KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA

DEPARTMENT OF MATHEMATICAL SCIENCES

A Research Proposal entitled

"COINCIDENCES AND APPROXIMATION OF NON-COMMUTING
MULTIVALUED MAPS WITH APPLICATIONS "

Proposed by

Dr. A. R. Khan
Dr. N. Hussain

November 1, 2006

DEANSHIP OF SCIENTIFIC RESEARCH

RESEARCH PROJECT NO. …………………………

Title of Proposal:

"COINCIDENCES AND APPROXIMATION OF NON-COMMUTING MULTIVALUED MAPS WITH APPLICATIONS "

Duration of Project (in months)	**: 12**
Proposed Starting Date	**: April, 2007**
Ending Date	**: March, 2008**
Total Project Cost	**: (SR) 38,900**

Submitted by:

Dr. A. R. Khan, Associate Professor, Mathematical Sciences

Dr. N. Hussain, Assistant Professor, Department of Mathematics, KAAU, Jeddah

Date: **November 1, 2006**

APPROVALS:_____

Chairman:_____Date:_____

Department: **Mathematical Sciences**

Chairman,
Research Committee:_____Date: _____

Vice Rector for Graduate Studies
and Scientific Research: _____Date:_____

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2. <u>ABSTRACT</u>

The goal of this project is to study coincidence point , best approximation and iterative approximation results for compatible, weakly biased and subcompatible single valued as well as multivalued maps in Hilbert spaces, locally convex spaces and hyperconvex spaces. We will mainly focus on deterministic and random common fixed points and approximation theorems of Fan, Sahab, Khan and Sessa, Jungck and Sessa, O'Regan and Hussain and Shahzad for pairs of compatible, weakly biased and subcompatible maps. We aniticpate that as applications of our results some well known theorems in current literature would follow as corollaries to our results – thus broaderning the scope of common fixed point, approximation results and iterative methods.

3. <u>INTRODUCTION AND LITERATURE REVIEW</u>

The fixed point theory is important both from academic as well as applications point of view. The fixed point results due to Browder [10], Ciric [11], Ciric and Ume [12], Deng and Yang [14], Jungck [34-36], Jungck and Rhoades [39], Nadler [60], Khan and Hussain [46-48], O'Regan [62], Rhoades [69], Shahzad and Hussain [77] and Tarafdar [86] for single valued and multivalued maps are of particular importance and play a fundamental role in this context. These results have applications in various fields; among others are numerical analysis and differential equations. On the other hand, metrizable locally convex spaces have been widely used in a number of disciplines (for instance, the reader of economics is referred to Yannelis and Prabhakar [90, Remark 7.4] and differential equation experts are referred to [67] and references there in for more details).

Approximation theory has applications in physical sciences, engineering and economics. It also plays an important role in optimization, probability theory and artificial neural net works etc. (see [15, 25, 59, 80, 81]). We also refer to the recent contributions by Al-Thagafi [1], Al-Thagafi and Shahzad [2], Hussain and Jungck [28], Jungck and Hussain [37], Jungck and Sessa [40], Khan [44], O'Regan and Hussain [63], O'Regan and Shahzad [64], Shahzad [76] and Singh [81,82] for more work on approximation theory.

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. The interest in this subject enhanced

after publication of the survey paper by Bharucha-Reid [8]. Since then various types of random fixed point theorems have been obtained by numerous mathematicians; see, for example, [4,5,18,32,52,57,65,75,77,87,89]. Recently, Shahzad and Latif [78] and Khan and Hussain [46] established random versions of the results of Latif and Tweddle [56] in the setting of Banach spaces and Frechet spaces, respectively. In [69], Rhoades has generalized some results of Latif and Tweddle [56] by replacing commutativity condition of maps with R-subcommuting condition whereas, Shahzad [75] has studied their results for the class of R-subweakly commuting maps. More recently, Shahzad and Hussain [77] and Khan et al. [45] have established deterministic and random coincidence point results for more general class of T-weakly commuting maps in the setting of normed spaces. Deng and Yong [14] have applied their coincidence theorems to solve minimax inequalities, section theorems, best approximation theorems, and to prove the existence theorems of weighted Nash equilibria.

The notion of "hyperconvexity" in a metric space has been extensively studied (see e.g. [42] and [54] where further references are also given). Recently, fixed point and approximation theory in hyperconvex spaces has been the focus of several researchers (see [19], [42], [43] and [79]).

