APPROXIMATING COMMON FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. We introduce three-step iterative schemes with errors for two and three nonexpansive maps and establish weak and strong convergence theorems for these schemes. Mann-type and Ishikawa-type convergence results are included in the analysis of these new iteration schemes. The results presented in this paper substantially improve and extend the results due to Khan and Fukhar-ud-din (2005), Shahzad (2005), Takahashi and Tamura (1995), Tan and Xu (1993) and Senter and Dotson (1974).

1. Introduction

Let C be a nonempty convex subset of a real Banach space E. A map $T : C \to C$ is called : (i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;(ii) quasi-nonexpansive if the set $F(T)$ of fixed points of T is nonempty and $\|Tx - Ty\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$.

Das and Debata [1] introduced the following iteration scheme:

$$\begin{cases} x_1 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T_2 x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 y_n, \qquad \text{for all } n \geq 1, \end{cases} \tag{1.1}$$

where T_1, T_2 are quasi-nonexpansive selfmaps with compact domain and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$. They used the scheme (1.1) to approximate common fixed points of the maps when E is strictly convex. For $T_1 = T_2$, the scheme (1.1) was introduced by Ishikawa [2] (see also Mann [3]). The weak convergence of the Ishikawa sequence for a nonexpansive map in a uniformly convex Banach space with the Opial property (or whose norm is Fréchet differentiable) has been studied by many authors(see, eg., [4],[5],[6]). Takahashi and Tamura [7] proved weak convergence of the iterates $\{x_n\}$ defined by (1.1) in a uniformly convex Banach space E which satisfies the Opial property or whose norm is Fréchet differentiable and T_1, T_2 are nonexpansive selfmaps on a closed convex subset of E. Recently, Shahzad [8] extended Theorem 3.3 of Takahashi and Tamura [7] to a class of uniformly convex Banach spaces which neither satisfies the Opial property nor has a Fréchet differentiable norm.

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Goebel and Kirk [9] in 1972, introduced the notion of an asymptotically nonexpansive map. A map $T : C \to C$ is asymptotically nonexpansive(cf. [9]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$, for all $x, y \in C$ and for all $n \geq 1$; in particular, if $k_n = 1$ for all $n \geq 1$, it becomes nonexpansive. The map T is uniformly $L-$Lipschitzian if there exists some positive constant L such that $\|T^n x - T^n y\| \leq L \|x - y\|$, for all $x, y \in C$ and for all $n \geq 1$. They, also, established that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space and T is an asymptotically nonexpansive selfmap of C, then T has a fixed point. Bose [10] in 1978, initiated the study of iterative construction of asymptotically nonexpansive maps. Schu [11], in 1991, considered the following modified Mann iteration process (cf. Mann [3]) for an asymptotically nonexpansive map T on C and $\{\alpha_n\}$ a sequence in $[0, 1]$:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \qquad \text{for all } n \geq 1. \end{cases}$$

In 1994, Tan and Xu [12] studied the modified Ishikawa iteration process (cf. Ishikawa [2]) for an asymptotically nonexpansive map T on C, $\{\alpha_n\}$ in $[0, 1]$, $\{\beta_n\}$ bounded away from 1 and the scheme described as:

$$\begin{cases} x_1 \in C, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \qquad \text{for all } n \geq 1. \end{cases}$$

In 2002, Xu and Noor [13] introduced a three-step iterative scheme for an asymptotically nonexpansive map T on C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ sequences in $[0, 1]$, as follows:

$$\begin{cases} x_1 \in C, \\ z_n = (1 - \gamma_n)x_n + \gamma_n T^n x_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \qquad \text{for all } n \geq 1. \end{cases}$$

Recently Cho. *et al.* [14] and Liu and Kang [15] have studied weak and strong convergence of three-step iterations with errors for an asymptotically nonexpansive map in a uniformly convex Banach space.

Finding common fixed points of maps acting on a Hilbert space is a problem that often arises in applied mathematics. In fact, many algorithms have been introduced for different classes of maps with a nonempty set of common fixed points. Unfortunately, the existence results of common fixed points of maps are not known in many situations. Therefore, it is natural to consider approximation results for these classes of maps. Approximation of common fixed points of two or more nonexpansive maps and asymptotically nonexpansive maps by iteration has been studied by many authors (see, eg., [7,8,12,16-20]).

