \implies f(n) = 0)]\}.$$

It is straightforward to show using the syntactical viewpoint above that $\mathcal{S}(A, B)$ is a Π^0_1 class.

1.3 The Muchnik Lattice of Π^0_1 classes

If $A, B \subseteq 2^{\mathbb{N}}$ then A *is Muchnik reducible to* B, written $A \leqslant_w B$ if

$$\forall y \in B \exists x \in A \; y \geqslant_T x.$$

If $A \leqslant_w B$ and $B \leqslant_w A$ then we write $A \equiv_w B$ and say A and B are *Muchnik equivalent*. The relation $\leqslant_w$ is a pre-order on $2^{\mathbb{N}}$ and it can be made into a partial order in the familiar way.

The *Muchnik degree* of $A \subseteq 2^{\mathbb{N}}$ is the set

$$\deg_w(A) = \{B \subseteq 2^{\mathbb{N}} : B \equiv_w A\}.$$

$\deg_w(A) \leqslant \deg_w(B)$ if $A \leqslant_w B$. This relation is now a partial order on the collection of Muchnik degrees.

If $\mathbb{P}$ is the collection of non-empty Π^0_1 subclasses of $2^{\mathbb{N}}$ then the structure

$$\mathcal{P}_w = \langle \{\deg_w(P) : P \in \mathbb{P}\}, \leqslant \rangle$$

we call the *Muchnik lattice*. To show it is in fact a lattice it is necessary to demonstrate that every two Muchnik degrees have an infimum and supremum. They are as follows.

If $P, Q \in \mathbb{P}$ then define

$$P \vee Q = \{f \oplus g : f \in P \text{ and } g \in Q\},$$

$$P \wedge Q = \{\langle 0 \rangle^\frown f : f \in P\} \cup \{\langle 1 \rangle^\frown g : g \in Q\},$$

and then

$$\deg_w(P) \vee \deg_w(Q) = \deg_w(P \vee Q),$$
$$\deg_w(P) \wedge \deg_w(Q) = \deg_w(P \wedge Q).$$

These operations in $\mathcal{P}_w$ are distributive over each other as can be easily confirmed. Futhermore $\mathcal{P}_w$ has maximum and minimum elements denoted $\mathbf{1}_w$ and $\mathbf{0}_w$ respectively. Any Π^0_1 class with a computable element is a representative of $\mathbf{0}_w$. One representative of the maximum Muchnik degree is

$$\text{DNR}_2 = \{f \in 2^{\mathbb{N}} : \forall n \{n\}(n) \neq f(n)\}.$$

This is not immediately obvious but it is proved in [18].

A similar reducibility relation on $\mathbb{P}$ is called *Medvedev reducibility* (sometimes *strong* reducibility). If $P, Q \in \mathbb{P}$ and if there is a computable functional $\Phi : P \longrightarrow Q$ then Q is said to be *Medvedev reducible* to P, written $P \geqslant_M Q$. This gives rise in the same manner as above to the *Medvedev lattice* $\mathcal{P}_M$ which is also distributive. If $P, Q \in \mathbb{P}$ then $P \vee Q$ and $P \wedge Q$ are also representatives of the supremum and infimum of their Medvedev degrees. Futhermore, DNR_2 is a representative of $\mathbf{1}_M$.

2 Five computably topological properties

We now define the five properties mentioned in the introduction and prove that they are invariant under computable homeomorphisms.

Definition 2.1. *A* Π^0_1 P *class is* small *if* $\mathrm{Br}(P)$ *is hyperimmune.*

Definition 2.2. *A* Π^0_1 *class* P *is* pathwise hyperimmune (p.h.i.) *if, for some* $X \in P$, $\mathrm{Br}_X(P)$ *is hyperimmune.*

Definition 2.3. *A* Π^0_1 *class is* everywhere pathwise hyperimmune (e.p.h.i.) *if, for all* $X \in P$, $\mathrm{Br}_X(P)$ *is hyperimmune.*

Definition 2.4. *A* Π^0_1 *class is* uniformly pathwise hyperimmune (u.p.h.i.) *if there is no computable function* ϕ *such that for all* $X \in P$, ϕ *dominates* $\mathrm{Br}_X(P)$.

There is a direct counterpart in $2^{\mathbb{N}}$ to the notion of disjoint strong array. If D_n is a finite set of (the Gödel numbers of) finite binary strings then we define

$$\begin{aligned} D_n^* &= \{g \in 2^{\mathbb{N}} : \exists \sigma \in D_n \; g \supset \sigma\} \\ &= \bigcup \{U_\sigma : \sigma \in D_n\}. \end{aligned}$$

n is then the *canonical index* of the clopen set D_n^*.

We now define a property most directly analogous to the property of hyperimmunity of susets of $\mathbb{N}$.

Definition 2.5. *A* Π^0_1 *class* P *is* hyperimmune (h.i.) *if there is no disjoint strong array* $\langle D^*_{f(n)} \rangle$ *such that for all* $n, P \cap D^*_{f(n)} \neq \emptyset$.

To further emphasise the relatedness of hyperimmunity in $2^{\mathbb{N}}$ and hyperimmunity in $\mathbb{N}$ we make the following observation. If f is a computable function, then we call $\langle D_{f(n)} \rangle$ an *incomparable strong array* if for all n $D_{f(n)} \subseteq 2^{<\mathbb{N}}$ and for all $\sigma, \tau \in \bigcup_n D_{f(n)}$, if $\sigma \neq \tau$, then σ and τ are incomparable.

We make the following definition now which will be useful later on.

Definition 2.6. *If* $C \subseteq 2^{\mathbb{N}}$ *is clopen then the* root set of C, $\mathrm{rt}(C)$, *is the unique finite subset of* $\mathrm{Br}(C)$ *of smallest cardinality such that* $C = \{f \in 2^{\mathbb{N}} : \exists \sigma \in \mathrm{rt}(C) \; f \supset \sigma\} = \bigcup \{U_\sigma : \sigma \in \mathrm{rt}(C)\}$.

Theorem 2.7. *P is an h.i. Π^0_1 class if and only if there is no incomparable strong array $\langle D_{f(n)}\rangle$ such that for all n*

$$\mathrm{Ext}(P) \cap D_{f(n)} \neq \emptyset.$$

Proof. This is straightforward using the fact that two clopen subsets of $2^{\mathbb{N}}$ are disjoint if and only if their two root sets are pairwise incomparable. □

The property of smallness has other quite natural characterisations as shown in the next theorem. The two following definitions will be useful.

Definition 2.8. *If P is a perfect closed subset of $2^{\mathbb{N}}$ then let Φ_P be the canonical, order-preserving map from $2^{<\mathbb{N}}$ onto $\mathrm{Br}(P)$. That is,*

$\Phi_P(\emptyset)$ = the unique element of $\mathrm{Br}(P)$ of minimum length

$\Phi_P(\sigma^\frown\langle 0\rangle)$ = the unique element of $\mathrm{Br}(P)$ of minimum length extending $\Phi_P(\sigma)^\frown\langle 0\rangle$

$\Phi_P(\sigma^\frown\langle 1\rangle)$ = the unique element of $\mathrm{Br}(P)$ of minimum length extending $\Phi_P(\sigma)^\frown\langle 1\rangle$

Definition 2.9. *The* set of branching levels *of a Π^0_1 class P is the set*

$$\mathrm{Brl}(P) = \{n : \exists\sigma \in \mathrm{Br}(P),\ |\sigma| = n\}.$$

Theorem 2.10. *The following are equivalent:*

1. *P is small,*
2. *the function from $\mathbb{N}$ to $\mathbb{N}$ given by*

$$n \mapsto \min\{|\Phi_P(\sigma)| : |\sigma| = n\}$$

is not dominated by any computable function,

3. *$\mathrm{Brl}(P)$ is hyperimmune,*
4. *there is no computable function f such that $\forall n\ \|P[f(n)]\| \geqslant n$,*
5. *there is no computable function f such that*

$$\forall n \exists\sigma \in \mathrm{Br}(P)\ f(n) \leqslant |\sigma| < f(n+1).$$

Proof. The proofs of most of the above can be found in [3]. The remaining part is straightforward. □

Uniform pathwise hyperimmunity also has the following alternative characterisations which emphasise its relationship to smallness.