Common fixed point theorems for families of commuting contraction maps have been a popular area of research (see, e.g. Al-Thagafi [1] and Belluce and Kirk [6]). In 1982, Sessa [73] introduced the concept of weakly commuting maps to generalize commutativity. Pant [66] and Jungck [36] generalized weak commutativity to the notions of R-weakly commuting and compatible maps, respectively. In 1995, Jungck and Pathak [38] introduced the more general concepts of biased and weakly biased maps. Since then, many interesting common fixed point theorems for compatible, weakly compatible and weakly biased maps under contractive conditions have been obtained by a number of authors. Many results which are closely related to Gregus's Theorem have appeared in recent years ([11,16,21,45,50]). The existence of fixed points from the set of best approximations has been studied by various authors; see Al-Thagafi [1], Hussain and Rhoades [31], Kamran [41], O'Regan and Hussain [63], O'Regan and Shahzad [64], Sahab et al. [72], Shahzad [76], and Singh [82]. Hussain and Khan [29] and Hussain et al. [30] have recently, obtained results on invariant approximation in the context of Hausdorff locally convex spaces which extend many known results including the main result of Sahab, Khan and Sessa [72].

Assuming the existence of a "contractive and jointly continuous family of functions from (0,1) to (0,1) associated with a subset X of a Hausdorff locally bounded topological vector space", a concept due to Dotson [17], Hussain [26], and Hussain et al. [30] have proved the existence of common fixed points for generalized I-nonexpansive maps on X (without convexity argument); the authors also obtained Brosowski-Meinardus type theorems on invariant best approximation in this setup.

In 1967, Browder [9] and Halpern [24] proved strong convergence theorems in the framework of Hilbert spaces for implicit and explicit iteration schemes, respectively. These results have been extended in various directions by Reich [68], Xu [88] and many others (see [7,49,54, 70]). The class of asymptotically nonexpansive mappings was introduced by Goeble and Kirk [22] and further studied by various authors (see [7] and [54]). Ishikawa iteration (a two-step iteration) and Mann iteration (a one-step) processes for asymptotically nonexpansive self-mappings have been studied by various authors. For example, see [13,53,74,83,84]. Recently, Beg et al. [3], have proved strong convergence of $Ix_n = x_n = (1 - k_n)q + k_n T^n x$ to a common fixed point of uniformly R-subweakly commuting pair (I,T).

The purpose of this project is to establish new coincidence and common fixed point results for some non-commuting single valued as well as multivalued maps satisfying Lipschitz type conditions on locally convex spaces, metrizable topological vector spaces and hyperconvex spaces. As applications, we obtain: the existence of common fixed points of the maps from the set of best approximations, existence of best approximation results of Thagafi type and study certain minimax inequalities and section theorems due to Ding and Yang [14]. We intend to explore certain aspects of Ciric's and Jungck's extensions of Gregus's Theorem in this setting. We shall also consider strong and weak convergence theorems for asymptotically I-nonexpansive mappings in a uniformly convex Banach space. Ky Fan's approximation result in ordered Banach spaces will also be studied. Our work would improve many well-known results in current literature.

We expect that our investigations would unify and strengthen several well-known existing results about coincidences, common fixed points, best approximations and iterative approximation of fixed points.

We shall use Berinde [7], Joshi and Bose [33], Kirk and Sims [54], Kothe [55], Rudin [71] and Zeidler [91] as our general references.

4. **PROJECT OBJECTIVES**

Main goal is to study common fixed point, approximation theory and iterative techniques in locally convex spaces, metrizable topological vector spaces and hyperconvex spaces for non-commuting class of maps with a particular reference to the Gregus's Theorem, iterative approximation techniques and Ky Fan's best approximation theorem. The topics of our interest will include

 (i)   Generalized I-nonexpansive multivalued maps,

 (ii)   Gregus type multivalued maps,

(iii)   compatible, subcompatible and weakly biased maps (see [34,36,38] for closely related concepts),

(iv)   iterative processes and their strong and weak convergence,

 (v)   Subbiased maps and approximation theory,

(vi)   Study of Roberts spaces as a model (cf. [61]).

5. **DESCRIPTION OF THE PROBLEM AND PROPOSED APPROACH FOR ITS SOLUTION**

We intend to establish

 (i)   Deterministic and random coincidence results with applications.
 (ii)   Common fixed point and approximation results in locally convex spaces.
(iii)   Ciric and Jungck type extensions of Gregus's Theorem.
(iv)   Iterative approximation theorems similar to those by Beg et al. [3],   Rhoades [70], and Xu [88].
 (v)   Some Ky Fan-type approximation results for 1-set contractive maps on a closed convex subset of an ordered Banach space.
(vi)   Some properties of subbiased maps.