For three maps $T_i : C \to C (i = 1, 2, 3)$, we define the following three-step iteration scheme with errors (cf. [17] and reference therein; see also [13]):

$$\begin{cases} x_1 \in C, \\ z_n = \alpha_n^{(3)} x_n + \beta_n^{(3)} T_3 x_n + \gamma_n^{(3)} u_n^{(3)}, \\ y_n = \alpha_n^{(2)} x_n + \beta_n^{(2)} T_2 z_n + \gamma_n^{(2)} u_n^{(2)}, \\ x_{n+1} = \alpha_n^{(1)} x_n + \beta_n^{(1)} T_1 y_n + \gamma_n^{(1)} u_n^{(1)}, \qquad \text{for all } n \geq 1, \end{cases} \tag{1.2}$$

where$\{u_n^{(j)}\}$ is a bounded sequence in C for each $j = 1, 2, 3$ and $\{\alpha_n^{(j)}\}, \{\beta_n^{(j)}\}$ and $\{\gamma_n^{(j)}\}$ are sequences in $[0, 1]$ satisfying:

$$\alpha_n^{(j)} + \beta_n^{(j)} + \gamma_n^{(j)} = 1 \text{ for all } n \geq 1 \text{ and each } j = 1, 2, 3.$$

If we choose $T_1 = T_3$ in (1.2), it reduces to the following three-step iteration scheme of two maps:

$$\begin{cases} x_1 \in C, \\ z_n = \alpha_n^{(3)} x_n + \beta_n^{(3)} T_1 x_n + \gamma_n^{(3)} u_n^{(3)}, \\ y_n = \alpha_n^{(2)} x_n + \beta_n^{(2)} T_2 z_n + \gamma_n^{(2)} u_n^{(2)}, \\ x_{n+1} = \alpha_n^{(1)} x_n + \beta_n^{(1)} T_1 y_n + \gamma_n^{(1)} u_n^{(1)}, \qquad \text{for all } n \geq 1. \end{cases} \tag{1.3}$$

The choice $\alpha_n^{(3)} = 1$ in (1.2), leads to the following iterative scheme [17] :

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n^{(2)} x_n + \beta_n^{(2)} T_2 x_n + \gamma_n^{(2)} u_n^{(2)}, \\ x_{n+1} = \alpha_n^{(1)} x_n + \beta_n^{(1)} T_1 y_n + \gamma_n^{(1)} u_n^{(1)}, \qquad \text{for all } n \geq 1. \end{cases} \tag{1.4}$$

In case $\beta_n^{(3)} = 0$ and $\gamma_n^{(j)} = 0$ in (1.2), we get (1.1).

We study the iteration schemes (1.2) and (1.3) and prove their weak convergence to a common fixed point of nonexpansive maps in a uniformly convex Banach space. Our weak convergence result applies not only to Hilbert spaces and L^p spaces $(1 < p < \infty)$ but also to the rather large class of spaces admitting the Kadec-Klee property(cf. [21, p. 573]). We also discuss strong convergence of these schemes. It is remarked that the results presented in this paper are new even for nonexpansive maps. Our convergence theorems improve, unify and generalize many important results in the current literature.

2. Prliminaries and Notations

Recall that a Banach space E is said to be uniformly convex if for each $r \in [0, 2]$, the modulus of convexity of E given by:

$$\delta(r) = \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq r \right\}$$

satisfies the inequality $\delta(r) > 0$ for all $r > 0$.

For sequences, the symbol $\rightarrow$ (resp.$\rightharpoonup$) indicates norm (resp. weak) convergence. Let $S=\{x\in E:\|x\|=1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The space E has : (i) Gâteaux differentiable norm [5] if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each x and y in S; (ii) Fréchet differentiable norm [5] if for each x in S, the above limit exists and is attained uniformly for y in S and in this case, it has been shown in [5] that

$$\langle h, J(x)\rangle+\frac{1}{2}\|x\|^2 \quad\leq\quad \frac{1}{2}\|x+h\|^2\leq\langle h, J(x)\rangle+\frac{1}{2}\|x\|^2+b(\|h\|) \qquad (2.1)$$

for all x,h in E, where J is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x\in X,\langle .,.\rangle$ is the pairing between E and E^*, and b is a function defined on $[0,\infty)$ such that $\lim_{t\downarrow 0}\frac{b(t)}{t}=0$; (iii) Opial property [22] if for any sequence $\{x_n\}$ in $E, x_n\rightharpoonup x$ implies that $\limsup_{n\to\infty}\|x_n-x\|<\limsup_{n\to\infty}\|x_n-y\|$ for all $y\in E$ with $y\neq x$ and (iv) Kadec-Klee property if for every sequence $\{x_n\}$ in $E, x_n\rightharpoonup x$ and $\|x_n\|\rightarrow\|x\|$ together imply $x_n\rightarrow x$ as $n\rightarrow\infty$.

A mapping $T:C\rightarrow E$ is demiclosed at $y\in E$ if for each sequence $\{x_n\}$ in C and each $x\in E,\ x_n\rightharpoonup x$ and $Tx_n\rightarrow y$ imply that $x\in C$ and $Tx=y$.

We recall the following useful lemmas for the development of our results.

Lemma 2.1 [5, Lemma1]. *Let $\{s_n\}$ and $\{t_n\}$ be two nonnegative real sequences such that*

$$s_{n+1}\leq s_n+t_n \text{ for all } n\geq 1.$$

If $\sum_{n=1}^{\infty}t_n<\infty$, then $\lim_{n\to\infty}s_n$ exists.