Theorem 2.11. *The following are equivalent:*

1. *P is u.p.h.i,*
2. *the function from $\mathbb{N}$ to $\mathbb{N}$ given by*

$$n \mapsto \max(\{|\Phi_P(\sigma)| : |\sigma| = n\})$$

is not dominated by any computable function (compare 2.10 2.),

3. *There is no computable function f such that*

$$\forall n \forall \sigma \in P[f(n)] \exists \tau \in \mathrm{Br}(P)\ \tau \supseteq \sigma \text{ and } |\tau| < f(n+1)$$

(compare 2.10 5.).

Proof. The proofs are similar to the proofs of Theorem 2.10. □

It will also be useful to note the following characterisation of e.p.h.i. and p.h.i. analogous to 2.10 5.

Theorem 2.12. *A Π^0_1 class P is p.h.i. (e.p.h.i.) if and only if for some (all) $X \in P$ there is no strictly increasing computable function f such that for all n there is a m such that $f(n) \leqslant m < f(n+1)$ and $X|_m \in \mathrm{Br}(P)$.*

Proof. See [3] Theorem 2.27 ($\Leftarrow$). □

2.1 Invariance under computable homeomorphisms

Now that these properties are defined, we will prove that they are all computable topological properties.

Theorem 2.13. *Smallness is a computably topological property.*

This was proved in [4]. In fact the stronger result was proved that if P and Q are Π^0_1 classes and $\{e\} : P \longrightarrow Q$ is surjective, then if P is small so is Q. This stronger property is shared by h.i. Π^0_1 classes but for p.h.i., e.p.h.i. and u.p.h.i. Π^0_1 classes injectivity seems to be needed. The necessity of injectivity in these cases has yet to be established however.

Theorem 2.14. *Hyperimmunity is a computably topological property.*

Proof. Suppose $P, Q \subseteq 2^{\mathbb{N}}$ are Π^0_1, $\{e\} : P \longrightarrow Q$ is a computable surjection and Q is not hyperimmune. Let $\langle D^*_{f(n)} \rangle$ witness this last fact. We will find a computable function g so that $\langle D^*_{g(n)} \rangle$ witnesses the fact that P is not hyperimmune.

We first define two functions that will be useful later. Let $n \in \mathbb{N}$ and $n \mapsto l_n = l$ and $n \mapsto t_n = t$ be two functions with the property that $\forall \tau \in P_t[l]\ \{e\}^\tau_t[n] \in Q_t[n]$. Such numbers l and t exist because P is compact. Thus l and t can be found by a computable search, and the functions $n \mapsto l_n$ and $n \mapsto t_n$ can be taken to be computable. We also assume for later purposes that l_n is strictly increasing.

Now let $m = m(n) = \max\{|\sigma| : \sigma \in D_{f(n)}\}$; $l = l_m, t = t_m$, and let $D_{g(n)} = \{\tau \in P_t[l] : \exists \sigma \in D_{f(n)}\ \{e\}^\tau_t \supseteq \sigma\}$. g is computable and $\langle D^*_{g(n)} \rangle$ is pairwise disjoint because $\langle D^*_{f(n)} \rangle$ is. □

This next lemma is key to a lot of what follows.

Lemma 2.15. *Suppose P and Q are Π^0_1 classes and $\{e\} : P \longrightarrow Q$ is a computable homeomorphism. For all strictly increasing $f \in \mathbb{N}^{\mathbb{N}}$, there exists a strictly increasing $g \leqslant_T f$ with the property: for all $X \in P$ and $Y \in Q$ such that $Y = \{e\}^X$, and for all $n \in \mathbb{N}$*

$$\mathrm{Br}_Y(Q) \cap [f(n), f(n+1)) \neq \emptyset \implies \mathrm{Br}_X(P) \cap [g(n), g(n+1)) \neq \emptyset.$$

Proof. Let $n \in \mathbb{N}$ and $Y = \{e\}^X$ be arbitrary. Let l_n and t_n be as in Theorem 2.14. n is understood when we drop the subscripts. We describe a quotient-like structure by pulling back $\{e\}$. If $\sigma \in P_t[l]$, then denote by $[\sigma]_n$ the set

$$\{\tau \in P_t[l] :\ \{e\}^\tau_t[n] = \{e\}^\sigma_t[n]\}.$$

We write $[\sigma]_n \succ [\tau]_m$ if $n > m$ and, for some $\gamma \in [\tau]_n$, $\sigma \supsetneq \gamma$.

Suppose now that $f \in \mathbb{N}^{\mathbb{N}}$ is strictly increasing. We will construct the required g computable in f.

First let $g(0) = l_{f(0)}$.

Now suppose $g(m)$ is known. Let n be the least number such that $l_{f(n)} > g(m)$. n exists as f and l are strictly increasing. Let $\sigma = X[l_{f(n)}]$ and consider $[\sigma]_{f(n)}$. There are three cases in total. It will be in general impossible to effectively decide which case pertains but it also will become clear that an f-computable choice for $g(m+1)$ will be sufficient for all three cases. Suppose $\mathrm{Br}_Y(Q) \cap [f(n), f(n+1)) \neq \emptyset$.

Case 1. $[\sigma]_{f(n)} \cap \mathrm{Ext}(P) = \{\sigma\}$. Because $\mathrm{Br}_Y(Q) \cap [f(n), f(n+1)) \neq \emptyset$, we know that there is a $\gamma \in Q[f(n+1)]$ such that $\gamma \supsetneq Y[f(n)]$

and $\gamma \neq Y[f(n+1)]$. Therefore we know that there is a $\tau \in P[l_{f(n+1)}]$ such that $[\tau]_{f(n+1)} \succ [\sigma]_{f(n)}$ and $\tau \notin \big[X[l_{f(n+1)}]\big]_{f(n+1)}$. We must have $\tau \supsetneq \sigma$ (as $[\sigma]_{f(n)} \cap \mathrm{Ext}(P)$ is a singleton) and $\tau \neq X[l_{f(n+1)}]$. So there is a branching node on X between $l_{f(n)}$ and $l_{f(n+1)}$.

For this case the choice $g(m+1) = l_{f(n+1)}$ clearly suffices.

Case 2. The class $[\sigma]_{f(n)} \cap \mathrm{Ext}(P)$ contains more than one element. As $\{e\}$ is one-to-one, and P is compact, we can effectively find natural numbers $u, v > \max\{l_{f(n)}, t_{f(n)}\}$ such that for every $\tau, \tau' \in P_u[v]$,

$$\tau[l_{f(n)}] \neq \tau'[l_{f(n)}] \Rightarrow \exists a \leqslant u \ \{e\}^{\tau}_u(a)\downarrow \neq \{e\}^{\tau'}_u(a)\downarrow .$$

Both f and l_n are strictly increasing in n so we can effectively find a k that will make $l_{f(k)}, t_{f(k)} \geqslant \max\{u, v\}$. We will then have, for all $\tau \in P_{t_{f(k)}}[l_{f(k)}]$,

$$\tau \not\supseteq \sigma \Rightarrow \tau \notin \big[X[l_{f(k)}]\big]_{f(k)}.$$

This leads to two sub-cases:

Case 2a. $\big[X[l_{f(k)}]\big]_{f(k)} \cap \mathrm{Ext}(P)$ is a singleton. This situation is similar to case 1 and $g(m+1) = l_{f(k+1)}$ will suffice.

Case 2b $\big[X[l_{f(k)}]\big]_{f(k)} \cap \mathrm{Ext}(P)$ has at least two elements. Every element of $\big[X[l_{f(k)}]\big]_{f(k)}$ extends σ so there must be at least two incompatible extendible nodes of $P[l_{f(k)}]$ extending σ and hence $\mathrm{Br}_X(P) \cap [l_{f(n)}, l_{f(k)}) \neq \emptyset$. The choice $g(m+1) = l_{f(k)}$ will then suffice.

In all three cases the choice $g(m+1) = l_{f(k+1)}$ suffices and we can find k by an effective search. Thus g is an f-computable fuction with the required property.

□

We are particularly interested in the situation when f in the previous lemma is computable. This gives immediately the following.

Theorem 2.16. *E.p.h.i, p.h.i, and u.p.h.i. are all computably topological properties.*

Proof. Suppose P and Q are computably homeomorphic Π^0_1 classes. If $Y \in Q$ and f a computable function such that

$$\forall n \ \mathrm{Br}_Y(Q) \cap [f(n), f(n+1)) \neq \emptyset.$$

Then Lemma 2.15 constructs a g, also computable, such that

$$\forall n \ \mathrm{Br}_X(P) \cap [g(n), g(n+1)) \neq \emptyset,$$

where X is the preimage of Y under the homeomorphism. This proves (using Theorem 2.12) that if Q is not e.p.h.i, (p.h.i, u.p.h.i) then neither is P. □

2.2 Lattice Operations

The final lemmas in this section will be useful later on for constructing Π^0_1 classes with required properties.