Main tools of our investigation will be

> (a) techniques similar to those of Jungck and Hussain [37], Jungck and Rhoades [39], Khan et al. [45] and Shahzad and Hussain [77] in our frame-work,
>
> (b) Jungck's [35] extension of Gregus's Theorem,
>
> (c) fixed point theorem(s) due to Jungck [34,36], Khamsi, Kirk and Yanez [43], Lin [57], O'Regan and Hussain [63] and Pant [66] .

(d) Iterative approximation techniques due to Beg et al. [3], Chidume et al. [13], Rhoades [70], and Xu [88].

6. RELEVANCE OF THE PROJECT FOR MATHEMATICAL SCIENCES DEPT. (KFUPM)

Fixed point theory, best approximations and iterative processes are the active areas of research with wide range of applications in various fields (see [7, 54, 59, 91, 92]). We hope that the results obtained in this project would contribute to research activities in this area. Further, collaboration in this research project would be useful for the graduate program of the department.

7. PLAN OF WORK

The project would be spread over twelve (12) months according to the following scheme:

Phase I (3 months)
During this phase, we will study the relationship among various classes of non-commuting maps on locally convex spaces, metrizable topological vector spaces and hyperconvex spaces.

Phase II (3 months)
During this phase, we will work on coincidences and approximation theorems for non-commuting maps such as compatible and subcompatible, weakly biased on metrizable topological vector spaces and hyperconvex spaces.

Phase III (3 months)
During this phase, we intend to seek extensions of our results to more general domains and consider the random case. We shall concentrate on iterative approximation of common fixed points based on the results developed earlier. Also, we shall try to find a suitable class of topological vector spaces as a model for the illustration of our results.

Phase IV (3 months)
During this phase, we shall consolidate our results and prepare a report on the project along with pre-print of our research paper(s).

8. SCHEDULING

i **Principal Investigator**: The involvement of Dr. A.R.Khan will be 40% during the academic year.

ii **Co-Investigator**: The involvement of Dr. N. Hussain will be 40% during the academic year.

9. <u>PUBLICATION PLAN</u>

We expect three research papers as an outcome of this project. We will send these papers to professional refereed journals for publication.

10. <u>REFERENCES</u>

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11. BUDGET

    A. Principal Investigator:
        Dr. A.R.Khan(ID # 6970893)
        Project Duration: 12 months
        Compensation @ SR1,200/- per month   14,400

    B.   Conference Attendance   10,000

    C. High -speed Computer and Printer   11,000

    D. Others (typing, stationery, copying, etc.)   1,500

    E. Inland Travel Expenses   2,000

        TOTAL   SR   38,900

12. LIST OF REFEREES

[1]   Prof. S.P. Singh,
    Department of Mathematics and Statistics
    Memorial University of Newfoundedland, St. John's NF
    Canada A1C 5S7
    E-mail : spsingh@math.mun.ca

[2].   Prof. G. Jungck,
    Department of Mathematics, Bradley University,

    Peoria, Illinois 61625, USA

    E-mail : gfj@hilltop.bradley.edu

[3]   Prof. V. Berinde,
    Department of Mathematics and Computer Science
    Faculty of Sciences North University of Baia Mare
    Victoriei Nr. 76, 430122 Baia Mare, ROMANIA
    E-mail : vberinde@ubm.ro

[4]   Prof. B. E. Rhoades,

    Department of Mathematics, Indiana University,

    Bloomington, IN 47405-7106, USA.

    E-mail : rhoades@indiana.edu

[5]   Prof. M. A. Khamsi,
    Department of Mathematical Sciences
    University of Texas at El Paso\\
    El Paso, TX 79968-0514 \\
    E-mail mohamed@math.utep.edu

[6]   Prof. M.S. Khan
    Department of Mathematics and Statistics
    College of Science, Sultan Qaboos University
    P.O.Box 36Al-Khoud 123
    Muscat ,Oman
    E-mail : mohammad@squ.edu.om

Nationality	Pakistani
Date of Birth	January 1, 1953
Present Address	Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, (KFUPM), Dhahran 31261, Saudi Arabia.

Academic Record

Ph.D.	University of Wales, UK, 1983
M.Sc.	University of the Punjab, Lahore, Pakistan, 1974
B.Sc.(Hons.)	University of the Punjab, Lahore, Pakistan, 1973
Special Field	Functional Analysis
Minor Fields	Approximation Theory and its Applications to Fixed Point Theory, Measure Theory and Integration.

Professional Experience

Associate Prof.	Department of Mathematical Sciences, KFUPM, Dhahran , Saudi Arabia. September 1997 to date.
Professor	Center for Advanced Studies In Pure and Applied Mathematics (CASPAM), Bahauddin Zakariya (B.Z.) University, Multan, Pakistan. January 1996 - August 1997.
Associate Prof.	CASPAM, B.Z. University, Multan, Pakistan. May 1988 - December 1995.
Assistant Prof.	CASPAM, B.Z. University, Multan, Pakistan. August 1984 - May 1988.
Lecturer	Department of Mathematics, University of Multan, Pakistan. October 1976 - August 1984.

Research Activities

Research Project completed At KFUPM (Research Project: FT/2002-01)

Deterministic and random versions of Ky Fan,s approximation theorem with applications

Ph.D.Thesis supervised at KFUPM

A. A. Domlo(ID #200652)   Fixed points of some nonlinear   maps with applications ,2006 .

(b) Theses supervised at B.Z. University, Multan, Pakistan

(i)   Ph.D. Thesis Supervised

(1) Nawab Hussain   Some types of best approximation and their applications, 2002.
(2) Arjamand Bano   Fixed points for multivalued maps,2004.
(3) M.Arif Rafiq   Fixed point theorems in generalized metric and Banach spaces, 2004.

(ii)   M. Phil. Thesis Supervised

1.	Attiya Qaseem Siddiqui	The strict convexity and fixed point theorems, 1987.
2.	M.K. Mehmood	The fixed points of non-expansive mappings,1988 (Review).
3.	Naseem Gul	The Hewitt-Yosida decomposition, 1988 (Review).
4.	Safeer Hussain	Group-valued order continuous submeasures, 1990.
5.	Mujahid Abbas	Multifunctions and the existence of best approximation, 1992.
6.	M. Shahid Ashraf	Strongly unique best approximation in metrizable topological vector spaces, 1993.
7.	M. Aslam	Best approximation and fixed point theorems in locally convex spaces, 1995.
8.	Hamid Ayaz	Some aspects of strong uniqueness, 1995.
9.	Gulam Mustafa	Fixed point theorems for multivalued mappings, 1996.
10.	M. Akram	Some studies in open mapping and closed graph theorems, 1996.

(c) List of M. Phil. theses examined at Quaid-e-Azam University, Islamabad, Pakistan.

1. Fixed point theorems, 1986.
2. Integral representation of linear operators, 1987.
3. Decompositional properties of Banach spaces, 1988.
4. Best approximation in function spaces, 1990.
5. P. Commutative topological algebras, 1991.
6. Compact and precompact operators on topological vector spaces, 1991.
7. Gelfand representation theory of topological algebras, 1991.
8. Fixed point theorems in probabilistic spaces, 1991.
9. Decomposition of C*-algebras relative to a functional equation, 1991.
10. Weak resolvents of operators, 1992.

Selected Research Publications (In refereed Journals)

1. A.R. Khan (with K. Rowlands), A decomposition theorem for submeasures, Glasgow Math. J. 26, no.1, 1985, 67-74.

2. A.R. Khan, On group-valued sbmeasures, Punjab Univ. J. Math. VXII-VXIII, 1984-85, 11-22.

3. A.R. Khan, A note on semigroup-valued measures, Math. Japonica, 30, no.3, 1986, 399-403.

4. A.R. Khan (with K. Rowlands), On a theorem of Danes and the principle of equicontinuity, Bolletino UMI (6) 5-A, 1986, 211-215.

5. A.R. Khan, On the theorem of Helson and the principle of equicontinuity, J. Natural Sci. and Math. 27, no.1, 1987, 15-19.

6. A.R. Khan (with A.Q. Siddiqui) Fixed points in metrizable topological vector spaces, Math. Japonica, 36, no.6, 1991, 1129-1134.

7. A.R. Khan (with S.H. Khan), Principle of equicontinuity for topological groups, Matema. Bech. 44, 1992, 7-10.

8. A.R. Khan (with Zahida and M. Abbas), Fixed point theorems for set-valued mappings in a semi-convex setting, Southeast Asian Bull. Math., World Scientific Publishing Co., special issue 1993, 43-47.