Theorem 2.17. *If $P, Q \subseteq 2^\omega$ are Π^0_1, then $P \vee Q$ is small if and only if $P \wedge Q$ is small if and only if both P and Q are small.*

Proof. The proof of this is in [3]. □

Lemma 2.18. *If $X \subseteq \omega$ and $Y \subseteq \omega$ are co-c.e. then $X \oplus Y = \{2x : x \in X\} \cup \{2x+1 : x \in Y\}$ is hyperimmune if and only if both X and Y are.*

Proof. If X or Y were not h.i, it would be straightforward to construct a disjoint strong array witnessing the fact that $X \oplus Y$ were not h.i. So suppose that $X \oplus Y$ was not h.i. Let f be a computable function such that for all n $D_{f(n)} \cap X \oplus Y \neq \emptyset$. Let $\left(D_{f(n)}\right)_0 = \{m/2 : m \text{ is even and } m \in D_{f(n)}\}$, and let $\left(D_{f(n)}\right)_1 = \{(m-1)/2 : m \text{ is odd and } m \in D_{f(n)}\}$. For every n either $\left(D_{f(n)}\right)_0 \cap X \neq \emptyset$ or $\left(D_{f(n)}\right)_1 \cap Y \neq \emptyset$. Therefore, if for infinitely many n $\left(D_{f(n)}\right)_0 \cap X = \emptyset$, then for infinitely many n $\left(D_{f(n)}\right)_1 \cap Y \neq \emptyset$ and an infinite sequence of such n's could be computed (because X is co-c.e.), contradicting the hyperimmunity of Y. So for some N, and for all $n \geqslant N$, $\left(D_{f(n)}\right)_0 \cap X \neq \emptyset$ contradicting the hyperimmunity of X. □

Theorem 2.19. *If P and Q are Π^0_1 classes and both are e.p.h.i. (p.h.i, u.p.h.i, h.i.), then so is $P \wedge Q$.*

Proof. The proofs for e.p.h.i, p.h.i, and u.p.h.i. are very straightforward. The proof for h.i. is analogous to the proof of Lemma 2.18. □

Theorem 2.20. *If $P \wedge Q$ is e.p.h.i. (h.i.), then so are P and Q. This is not the case for u.p.h.i. and p.h.i.*

Proof. The first part is immediate. The second is done by noticing that $S \wedge 2^{\mathbb{N}}$ is both u.p.h.i and p.h.i. if S is small. □

Theorem 2.21. *If S is a small Π^0_1 class and P a u.p.h.i. Π^0_1 class, then $S \vee P$ is u.p.h.i.*

Proof. We show the contrapositive. Let f be a computable even-valued function witnessing the fact that $S \vee P$ is not u.p.h.i. Then for all n and $\sigma \oplus \tau \in S \vee P[f(n)]$ and for all $X \oplus Y \in S \vee P(\sigma \oplus \tau)$ either $\mathrm{Br}_X(S) \cap \big[f(n)/2, f(n+1)/2\big) \neq \emptyset$ or $\mathrm{Br}_Y(P) \cap \big[f(n)/2, f(n+1)/2\big) \neq \emptyset$. As S is a small Π^0_1 class, there must be a infinite computable set $\{n_i : i \in \mathbb{N}\}$ such that for all i $\mathrm{Brl}(S) \cap \big[f(n)/2, f(n+1)/2\big) = \emptyset$. Therefore for all i and for all $\sigma \in P[f(n_i)]$ there is a $\tau \supseteq \sigma$ such that $\tau \in \mathrm{Br}(P)$ and $|\tau| \leqslant f(n+1)/2$. Hence $f(n_i)/2$ witnesses the fact that P is not u.p.h.i. Contradictiction. □

Theorem 2.22. *If P and Q are Π^0_1, then P and Q are (e.)p.h.i. if and only if $P \vee Q$ is.*

Proof. Straightforward using Lemma 2.18. □

Theorem 2.23. *If $P \vee Q$ is h.i, then both P and Q are.*

Proof. Without losing generality assume that P is not h.i. If f is computable and if $\forall n\ D^*_{f(n)} \cap P \neq \emptyset$, then $\forall n\ (D^*_{f(n)} \vee 2^{\mathbb{N}}) \cap P \vee Q \neq \emptyset$. □

The converse to the previous theorem has not been proved. It is analogous to the theorem that the disjoint union of two co-c.e. and hyperimmune subsets of $\mathbb{N}$ is hyperimmune. We conjecture that it is false in this context. We also conjecture that the join of two u.p.h.i. Π^0_1 classes is not necessarily u.p.h.i.

2.3 Comparisons to measure and each other

In [4] it is shown that all small Π^0_1 classes have measure zero. Here we improve this result to that it holds for all e.p.h.i. classes.

Theorem 2.24. *If P is an e.p.h.i. Π^0_1 class then $\mu(P) = 0$*

Proof. Suppose P is a Π^0_1 class and $\mu(P) > 0$. We will describe an $X \in P$ and a computable function f with f dominating $\mathrm{Br}_X(P)$. Let k be the least positive integer such that $1/2^k < \mu(P)$. There must be at least two extendible nodes on $P[k]$ (or else $1/2^k \geqslant \mu(P)$) and so there must be a branching node of length strictly less than k. There also must exist a $\sigma \in P[k]$ such that $2^k \cdot \mu(P(\sigma)) \geqslant \mu(P)$. Let this

σ be σ_1. Iterating the process, there is a $\sigma \in P[(n+1)k]$ extending σ_n such that $2^k \cdot \mu(P(\sigma)) \geqslant \mu(P(\sigma_n))$. Let this σ be σ_{n+1}. As above, there must be a branching between σ_n and σ_{n+1}.

Then $X = \bigcup_{n=1}^{\infty} \sigma_n$ and $f(n) = nk$ are as required. $\square$

However u.p.h.i. Π^0_1 classes need not have measure zero. For example, if S is small and $\mu(Q) > 0$, then $S \wedge Q$ is u.p.h.i. and $\mu(S \wedge Q) > 0$. H.i. Π^0_1 classes are also not necessarily of measure zero as the following shows.

Theorem 2.25 (Simpson)**.** *Every Π^0_1 class of positive measure contains an h.i. Π^0_1 class of positive measure.*

Proof. Suppose $P \subseteq 2^{\mathbb{N}}$ is Π^0_1 and $\mu(P) \geqslant m > 0$ for some computable real m. We will diagonalise against the class of disjoint strong arrays to create an h.i. subclass. Let d be a partial computable function such that , for a given $e \in \mathbb{N}$, $\mu(D^*_{\{e\}(d(e))}) < m/2^{e+1}$. $d(e)$ is defined if (but not only if) the range of $\{e\}$ is infinite. Let $P' = P \smallsetminus \bigcup_{e\in\mathbb{N}} D^*_{\{e\}(d(e))}$. P' is Π^0_1 as $\bigcup_{e\in\mathbb{N}} D^*_{\{e\}(d(e))}$ is Σ^0_1 and it has positive measure because

$$\mu(\bigcup_{e\in\mathbb{N}} D^*_{\{e\}(d(e))}) \leqslant \sum_{e\in\mathbb{N}} m/2^{e+1} \leqslant m/2 < m.$$

It is h.i. because for all e, $D^*_{\{e\}(d(e))} \cap P' = \emptyset$. $\square$

Theorem 2.26. *Small $\Rightarrow$ e.p.h.i. $\Rightarrow$ p.h.i. $\Rightarrow$ u.p.h.i.*

Proof. From the definitions it is clear that e.p.h.i. $\Rightarrow$ p.h.i. $\Rightarrow$ u.p.h.i. For the first implication suppose P were Π^0_1 and not e.p.h.i. — witnessed by $X \in P$ and computable function f dominating $\mathrm{Br}_X(P)$. $\mathrm{Br}_X(P) \subseteq \mathrm{Brl}(P)$ and so f also dominates $\mathrm{Brl}(P)$. Therefore P is not small. $\square$

Theorem 2.27. *U.p.h.i. $\not\Rightarrow$ p.h.i.*

Proof. We denote by $\mathbf{1}^n$ and $\mathbf{0}^n$ the strings of n ones and zeroes respectively, with the understanding that $\mathbf{1}^0 = \mathbf{0}^0 = \emptyset$. Let f be the principal function of some hyperimmune Π^0_1 subset of $\mathbb{N}$. Let T be the tree generated by the set $\{\mathbf{0}^{i\frown}\mathbf{1}^{f(i)+1\frown}\gamma : i \in \mathbb{N}, \gamma \in 2^{<\mathbb{N}}\}$ and let $P = [T]$. P is Π^0_1 by inspection. For every $X \in P$, $\mathrm{Br}_X(P)$ is cofinite so P is clearly not p.h.i. However if Φ_P is the function from 2.8 then for every $n > 0$, $\max\{|\Phi_P(\sigma)| : |\sigma| = n\} \geqslant f(n-1)$ which is not dominated by any computable function. So P is u.p.h.i. $\square$

The P constructed in the previous theorem has a computable path (namely $\mathbf{0}^\infty$) and so has trivial Muchnik degree. As we will be interested in the Muchnik degrees of the Π^0_1 classes we create this could be a problem, however as the next theorem shows, we needn't worry.

Theorem 2.28. *There exists a Π^0_1 class with no computable path (and hence perfect) that is u.p.h.i but not p.h.i.*

Proof. Let P be as constructed in the previous theorem, and let S be any small Π^0_1 class. Then Lemma 2.21 says that $P \vee S$ will be u.p.h.i. and Theorem 2.22 says that it will not be p.h.i. □

Theorem 2.29. *P.h.i. $\not\Rightarrow$ e.p.h.i. and h.i. $\not\Rightarrow$ e.p.h.i.*

Proof. Any p.h.i. or h.i. class of positive measure illustrates this. □

The following is based on a construction by Lerman.

Theorem 2.30. *E.p.h.i. $\not\Rightarrow$ h.i.*

Proof. We construct an e.p.h.i. class P which is not h.i. by describing a computable sequence T_s of nested computable trees such that $T = \bigcap_s T_s$ and $P = [T]$. P will be countable with exactly one non-isolated path X. We will find a perfect Π^0_1 class with the required properties in a corollary. We adopt the $\mathbf{0}^n$ notation from Theorem 2.27. To build T_s we construct a sequence of natural numbers $0 = l_0 \leqslant l_1 \leqslant l_2 \dots$ with $\lim_{s\to\infty} l_s = \infty$. At each stage s we have

i. $T_s[l_s] = T[l_s]$

ii. $\forall \sigma \in T_s[l_s]$, $[\tau \supseteq \sigma \implies \tau \in T_s]$.

To ensure that T has the required properties, we construct concurrently with T_s two double sequences of non-negative integers

$$e_{0,s} < e_{1,s} < \dots < e_{n_s,s}$$

and

$$u_{0,s}, u_{1,s}, \dots u_{n_s,s}$$

with the following properties:

A1: $\lim_s n_s = \infty$.

A2: $\forall i$ $\lim_s u_{i,s}$ and $\lim_s e_{i,s}$ exist and are denoted u_i and e_i.

A3: The unique non-isolated path of P is

$$X = \mathbf{1}^{e_0} {}^\frown \mathbf{0}^{u_0} {}^\frown \mathbf{1}^{e_1} {}^\frown \mathbf{0}^{u_1} {}^\frown \dots$$

A4: If we let $\tau_{i,s}$ denote the string

$$\mathbf{1}^{e_{0,s}}\frown\mathbf{0}^{u_{0,s}}\frown\mathbf{1}^{e_{1,s}}\frown\mathbf{0}^{u_{1,s}}\frown\ldots\frown\mathbf{0}^{u_{i,s}},$$

then $\forall s\ \tau_{n_s,s} \in T_s[l_s]$. $\tau_{n_s,s}$ is to be considered an approximation to the path X.

If s is a stage at which $T_{s+1} \neq T_s$ then $l_{s+1} > l_s$ and

$$T_{s+1}[l_{s+1}] = \{\sigma\frown\mathbf{0}^{l_{s+1}-l_s} : \sigma \in T_s[l_s]\} \cup \{\tau_{n_s,s}\frown\mathbf{1}^p\frown\mathbf{0}^{l_{s+1}-l_s-p} : 0 < p \leqslant l_{s+1} - l_s\}. \quad (1)$$

T_{s+1} is then any string extending or extended by an element of $T_{s+1}[l_{s+1}]$.

All that remains in the construction is to describe the sequences $\langle e_{i,s}\rangle$, $\langle u_{i,s}\rangle$ and $\langle l_s\rangle$ and to determine the stages at which $T_{s+1} \neq T_s$.

At stage $s = 0$ we set $n_s = 0$ and $e_{n_s,s} = u_{n_s,s} = 0$. This gives $l_0 = 0$ and $\tau_{0,0} = \emptyset$ by definition. Now let s be arbitrary and suppose n_s and l_s are defined. Also suppose that $e_{i,s}$ and $u_{i,s}$ are defined for all $i \leqslant n_s$. For convenience we begin indexing the partial computable functions at 1. Let e be the least positive integer such that

B1: $e \neq e_{i,s}$ for any $i \leqslant n_s$,

B2: for some $0 < k \leqslant s$, if j is the largest integer such that $e_{j,s} < e$, then

$$|\tau_{j,s}| + e + k \leqslant \{e\}_s(|\tau_{j,s}| + e + k)\downarrow < \{e\}_s(|\tau_{j,s}| + e + k + 1)\downarrow \quad (2)$$

Then we set:

C1: $n_{s+1} = j + 1$,
C2: $e_{n_{s+1},s+1} = e$
C3: $e_{i,s+1} = e_{i,s}$ and $u_{i,s+1} = u_{i,s}$ for all $i \leqslant j$,
C4: $l_{s+1} = \max\{l_s + 1, \{e\}(|\tau_{j,s}| + e + k + 1)\}$,
C5: $u_{n_{s+1},s+1} = l_{s+1} - |\tau_{j,s}| - e$
(this to ensure that $|\tau_{n_{s+1},s+1}| = l_{s+1}$).

If no such e exists then all values are unchanged. The point is that if $\{e\}$ appears at stage s to be a total increasing function, then we ensure that

$$\mathrm{Br}_X(P) \cap \big[\{e\}(|\tau_{j,s}| + e + k), \{e\}(|\tau_{j,s}| + e + k + 1)\big) = \emptyset.$$

As X is the only element of P that has infinitely many branching nodes on it, this ensures that P is e.p.h.i. via theorem 2.12.

It remains to show that P is e.p.h.i. and not h.i. This is done in the next few lemmas.

Lemma 2.31. *For all $\tau \in T$, $\tau^\frown \mathbf{0}^\infty \in P$.*

Proof. If $\{e\}$ is a total increasing function then there will be a stage s for which 2 is satisfied and $l_{s+1} > l_s$. Thus $\lim_s l_s = \infty$. Let $\tau \in T$ be arbitrary and s such that $l_s \leqslant |\tau| < l_{s+1}$. Then τ is of the form $\sigma^\frown \mathbf{1}^n{}^\frown \mathbf{0}^m$ for some $0 \leqslant n, m < l_{s+1} - l_s$ and $\sigma \in P[l_s]$. An inspection of Equation (1) above taking $p = n$ (if necessary) then gives the result. □

Lemma 2.32. *For all s $\tau_{n_s,s} \in T$.*

Proof. By induction. Firstly, $\tau_{n_0,0} = \emptyset \in T$. Now let s be arbitrary and suppose $\tau_{i,s} \in T$ for all $i \leqslant n_s$. We can assume $T_s \neq T_{s+1}$.There are two cases.

Case 1. $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$. In this case j from (2) is just n_s and

$$\tau_{n_{s+1},s+1} = \tau_{n_s,s}{}^\frown \mathbf{1}^{e_{n_{s+1},s+1}}{}^\frown \mathbf{0}^{u_{n_{s+1},s+1}}.$$

So using equation (2) and the definition of l_{s+1} we have

$$0 < e_{n_{s+1},s+1} \leqslant l_{s+1} - l_s.$$

Take $p = e_{n_{s+1},s+1}$ in (1). We can do this because $l_{s+1} - l_s = l_{s+1} - |\tau_{n_s,s}| > e_{n_{s+1},s+1}$ by A4, C4 and (2) above.