9. A.R. Khan (with M.A. Shahid and N.Hussain), Strong uniqueness in metrizable topological vector spaces, Bull. Malaysian Math. Soc. (Second series) 17, 1994, 21-27.

10. A.R. Khan (with L.A. Khan), An extension of Brosowski-Meinardus theorem on invariant approximation, Approximation Theory and its Applications 11, no.4, 1995, 1-5.

11. A.R. Khan (with M. Aslam & N. Hussain), Some best approximation results in locally convex spaces, Approximation Theory and its Applications 12, no.3, 1996, 29-36.

12. A.R. khan (with N. Hussain), Mann iterative construction of fixed points in locally convex spaces, J. Natural Sci. and Math. 26, no.2, 1996, 155-159.

13. A.R. Khan (with N. Hussain & L.A. Khan), A note on Kakutani type fixed point theorems, Internat. J. Math. & Math. Sci. 24, no.4, 2000, 231-235.

14. A.R. khan (with N.Hussain), Best approximation and fixed point results, Indian J. Pure Appl. Math. 31(8), 2000, 983-987.

15. A.R. Khan (with N. Hussain & A.B. Thaheem), Applications of fixed point theorems to invariant approximation, Approximation Theory and its Applications 16, no. 3, 2000, 48-55.

16. A.R. Khan (with N. Hussain), Fixed point and best approximation theorems for *-nonexpansive maps, Punjab Univ. J. Math. XXXIII, 2000, 135-144.

17. A. R. Khan (with N. Hussain), Random fixed points for *-nonexpansive random operators, J. Appl. Math. Stoch. Anal. 14, no.4. 2001, 341-349.

18. A.R. Khan (with N. Hussain), Iterative approximation of fixed points of nonexpansive maps, Sci. Math. Japon. 54, no.3, 2001, 503-511.

19. A.R. Khan (with A.B. Thaheem), On some properties of Banach operators, Internat. J. Math & Math. Sci. 27, no.3, 2001, 149-153.

20. A.R. Khan (with S.H. Khan), Group-valued submeasures and the range of measures, Scientific Annals, of "A1. I. Cuza", University of Iasi, XLVII, Mate., 2001, 35-42.

21. A.R. Khan (with N. Hussain), An extension of a theorem of Sahab, Khan and Sessa, Internat J. Math. & Math. Sci. 27, no.11, 2001, 701-706.

22. A.R. Khan (with A. Latif and A. Bano), Some results on multivalued s-nonexpansive maps, Radovi Mat. 10, no.1, 2001, 195-201.

23. A.R. Khan (with N. Hussain), Random fixed point theorems for *-nonexpansive operators in Frechet spaces, J. Korean, Math. Soc. 39, no.1, 2002, 51-60.

24. A.R. Khan (with A.B. Thaheem and N. Hussain), Random fixed points and random approximations in nonconvex domains, J. Appl. Math. Stoch. Anal., 15, no.3, 2002, 263-270.

25. A.R. Khan (with A. Bano and N. Hussain), Common fixed points in best approximation theory, Internat. J. Pure and Applied Math., 2, no.4, 2002, 411-426.

26. A.R. Khan (with N. Hussain), Random approximations and random fixed points for *-nonexpansive maps, Math. Sci. Res. J., 6, no.4, 2002, 174-182.

27. A.R. Khan (with A. Latif, A. Bano and N. Hussain), Coincidence point results in locally convex spaces, Internat, J. Pure and Applied Math. 3, no.4, 2002, 413-423.

28. A.R. Khan (with N. Hussain), Common fixed point results in best approximation theory, Applied Mathematics Letters, 16, no.4, 2003, 575-580.

29. A.R. Khan (with I. Beg and N. Hussain), Fixed point, almost fixed point and best approximation of nonexpansive multivalued mappings in Banach Spaces, Adv. Math. Sci. Appl. 13, no.1, 2003, 83-111.

30. A.R. Khan (with N. Hussain), Random fixed points of multivalued *-nonexpansive maps, Random Operators and Stochastic Equations, 11, no.3, 2003.

31. A.R. Khan (with N. Hussain), Applications of the best approximation operator to *-nonexpansive maps in Hilbert spaces, Numer. Funct. Anal. and Optimiz. 24, no.3& 4, 327-338.