Case 2. Let $j < n_s$ be the largest integer such that $\tau_{n_{s+1},s+1} \supsetneq \tau_{j,s}$ and

$$\tau_{n_{s+1},s+1} = \tau_{j,s}{}^\frown \mathbf{1}^{e_{n_{s+1},s+1}}{}^\frown \mathbf{0}^{u_{n_{s+1},s+1}}.$$

By definition $e_{n_{s+1},s+1} < e_{j+1,s}$ so

$$\tau_{j,s}{}^\frown \mathbf{1}^{e_{n_{s+1},s+1}} \subsetneq \tau_{j,s}{}^\frown \mathbf{1}^{e_{j+1,s}} \subseteq \tau_{n_s,s} \in T.$$

Therefore $\tau_{j,s}{}^\frown \mathbf{1}^{e_{n_{s+1},s+1}} \in T$, and so by lemma 2.31 $\tau_{n_{s+1},s+1} \in T$. □

Lemma 2.33. *For all s such that $T_s \neq T_{s+1}$, $|\tau_{n_{s+1},s+1}| > |\tau_{n_s,s}|$. Either $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$ or $\tau_{n_{s+1},s+1}$ is less than $\tau_{n_s,s}$ lexicographically.*

Proof. $|\tau_{n_s,s}| = l_s$ for all s and l_s is increasing in s whenever $T_s \neq T_{s+1}$. Assume that it is not the case that $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$. If j is as (2), then $\tau_{n_{s+1},s+1} \supseteq \tau_{j,s}^\frown \mathbf{1}^{e_{n_{s+1},s+1}}{}^\frown \mathbf{0}^1$ and $\tau_{n_s,s} \supseteq \tau_{j,s}^\frown \mathbf{1}^{e_{j+1,s}}$. As $e_{n_{s+1},s+1} < e_{j+1,s}$, the result follows. □

Lemma 2.34. *P is not h.i.*

Proof. For convenience we (computably) re-index the sequence $\langle T_s \rangle$ so that $T_{s+1} \neq T_s$ for all s. Now consider the disjoint strong array given by $D_{f(s)} = \{\tau_{n_s,s}{}^\frown \mathbf{1}^{l_{s+1}-l_s}\}$ for each s. First notice that $\tau_{n_s,s}{}^\frown \mathbf{1}^{l_{s+1}-l_s} \in T_{s+1}$ for all s (take $p = l_{s+1} - l_s$ in (1)). We claim that the sequence is increasing in length and strictly decreasing in lexicographical order. Hence it is pairwise incomparable. Lemma 2.31 then guarantees that $D_{f(s)} \cap \mathrm{Ext}(P) \neq \emptyset$ for all s and therefore P is not h.i. by Theorem 2.7.

To prove the claim consider two cases.

Case 1. $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$. Then $|\tau_{n_{s+1},s+1}| = l_{s+1} = |\tau_{n_s,s}{}^\frown \mathbf{1}^{l_{s+1}-l_s}|$ and $\tau_{n_{s+1},s+1} = \tau_{n_s,s}{}^\frown \mathbf{1}^{e_{n_{s+1},s+1}}{}^\frown \mathbf{0}^{u_{n_{s+1},s+1}}$. But

$$\begin{aligned} u_{n_{s+1},s+1} &= l_{s+1} - |\tau_{j,s}| - e_{n_{s+1},s+1} \\ &> |\tau_{j,s}| + e_{n_{s+1},s+1} + k - |\tau_{j,s}| - e_{n_{s+1},s+1} \\ &\quad \text{from (2) and the definition of } l_{s+1} \\ &> 0. \end{aligned}$$

Therefore $\tau_{n_{s+1},s+1}$ is lexicographically less than $\tau_{n_s,s}{}^\frown \mathbf{1}^{l_{s+1}-l_s}$.

Case 2. If it is not the case that $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$, then by Lemma 2.33 $\tau_{n_{s+1},s+1}$ is lexicographically less than $\tau_{n_s,s}$. As $|\tau_{n_{s+1},s+1}| > |\tau_{n_s,s}|$, the two strings must be incomparable. Therefore any extension of $\tau_{n_{s+1},s+1}$ must be lexicographically less than any extension of $\tau_{n_s,s}$. The result follows *a fortiori.* □

Lemma 2.35. *X is the only non-isolated path in P.*

Proof. It is immediate from the construction that $e_{i,s+1} \leqslant e_{i,s}$ for all i and s. So e_i exists for all i. $u_{i,s} \neq u_{i,s+1}$ only when $e_{i,s} \neq e_{i,s+1}$ so u_i exists as well. And for all i, $\tau_i = \lim_s \tau_{i,s} = \mathbf{1}^{e_0}{}^\frown \mathbf{0}^{u_0}{}^\frown \mathbf{1}^{e_1}{}^\frown \mathbf{0}^{u_1}{}^\frown \ldots {}^\frown \mathbf{0}^{u_i}$ exists. But for each s $\tau_{i,s} \in T$ and so $\tau_i \in T$. $X = \bigcup_i \tau_i$ and so $X \in P$. Furthermore, τ_i is a branching node for all i (as $\tau_i^\frown \mathbf{0}^1 \in T$ by Lemma 2.31 and $\tau_i^\frown \mathbf{1}^1 \in T$ as it is extended by τ_{i+1}). So there are infinitely many branching nodes along X and X is not isolated.

Let $i > 0$ be arbitrary and let s be such that $\tau_{i,s} = \tau_i$ and $n_t > i$ for all $t \geqslant s$. Then if $Y \in P$ such that $Y \not\supset \tau_i$ then $Y \not\supset \tau_{n_t,t}$ for all $t \geqslant s$. An inspection of (1) shows that for all $\sigma \supsetneq Y[l_s]$, $\sigma = Y[l_s]^\frown \mathbf{0}^{|\sigma|-l_s}$ and hence Y is isolated. □

Lemma 2.36. *If $\sigma \in T$ and σ is of the form*

$$\mathbf{1}^{e_0\frown}\mathbf{0}^{u_0\frown}\ldots\mathbf{1}^{e_1\frown}\mathbf{0}^{q}$$

where $0 < q < u_i$, then $\sigma \notin \mathrm{Br}(P)$.

Proof. By (1) above, if $\sigma \in T$ then $\sigma^\frown\langle 1\rangle \in T$ only if σ is of the form $\tau_{n_s,s}^\frown \mathbf{1}^q$ for some s and $0 \leqslant q < l_{s+1} - l_s$. This is inconsistent with being of the above form. □

Lemma 2.37. *P is e.p.h.i.*

Proof. Let $\{e\}$ be any strictly increasing total computable function - a candidate for witnessing the fact that P is not e.p.h.i. Let s be a stage such that

i. for all $e_i < e$, $e_{i,s} = e_i$.

For all stages $t \geqslant s$ and for all $e_i < e$, $\tau_{i,t} = \tau_i$. In particular if j is as in (2) then $e_j < e$ and $|\tau_{j,t}|$ is constant for all $t \geqslant s$. We can also assume that s is so large that it also satisfies:

ii. there exists a $0 < k \leqslant s$ such that

$$|\tau_{j,s}| + e + k \leqslant \{e\}_s(|\tau_{j,s}| + e + k)\downarrow < \{e\}_s(|\tau_{j,s}| + e + k + 1)\downarrow .$$

The construction then ensures that $X \supset \tau_j$ and the choice of $u_{n_{s+1},s+1}$ guarantees that

$$\mathrm{Br}_X(P) \cap \big[\{e\}(|\tau_{i,s}| + e + k), \{e\}(|\tau_{i,s}| + e + k + 1)\big) = \emptyset,$$

and so $\{e\}$ does not witness the fact that P is not e.p.h.i. As e was arbitrary, P is e.p.h.i. □

□

Theorem 2.38. *Small ⇒ h.i.*

Proof. If we assume that a Π^0_1 class P is not h.i. witnessed by $\langle D^*_{f(n)}\rangle_n$, then the computable function

$$n \mapsto \max\{|\sigma| : \sigma \in D_{f(n)}\}$$

witnesses the fact that P is not small via the characterisation 2.10 4. □

Corollary 2.39. *There is a Π^0_1 class with no computable elements that is e.p.h.i. but not h.i.*

Proof. Let S be any small Π^0_1 class with no computable path and P as in Theorem 2.30. By Lemma 2.22 $P \vee S$ is e.p.h.i. But $P \vee S$ is not h.i. by Lemma 2.23, and it does not have a computable elements. □

Corollary 2.40. *There is a Π^0_1 class with no computable elements that is e.p.h.i. but not small.*

Proof. $P \vee S$ from Corollary 2.39 is e.p.h.i. but not h.i. By the above theorem it cannot be small. □

Theorem 2.41. *h.i.$\not\Rightarrow$ u.p.h.i. In fact any Π^0_1 class of positive measure contains an h.i. Π^0_1 class of positive measure that is not u.p.h.i.*

Proof. Let P be any Π^0_1 class of measure $m > 0$. We will create the required $Q \subseteq P$ by adapting the construction of Theorem 2.25. Let $k \in \mathbb{N}$ be such that $m > 2^{-k}$. We will ensure that for all $\sigma \in \mathrm{Ext}(Q)$, $\mu(Q(\sigma)) \geqslant 2^{-2|\sigma|-k-2}$. The function defined recursively by

$$f(0) = 0$$

$$f(n+1) = 2f(n) + k + 3$$

will then witness the fact that Q is not u.p.h.i. via the characterisation 2.11 3. This is straightforward to see because for any n and any $\sigma \in Q[f(n)]$, the measure of $Q(\sigma)$ is no less than $2^{-2f(n)-k-2}$. So there must be a branching node above σ of length less than $2f(n)+k+3$ — if there were not, then the measure of $Q(\sigma)$ could be no greater than $2^{-2f(n)-k-3}$.

Now to construct Q. As usual $Q = \bigcap_s Q_s$ — a computable intersection of clopen sets. Let P' be the Π^0_1 class from Theorem 2.25 and suppose $P' = \bigcap_{s=0}^{\infty} P'_s$ — a computable intersection of clopen sets. Let $Q_0 = 2^{\mathbb{N}}$. Suppose Q_s is defined, let

$$Q_{s+1} = (P'_s \cap Q_s) \setminus \bigcup\{Q_s(\sigma) : \mu(Q_s(\sigma)) < 2^{-2|\sigma|-k-2}\}.$$

Then Q is equal to

$$P' \setminus \bigcup_{s=0}^{\infty} \bigcup \{Q_s(\sigma) : \mu(Q_s(\sigma)) < 2^{-2|\sigma|-k-2}\}$$

and it is h.i. as it is a subset of P'. To prove that $\mu(Q) > 0$ first notice that

$$\begin{aligned}
\mu(\bigcup_{s=0}^{\infty} \bigcup \{Q_s(\sigma) : \mu(Q_s(\sigma)) < 2^{-2|\sigma|-k-2}\} &\leqslant \sum_{n=0}^{\infty} \sum_{|\sigma|=n} 2^{-2|\sigma|-k-2} \\
&= \sum_{n=0}^{\infty} 2^n 2^{-2n-k-2} \\
&< m \sum_{n=0}^{\infty} 2^{-n-2} \\
&= m/2.
\end{aligned}$$

But $\mu(P') \geqslant m/2$ from Theorem 2.25, so $\mu(Q) > m/2 - m/2 = 0$.

Finally, assume for a contradiction that $\tau \in \mathrm{Ext}(Q)$ and $\mu(Q(\tau)) < 2^{-2|\tau|-k-2}$. Then there must exist a t such that $\mu(Q_t(\tau)) < 2^{-2|\tau|-k-2}$. But then $Q_t(\tau) \cap Q_{t+1} = \emptyset$ and $\tau \notin \mathrm{Ext}(Q_{t+1}) \supseteq \mathrm{Ext}(Q)$. Contradiction. □

It will be useful later to note the following.

Theorem 2.42. *If $S = \mathcal{S}(A, B)$ is a separating Π^0_1 class then S is u.p.h.i. if and only if it is small. That is, small, e.p.h.i, p.h.i, and u.p.h.i. are equivalent in the case of separating classes.*

Proof. All separating Π^0_1 classes S have the property $\forall n \in \mathrm{Brl}(S) \forall \sigma \in S[n]\ \sigma \in \mathrm{Br}(S)$. By an easy induction argument it can be seen that for all $n \in \mathbb{N}$

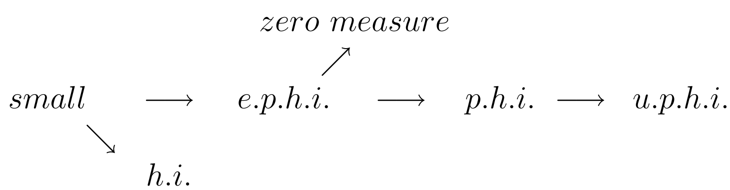
$$\min\{|\Psi_P(\sigma)| : |\sigma| = n\} = \max\{|\Psi_P(\sigma)| : |\sigma| = n\}.$$

Then the characterisations 2.10 2. and 2.11 2. show that any separating u.p.h.i. class is small.

□

The following diagram sums up the results in this section. The lack of an arrow between two properties indicates that a Π^0_1 counterexample with no computable paths is known.

Figure 1:



3 Thinness and other strengthenings

A similar analysis can be carried out using stronger notions than that of hyperimmunity. In this section we briefly consider the notions of *dense immunity* and *(co-)maximality*. For convenience I give their definitions here, but also see [19].

Definition 3.1. *$X \subseteq \mathbb{N}$ is* dense immune *if p_X dominates every computable function.*

Theorem 3.2. *$X \subseteq \mathbb{N}$ is dense immune if and only if for all strong arrays $\langle D_{f(n)} \rangle$ there are at most finitely many n such that*

$$\Big\| \bigcup_{i=0}^{n} D_{f(i)} \cap X \Big\| \geqslant n.$$

Notice that there is no requirement of disjointness in Theorem 3.2.

Definition 3.3. *$X \subseteq \mathbb{N}$ is* maximal *if it is coinfinite and for every c.e. set $Y \supseteq X$ either Y is co-finite or $Y \setminus X$ is finite.*

We use these well-established ideas to define analogous properties in $2^{\mathbb{N}}$ in the fashion of Section 2. Dense immunity turns out to be the most similar. We define a Π^0_1 class to be *very small (v.small)* in the same way as we defined smallness — but with "dense immunity" replacing "hyperimmunity". This was done in [3] in detail. We shall also define *everywhere pathwise dense immunity (e.p.d.i.)*, *pathwise dense immunity (p.d.i.)*, *uniform pathwise dense immunity (u.p.d.i.)* in the obvious way. In order to define *dense immunity (d.i.)* for subsets of $2^{\mathbb{N}}$ we will use the alternative characterisation in Theorem 3.2 and follow Theorem 2.7. That is, recalling Definition 2.6,

Definition 3.4. *A* Π^0_1 *class* P *is* dense immune *(d.i.) if it is infinite and there is no strong array* $\langle D^*_{f(n)}\rangle$ *such that for infinitely many* n

$$\|\mathrm{rt}\Big(\bigcup_{i=0}^{n} D^*_{f(i)}\Big) \cap \mathrm{Ext}(P)\| \geqslant n. \tag{3}$$

It is necessary to establish that these are all invariant under computable homeomorphisms. This is straightforward. The proof for d.i. Π^0_1 classes is similar to 2.14, the proof for v.small is in [3] and the rest of the proofs follow from Lemma 2.15.

To see how these new properties compare to the ones defined using hyperimmunity we first notice that every dense immune subset of $\mathbb{N}$ is hyperimmune so the following table is evident.

Figure 2:

$$\begin{array}{ccccccc} \textit{small} & \longrightarrow & \textit{e.p.h.i.} & \longrightarrow & \textit{p.h.i.} & \longrightarrow & \textit{u.p.h.i.} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \textit{v.small} & \longrightarrow & \textit{e.p.d.i.} & \longrightarrow & \textit{p.d.i.} & \longrightarrow & \textit{u.p.d.i.} \end{array}$$

It will be shown now that no diagonal arrows exist on the diagram (apart from the immediately necessary ones) and hence that the arrows on the bottom row are non-reversible. To see there are no unnecessary diagonal implications it is sufficient to establish the following four lemmas.

Theorem 3.5. *There is a small Π^0_1 class with no computable path that is not u.p.d.i.*

Proof. Let S be a small separating Π^0_1 class that is not v.small. Such an S exists by Theorem 3.16 of [3]. If S were u.p.d.i. then it would be v.small (using an analogous result to Theorem 2.42 and the fact that S is separating). □

Lemma 3.6. *E.p.d.i.$\not\Rightarrow$ small.*

Proof. (Sketch.)