32. A.R. Khan (with I. Beg and N. Hussain), Approximation of *-nonexpansive random multivalued operators on Banach Spaces, J. Australian Math. Soc. 76, 2004, 51-66.

33. A.R. Khan (with N. Hussain), Common fixed point and best approximation in P-normed spaces, Demonstratio Mathematica 36, no.3, 2003, 675-681.

34. A.R. Khan (with N. Hussain and A.B. Thaheem), Random fixed points and random approximations, Southeast Asian Bull. Math. 27, 2003, 1-6.

35. A.R. Khan (with N. Hussain), Random Coincidence point theorem in Frechet spaces with applications, Stoch. Anal. Appl. 22, no.1, 2004, 155-168.

36. A.R.Khan (with N.Hussain and A.B.Thaheem), Some generalizations of Ky Fan,s best approximation theorem, Analysis in theory and Applications 20 ,no.2,2004,189-198.

37. A.R.Khan (with A.B.Thaheem) , On some properties of Banach operators II,Inter. J.Math.& Math. Sc.47 ,2004,2513-2515.

38. A.R.Khan (with H.Fukhar-ud-din),Convergence of implicit iterates with errors for mappings with unbounded domain in Banach spaces ,Inter.J.Math.&Math. Sc. 10,2005,1643-1653.

39. A.R.Khan (with A.Latif and N.Hussain), Some results on common fixed points and best approximation,Tamkang J.Math ,36,no.1,2005,33-38.

40. A.R.Khan ,Properties of fixed point set of a multivalued map,J.Appl.Math. Stoch. Anal.2005:3,2005,323-331.
41.A.R.Khan(with A.A. Domlo),Common fixed points of compatible maps on balls and eigenvalue problems, Nonlinear Analysis Forum 11,no1,2006,15-21.

42. A.R. Khan (with F.Akbar,N.Sultana and N.Hussain) ,Coincidence and invariant approximation theorems for generalized f- nonexpansive multivalued mappings,Internat. J. Math. Math. Sci.,2006 (2006),1-18.
43. A.R.Khan (with A.A. Domlo) ,Random fixed points of multivalued inward random operators,J.Appl.Math Stoch. Anal. Volume 2006,Article ID19428,pages1-8 .
44 . A.R. Khan (with K.Rowlands) , On locally solid topological lattice groups,Czechoslovak Mathematical Journal (accepted).
45. A.R.Khan , On principle of equicontinuity, Engineering Modeling (in press) .
46. A.R. Khan (with H.Fukhar-ud-din), Approximating common fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces (accepted) .

Other Publications

1. Introduction to Lebesgue Integration, Iimi Kitab Khana, Lahore, Pakistan, 1993 (M.Sc. Level Book).

2. Fixed point theorems for the sum of two maps in locally convex spaces, Proc. All Pakistan Mathematical Conference, 1997, 15-20.

3. Common fixed points from best approximation, Proc. 26th Summer Symposium in Real Analysis (Washington and Lee University), Real Analysis Exchange, 2002, 189-196.

4. Random coincidence points of multivalued contractive random operators, Proc.International Conference on Nonlinear Analysis and Convex Analysis ,Okinawa,2005,1-12.

Seminars / Invited Lectures

1. A measure-theoretic proof of the uniform boundedness principle, Department of Pure Mathematics, University College of Wales, Aberystwyth, UK (1979).

2. On the generalization of exhaustion principle for submeasures, Conference Scientific Society of Pakistan, University of Karachi, Pakistan (1983).

3. On a theorem of Danes and the principle of equicontinuity for topological groups, All Pakistan Mathematical Conference, Quaid-e-Azam University, Islamabad, Pakistan (1985).

4. On locally solid topological l-groups, accepted for presentation at International Congress of Mathematicians, Berkeley, California, USA (1986).

5. Decomposition theorems for group-valued submeasures, under the auspices of Research and Planners Group, B.Z. University, Multan, Pakistan (1986).

6. Some recent developments in functional analysis, Department of Mathematics, Govt. F.C. College, Lahore, Pakistan (1989).

7. Mathematical logic and reasoning, Summer School in science for talented students of Intermediate classes from all Boards of Education from the Punjab province arranged by the Board of Intermediate and Secondary Education Multan at Khanaspur, Pakistan (1991).

8. Fixed point theorems for set-valued mappings in a Semi-Convex setting, Manila International Conference on Functional analysis and Global analysis (1992).