The proof is similar to the proof of Theorem 2.30. A Π^0_1 class is constructed with exactly one non-isolated path X which has a dense-immune set of branching nodes on it. As before, every level of P is a branching level. □

Lemma 3.7. *P.d.i.$\not\Rightarrow$ e.p.h.i.*

Proof. Take a v.small Π^0_1 class P. $P \wedge 2^{\mathbb{N}}$ will be p.d.i. but not e.p.h.i. □

Lemma 3.8. *U.p.d.i.$\not\Rightarrow$ p.h.i.*

Proof. This is the same as Theorem 2.27 with f taken as the characteristic function of a dense immune Π^0_1 subset of $\mathbb{N}$. □

Of course other concepts of diminutiveness such as hyperhyperimmunity, r-maximality and so on (see [19] §X for example) could be studied in a similar way. We do not do this here and questions remain about whether the analogous properties would be computably topological in $2^{\mathbb{N}}$.

The collections of canonically indexed sets D_n and D_n^* form bases for the respective topologies on $\mathbb{N}$ and $2^{\mathbb{N}}$. But whereas $2^{\mathbb{N}}$ is compact in this topology, $\mathbb{N}$ is not. So some array definitions are possible in $2^{\mathbb{N}}$ that have no analogy (or rather no *interesting* analogy) in $\mathbb{N}$. For example we can require of a Π_1^0 class that for any disjoint strong array $\langle D^*_{f(n)}\rangle$ there are at most finitely many n such that $\langle D^*_{f(n)}\rangle \cap P \neq \emptyset$. The analogous property for subsets of $\mathbb{N}$ is equivalent to a set's being finite. However for subsets of $2^{\mathbb{N}}$ this property is equivalent to being *thin* in the sense of [6], [9], [10] and elsewhere. This is interesting to note because the usual definition of thinness (to follow) suggests that the correct analogy in $\mathbb{N}$ is maximality.

Definition 3.9. *A* Π_1^0 *class* P *is* thin *if its only* Π_1^0 *subclasses are its clopen subclasses (in the relative topology).*

It is not straightforward to see relationship between the previously defined properties and thinness but some have been established.

Theorem 3.10. *Thin* $\implies$ *d.i.*

Proof. Suppose P is not d.i. witnessed by the strong array $D^*_{f(n)}$. Let $D^*_{g(0)} = D^*_{f(0)}$ and

$$D^*_{g(n+1)} = \bigcup_{i=0}^{n+1} D^*_{f(i)} \smallsetminus \bigcup_{i=0}^{n+1} D^*_{f(i)}.$$

It is easy to see that $\langle D^*_{g(n)}\rangle$ is a disjoint strong array that also witnesses the fact that P is not d.i. As $\|\mathrm{rt}\Big(\bigcup_{i=0}^{n} D^*_{g(i)}\Big) \cap \mathrm{Ext}(P)\| \geqslant n$ for infinitely many n, it must be the case that $D^*_{g(n)} \cap P \neq \emptyset$ for infinitely many n. As $\langle D^*_{g(n)}\rangle$ is disjoint, P cannot be thin. □

Lemma 3.11. *v.small* $\implies$ *d.i.*

Proof. If $\langle D_{f(n)}\rangle$ witnesses the fact that P is not d.i. then the function

$$m(n) = \max\{|\sigma| : \sigma \in \mathrm{rt}\Big(\bigcup_{i=0}^{n} D_{f(i)}\Big)\}$$

witnesses the fact that $\mathrm{Br}(P)$ is not d.i. and hence that P is not v.small. □

Theorem 3.12. *D.i.* $\not\Longrightarrow$ *thin.*

Proof. If P is v.small then so is $P \vee P$ (see [3]), and by Lemma 3.11 $P \vee P$ is d.i. But $P \vee P$ is not thin as $\{f \oplus f : f \in P\}$ is a Π^0_1, non-clopen, proper subset of $P \vee P$. □

Theorem 3.13. *Thin* $\not\Longrightarrow$ *v.small.*

Proof. This is Theorem 4.3 in [3] and is a consequence of results in [9]. □

Theorem 3.14. *Thin* $\Longrightarrow$ *u.p.h.i.*

Proof. The proof is very similar to Simpson's proof that all thin Π^0_1 classes have zero measure. We prove the contrapositive. Suppose a Π^0_1 class P were not u.p.h.i. and this witnessed by the computable function f. That is,

$$\forall n \forall \tau \in P[f(n)] \exists \sigma \supseteq \tau \; f(n) \leqslant |\sigma| < f(n+1) \text{ and } \sigma \in \mathrm{Br}(P).$$

Define a sequence of elements of $\mathrm{Ext}(P)$ as follows ("left" and "right" here refer to the lexicographical ordering on $2^{\mathbb{N}}$):

$$\sigma_1 = \text{ the rightmost string on } P[f(1)]$$

$$\sigma_{n+1} = \text{ the rightmost string on } P[f(n+1)] \text{ to the left of } \sigma_n.$$

To prove that σ_n exists for all n, we use induction to prove that for all $n > 0$ there is a $\tau \in P[f(n)]$ such that τ is strictly to the left of σ_n. If τ is the rightmost such string in $P[f(n)]$, then σ_{n+1} will be the rightmost element of $P[f(n+1)]$ extending τ.

Base case: There is a branching node on P before level $f(1)$ so there must be a $\tau \in P[f(1)]$ strictly to the left of σ_1.

Induction: Suppose that τ is the rightmost element of $P[f(n)]$ strictly to the left of σ_n. There must be a branching node above τ before level $f(n+1)$ as P is u.p.h.i. Therefore there must be a τ' strictly to the left of σ_{n+1} defined as above.

The set $S = \bigcup_n U_{\sigma_n} \cap P$ is open in the relative topology of P, but it is not closed, as the set $\{U_{\sigma_n} : n \in \mathbb{N}\}$ is pairwise disjoint and P is compact. Furthermore, $\bigcup_n U_{\sigma_n}$ is Σ^0_1 so $P \smallsetminus S$ is a non-clopen Π^0_1 subclass of P, and P is not thin.

□

4 Muchnik and Medvedev Degrees

As we do for Turing degrees, if $\mathcal{C}$ is any property of Π^0_1 classes and $\mathbf{d}$ is a Muchnik degree then we say $\mathbf{d}$ has property $\mathcal{C}$ if $\mathbf{d}$ has a representative with property $\mathcal{C}$.

The questions arise now whether the properties defined in this paper describe different classes of Muchnik degrees. Also it can be asked where these classes of Muchnik degrees fit into the known structure of the Muchnik lattice. An analogous type of theorem in the Turing degrees is one of Dekker's that states that every c.e. Turing degree has a hyperimmune representative [8].

Here not as much is known as would be liked, but we present some basic results. Some conjectures and open questions are discussed in the following section.

The next two Lemmas are very useful in this area.

Lemma 4.1 (Simpson)**.** *For all Π^0_1 classes P and Q if $P \geqslant_w Q$, then there exists a Π^0_1 subclass $P' \subseteq P$ such that $P \geqslant_M Q$.*

Proof. See [17] or [3] □

Lemma 4.2 (Simpson)**.** *If P and Q are Medvedev complete Π^0_1 classes, then P is recursively homeomorphic to Q.*

Proof. See [18] □

Lemma 4.3. *DNR_2 is neither h.i. nor u.p.h.i.*

Proof. Let $e_0 < e_1 < e_2 \dots$ be a computable sequence of indices for the empty function. For every i define

$$E_i = \{f \in 2^{\mathbb{N}} : \forall j < i \ f(e_j) = 0 \text{ and } f(e_i) = 1\}.$$

Each E_i intersects DNR_2 as $0 \neq \{e_i\}(e_i) \neq 1$ for all i. They are also pairwise disjoint and so form a disjoint strong array. So DNR_2 is not h.i.

To see it is not *u.p.h.i.* first notice that $\mathrm{DNR}_2 = \mathcal{S}(A, B)$ where $A = \{e : \{e\}(e) \downarrow = 0\}$ and $B = \{e : \{e\}(e) \downarrow = 1\}$. It is therefore a separating class and if it were u.p.h.i. it would be small by 2.42. It is not small, however, because $e_0, e_1, e_2, \dots$ is a computable sequence of branching levels of DNR_2 (Theorem 2.10 3.)