9. Some aspects of Ky Fan best approximation theorem, Department of Mathematical Sciences, KFUPM, Dhahran, Saudi Arabia (1998).

10. Random fixed point results for *-nonexpansive operators and their applications, Department of Mathematical Sciences, KFUPM, Dhahran, Saudi Arabia (2000).

11. Common fixed points form best approximation,26th Summer symposium in Real Analysis, Washington and Lee University ,Lexington,U.S.A. (2002).

12. Noncommuting maps, Coincidence points and invariant approximation, Lahore University of Management Sciences (LUMS), Lahore, Pakistan, 2002.

13. Coincidence point theory: Existence and Applications, Department of Mathematical Sciences, KFUPM, Dhahran, Saudi Arabia (2003).

14. Random coincidence point of random multivalued operators, The fourth international conference on   Nonlinear Analysis and Convex Analysis , Okinawa,Japan,2005.

1. **Personal Data**

   Last Name: Abdullah   First Name: Nawab Hussain
   Date and Place of Birth: 22nd March, 1967 (District R.Y.Khan, Pakistan)
   Present Address: Department of Mathematics
                          King Abdulaziz University Jeddah 21589, Saudi Arabia

2. **Education:**

   M.Sc. 1986-88   B.Z. University, Multan, Pakistan.

   M. Phil. 1990-1992 Quaid-i-Azam University, Islamabad, Pakistan

   Ph.D. 1996-2001   B.Z. University, Multan, Pakistan.

   Specialization: Functional Analysis (Some Types of Best Approximation and their Applications).

   Other Research Interests: Approximation Theory and its Applications to Fixed Point Theory.

3. **Professional Experience**

Assistant Professor	Department of Mathematics, King Abdul Aziz University, Jeddah, Saudi Arabia, Jan.2004 to date
Assistant Professor	CASPAM, B. Z. University, Multan, Pakistan, January 2002 to January 2004.
Lecturer	CASPAM, B. Z. University, Multan, Pakistan, March 1994 to January 2002.

4. **Theses/Dissertations Supervised / Examined:**

Gulam Mustafa	Fixed point theorems for multivalued mappings, 1996.
M. Zahid	Deterministic and random common fixed point theorems, 2005.
N. Shafqat	Common fixed point results in certain topological spaces, 2005.

5. **Selected Research Publications (In refereed Journals):**

   1. A.R. Khan, N. Hussain and M.A. Shahid, Strong uniqueness in metrizable topological vector spaces, Bull Malaysian Mathematical Society (Second series) 17, 1994, 21-27.

   2. A.R. Khan, M. Aslam and N. Hussain, Some best approximation results in locally convex spaces, Approx. Theory and Appl., 12, 1996, 29-36.

3. A.R. Khan, N. Hussain and M. Aslam, Mann iterative construction of fixed points in locally convex spaces, J. Natural Sci. and Maths., 36, 1997, 155-159.

4. A.R. Khan, N. Hussain and M. Akram, On open mapping and closed graph theorems, Punjab University J. of Mathematics, 31, 1998, 95-102.

5. A.R. Khan, N. Hussain and L. A. Khan, A note on Kakutani type fixed point theorems, Internat. J Math. & Math. Sci. 24, 2000, 231-235.

6. A.R. Khan, N. Hussain and A.B.Thaheem, Applications of fixed point theorems to invariant approximation, Approx. Theory and Appl., 16, 2000, 48-55.

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15.   A. R. Khan, A. Bano and N. Hussain, Common fixed points in best approximation theory, Internat. J. Pure Appl. Math. 2, 2002, 411-426.

16.   A. R. Khan, A. Latif , N. Hussain and A. Bano, Coincidence point results in locally convex spaces, Internat. J. Pure and Appl. Math. 3, 2002, 413-423.

17.   N. Hussain and A.R. Khan, Common fixed point results in best approximation theory, Applied Math. Lett. 16, 2003, 575-580.

18.   I. Beg, N. Hussain and A.R. Khan, Fixed point, almost fixed point and best approximation of nonexpansive multivalued mapping in Banach spaces, Adv. Math. Sci. Appl. 13, 2003, 83-111.

19.   N. Hussain and A.R.Khan, Common fixed points and best approximation in
        p-normed spaces, Demonstratio. Math. 36, 2003, 675-681.

20.   A. R. Khan and N. Hussain. Characterizations of random approximations in
        locally convex spaces, Arch. Math.(BORNO) 39, 2003, 271-275.

21.   A.R. Khan, A.B. Thaheem and N. Hussain, Random fixed points and random
        Approximations, Southeast Asian Bull. Math. 27, 2003, 289-294.

22.   N. Hussain and A. R. Khan, Applications of the best approximation operator to
        *- nonexpansive maps in Hilbert spaces, Numer. Funct. Anal. Optimiz.
        24 (3-4), 2003, 327-338.

23.   N. Hussain and A. R. Khan, Random fixed points for *-nonexpansive multivalued
        maps, Random Oper. and Stoch. Eqs. 11, 2003, 243-254.

24.   A. R. Khan, A. B. Thaheem and N. Hussain, A stochastic version of Ky Fan's best
        approximation theorem, J. Appl. Math. Stoch. Anal. 16, 2003, 275-282.

25.   A.R. Khan and N. Hussain, Random coincidence point theorem in Frechet spaces with
        applications, Stoch. Anal.& Appl. 22, 2004, 155-167.

26.   I. Beg, A. R. Khan and N. Hussain, Approximation   of *-nonexpansive random
        multivalued operators on Banach spaces, J. Aust. Math. Soc. 76, 2004, 51-66.

27.   A. R. Khan, N. Hussain and A. B. Thaheem, Some generalizations of Ky Fan's
        best   approximation theorem, Analysis in Theory and Appl. 20, 2004, 189-198.

28.   A.R. Khan, A. Latif, A. Bano and N. Hussain, Some results on common fixed
        points and best approximation, Tamkang J. Math. 36, 2005, 33-38.

29.   N. Hussain, Donal O'Regan and Ravi P. Agarwal, Common fixed point and
        invariant approximation results on non-starshaped domains, Georgian Math. J. 12,
        2005, 659-669.

30.   N. Hussain, Common fixed point and invariant approximation results,
        Demonstratio Math. 39, 2006, 389-400.

31.   N. Hussain, Generalized I-nonexpansive maps and invariant approximation results
        in p-normed spaces, Analysis in Theory and Appl. 22, 2006, 72-80.

32.   N. Hussain, and V. Berinde, Common fixed point and invariant approximation
        results in certain metrizable topological vector spaces, Fixed Point Theory Appl.
        2006 (2006), 1-13.

33.   N. Hussain and G. Jungck, Common fixed point and invariant approximation
        results for noncommuting generalized (f, g)-nonexpansive maps, J. Math. Anal.
        Appl. 321(2006), 851-861.

34. N. Shahzad and N. Hussain, Deterministic and random coincidence results for f-nonexpansive maps, J. Math. Anal. Appl. 323, 2006, 1038-1046.

35. G. Jungck and N. Hussain, Compatible maps and invariant approximations, J. Math. Anal. Appl. 325, 2007, 1003-1012.

36. N. Hussain, Coincidence points for multivalued maps on non-starshaped domain, Demonstratio Math. (in press)

37. A.R. Khan, F. Akbar, N. Sultana and N. Hussain, Coincidence and invariant approximation theorems for generalized f-nonexpansive multivalued mappings, Internat. J. Math. Math. Sci. 2006(2006), 1-18.

38. Donal O'Regan and N. Hussain, Generalized I-contractions and pointwise R-subweakly commuting maps, Acta Math. Sinica (in press)

39. N. Hussain and B. E. Rhoades, C_q-commuting maps and invariant approximations, Fixed Point Theory Appl. 2006(2006), 1-9

40. N. Hussain and F. Akbar, Generalized I-nonexpansive maps and invariant approximation results, Southeast Asian Bull. Math. (in press)

6. SEMINARS / INVITED LECTURES

1. Common fixed points in best approximation theory, Functional Analysis Day, Department of Mathematics, Lahore University of Management Sciences (LUMS), Lahore, Pakistan, March, 2002.

2. Common fixed points from best approximation, Conference/Colloquium Series held at Govt. College University (G.C.U) Lahore, Pakistan, June, 2002.

3. Random fixed points and random approximations, 2-day Workshop on Computational Mathematics and related topics, Department of Mathematical Sciences, COMSATS Institute of Information Technology, Islamabad , Pakistan, July, 2002 .

4. Common fixed point and invariant AND approximation results on non-starshaped domains, **Summer School on Fixed point theory**, Department of Mathematics, Lahore University of Management Sciences (LUMS), Lahore, Pakistan, Aug., 2005.