□

Theorem 4.4. *If P is an h.i. or e.p.h.i. Π^0_1 class, then it is Muchnik incomplete.*

Proof. Suppose P were an h.i. Π^0_1 class and that $P \geqslant_w \mathrm{DNR}_2$. Then by Lemma 4.1 there would be a Π^0_1 $P' \subseteq P$ such that $P' \geqslant_M \mathrm{DNR}_2$. So P' is Medvedev complete. Hyperimmunity is closed under taking subsets so P' is also h.i. But DNR_2 is not h.i. by Lemma 4.3 and so no other Medvedev complete Π^0_1 class can be by Lemma 4.2 and Theorem 2.14.

The proof is identical for the e.p.h.i. case.

□

Theorem 4.5. *If P is u.p.h.i, then it is Medvedev incomplete.*

Proof. If it were Medvedev complete then DNR_2 would be u.p.h.i. by Lemma 4.2 and Theorem 2.14 □

It is currently an open question whether every u.p.h.i. or p.h.i. Π^0_1 class is Muchnik incomplete. We conjecture that it is so.

Theorem 4.6. *There is an h.i. Muchnik degree that is not less that any small Muchnik degree.*

Proof. There is a Π^0_1 class R consisting entirely of 1-random reals with the property that any Π^0_1 subclass of R has positive measure [14]. R also has the property that if M is any Π^0_1 class of positive measure then $R \geqslant_w M$ (see [17] for an exposition).

Lemma 2.25 implies that R must have an h.i. Π^0_1 subclass R' and the above implies that $R' \equiv_w R$. So R has h.i. Muchnik degree. But if S were any small Π^0_1 class such that $S \geqslant_w R$ then by Lemma 4.1 there would be a Π^0_1 $S' \subseteq S$ such that $S' \geqslant_M R$. That is, there would be a computable functional $\Phi : S' \longrightarrow R$. S' is small as it is a subclass of S and its image under Φ is also small by Theoerem 2.13. But every subset of R is of positive measure so the image of Φ must be small and of positve measure — contradicting Theorem 2.24.

□

Corollary 4.7. *The class of small Muchnik degrees is strictly contained in the class of h.i. Muchnik degrees.*

The problem of the density of the Muchnik lattice is still the outstanding problem in the area. Partial results have been obtained for example Corollary 3.17 in [3]. The next lemma by Simpson in [17] (Corollary 7.5) gives upward density for a large class of Muchnik degrees.

Lemma 4.8. *[Simpson] Let P, Q and S be Π^0_1 classes such that P is of positive measure and S is a separating class. Then*

$$P \vee Q \geqslant_w S \implies Q \geqslant_w S.$$

Proof. See [17]. Note that $P \ngeqslant_w S$ by Theorem 5.3 in [13]. The proof is a relativisation and generalisation of Theorem 5.3. □

Which gives the following theorem as a corollary.

Theorem 4.9. *If P is a small Π^0_1 class, then there is a Π^0_1 class Q such that*

$$P <_w Q <_w DNR_2.$$

Proof. By Lemma 2.38 P must be h.i. and therefore $P <_w DNR_2$ by Theorem 4.4. In the proof of Theorem 4.6 $R \nleqslant_w P$. Therefore $R \vee P >_w P$. But Lemma 4.8 also implies that $R \vee P <_w \mathrm{DNR}_2$ as DNR_2 is a separating class. □

5 Open Questions and Further Directions

• Does there exist a u.p.h.i. (e.p.h.i, p.h.i.) Muchnik degree that is not small (h.i)?

These problems can be solved by constructing a u.p.h.i. (e.p.h.i, p.h.i.) Π^0_1 class that has no small (h.i.) subclass. And then use an argument like the proof of Theorem 4.6.

It is not clear how to proceed in other questions of this type — for example does there exist a u.p.h.i. Muchnik degree that is not p.h.i?

• Is every u.p.h.i. (p.h.i.) Muchnik degree Muchnik incomplete?

An essential property in showing that every small Π^0_1 class (for example) is Muchnik incomplete is the property that every Π^0_1 subclass of a small Π^0_1 class is small. This property is not shared by u.p.h.i. or p.h.i. classes. Another method of showing incompleteness needs to be found.

There are many easily describable intermediate Muchnik degrees. For example in [17] Simpson defines a transfinite sequence of such degrees related to the diagonally non-recursive functions. It is unknown how the properties described in this paper relate to such degrees. For example the obvious question:

- Is DNR a small degree?

has not been answered. Here DNR is the Muchnik degree of the set $D = \{f \in \mathbb{N}^{\mathbb{N}} : \forall e f(e) \neq \{e\}(e)\}$. This set is a Π^0_1 subclass of $\mathbb{N}^{\mathbb{N}}$ rather that $2^{\mathbb{N}}$ and is in fact not even computably bounded - however there is a Π^0_1 class subclass of $2^{\mathbb{N}}$ whose elements have the same Turing degrees as those in D

- Is every small (thin) Muchnik degree thin (small)?

It is known that not every small Π^0_1 class is thin (see [3]) but is is not known if every thin class is small. Whether or not their Muchnik degrees coincide could be answered negatively if one were to construct a small (thin) Π^0_1 class with no thin (small) subclass.

References

[1] K. Ambos-Spies, G. H. Müller, and G. E. Sacks, editors. *Recursion Theory Week*. Number 1432 in Lecture Notes in Mathematics. Springer-Verlag, 1990. IX + 393 pages.

[2] Stephen Binns and Stephen G. Simpson. Embeddings into the Medvedev and Muchnik lattices of Π^0_1 classes. *Archive for Mathematical Logic*, 43:399–414, 2004.

[3] Stephen E. Binns. Small Π^0_1 classes. *Archive For Mathematical Logic*. to appear.

[4] Stephen E. Binns. *The Medvedev and Muchnik Lattices of* Π^0_1 *classes*. PhD thesis, The Pennsylvania State University, 2003. 80 pages.

[5] Douglas Cenzer and Jeffrey B. Remmel. Π^0_1 classes in mathematics. In [12], pages 623–821, 1998.

[6] Peter Cholak, Richard Coles, Rod Downey, and Eberhard Herrmann. Automorphisms of the lattice of Π^0_1 classes; perfect thin classes and ANC degrees. September 2000. Preprint, 33 pages.

[7] S. B. Cooper, T. A. Slaman, and S. S. Wainer, editors. *Computability, Enumerability, Unsolvability: Directions in Recursion*

Theory. Number 224 in London Mathematical Society Lecture Notes. Cambridge University Press, 1996. VII + 347 pages.

[8] J.C.E. Dekker. A theorem on hypersimple sets. *Proc. Amer. Math. Soc.*, 5:791–796, 1954.

[9] Rodney G. Downey, Carl G. Jockusch, Jr., and Michael Stob. Array nonrecursive sets and multiple permitting arguments. In [1], pages 141–174, 1990.

[10] Rodney G. Downey, Carl G. Jockusch, Jr., and Michael Stob. Array nonrecursive degrees and genericity. In [7], pages 93–105, 1996.

[11] H.-D. Ebbinghaus, G.H. Müller, and G.E. Sacks, editors. *Recursion Theory Week*. Number 1141 in Lecture Notes in Mathematics. Springer-Verlag, 1985. IX + 418 pages.

[12] Y. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel, editors. *Handbook of Recursive Mathematics*. Studies in Logic and the Foundations of Mathematics. North-Holland, 1998. Volumes 1 and 2, XLVI + 1372 pages.

[13] Carl G. Jockusch, Jr. and Robert I. Soare. Π^0_1 classes and degrees of theories. *Transactions of the American Mathematical Society*, 173:35–56, 1972.

[14] Antonín Kučera. Measure, Π^0_1 classes and complete extensions of PA. In [11], pages 245–259, 1985.

[15] Hartley Rogers, Jr. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967. XIX + 482 pages.

[16] S. G. Simpson, editor. *Reverse Mathematics 2001*. Lecture Notes in Logic. Association for Symbolic Logic, 2004. To appear.

[17] Stephen G. Simpson. Mass problems and randomness. *Bulletin of Symbolic Logic*, pages 29 pages, to appear.

[18] Stephen G. Simpson. Π^0_1 sets and models of WKL_0. In [16]. Preprint, April 2000, 29 pages, to appear.

[19] Robert I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic. Springer-Verlag, 1987. XVIII + 437 pages.