Lemma 2.2 [14, Lemma 1.7]. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space. Then there is a strictly increasing and continuous convex function $g:[0,\infty)\rightarrow[0,\infty)$ with $g(0)=0$ such that, for Lipschitzian continuous map $T:C\rightarrow X$ and for all $x,y\in C$ and $t\in[0,1]$, the following inequality holds:*

$$\|T(tx+(1-t)y)-(tTx+(1-t)Ty)\|\leq Lg^{-1}(\|x-y\|-L^{-1}\|Tx-Ty\|)$$

where $L\geq 1$ is the Lipschitz constant of T.

Note that the above lemma reduces to the corresponding lemma of Bruck [23] for $L=1$.

Lemma 2.3 [14, Lemma 1.6]**.** *Let C be a nonempty closed convex subset of a uniformly convex Banach space and let $T:C\rightarrow C$ be an asymptotically nonexpansive map; in particular, nonexpansive map. Then $I-T$ is demiclosed at 0.*

Lemma 2.4 [21, Lemma 2]. *Let E be a reflexive Banach space such that E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $x^*,y^*\in\omega_w(x_n)$(weak w-limit set of $\{x_n\}$). Suppose $\lim_{n\to\infty}\|tx_n+(1-t)x^*-y^*\|$ exists for all $t\in[0,1]$. Then $x^*=y^*$.*

Lemma 2.5 [11, Lemma 1.3]. *Suppose that E is a uniformly convex Banach space and $0<p\leq t_n\leq q<1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty}\|x_n\|\leq r$, $\limsup_{n\to\infty}\|y_n\|\leq r$ and $\lim_{n\to\infty}\|t_nx_n+(1-t_n)y_n\|=r$ hold for some $r\geq 0$. Then $lim_{n\to\infty}\|x_n-y_n\|=0$.*

In the sequel, $\cap_{i=1}^{2}F(T_i)$ or $\cap_{i=1}^{3}F(T_i)$ will be denoted by F.

3. Preparatory Lemmas

In this section, we prove some lemmas which play key role to establish weak and strong convergence results for the schemes (1.2) and (1.3).

Lemma 3.1. *Let C be a nonempty closed convex subset of a normed space E and let $T_i (i = 1, 2, 3)$ be nonexpansive selfmaps on C. Let $\{x_n\}$ be the sequence defined in (1.2) with $F \neq \phi$ and $\sum_{n=1}^{\infty}\gamma_n^{(j)} < \infty$ for $j = 1, 2, 3$. Then $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F$.*

Proof. Let $p \in F$. Since $\{u_n^{(j)}\}$ is bounded for each $j = 1, 2, 3$, there exists $M > 0$ such that $M = \max\{\sup_{n\geq 1}\left\|u_n^{(j)} - p\right\| : j = 1, 2, 3\}$ for any $p \in F$.

Now consider

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|\beta_n^{(1)}(T_1y_n - p) + \alpha_n^{(1)}(x_n - p) + \gamma_n^{(1)}(u_n^{(1)} - p)\right\| \\
&\leq \beta_n^{(1)}\|T_1y_n - p\| + \alpha_n^{(1)}\|x_n - p\| + \gamma_n^{(1)}\left\|u_n^{(1)} - p\right\| \\
&\leq \beta_n^{(1)}\|y_n - p\| + \alpha_n^{(1)}\|x_n - p\| + \gamma_n^{(1)}\left\|u_n^{(1)} - p\right\| \\
&\leq \beta_n^{(1)}\left\|\beta_n^{(2)}(T_2z_n - p) + \alpha_n^{(2)}(\,x_n - p) + \gamma_n^{(2)}(u_n^{(2)} - p)\right\| \\
&\quad + \alpha_n^{(1)}\|x_n - p\| + \gamma_n^{(1)}M \\
&\leq \beta_n^{(1)}\beta_n^{(2)}\|T_2z_n - p\| + \beta_n^{(1)}\gamma_n^{(2)}\left\|u_n^{(2)} - p\right\| \\
&\quad + (\alpha_n^{(1)} + \alpha_n^{(2)}\beta_n^{(1)})\|x_n - p\| + \gamma_n^{(1)}M \\
&\leq \beta_n^{(1)}\beta_n^{(2)}\|z_n - p\| + (\alpha_n^{(1)} + \alpha_n^{(2)}\beta_n^{(1)})\|x_n - p\| \\
&\quad + \beta_n^{(1)}\gamma_n^{(2)}\left\|u_n^{(2)} - p\right\| + \gamma_n^{(1)}M \\
&\leq \beta_n^{(1)}\beta_n^{(2)}\left\|\alpha_n^{(3)}(\,x_n - p) + \beta_n^{(3)}(T_3x_n - p) + \gamma_n^{(3)}(u_n^{(3)} - p)\right\| \\
&\quad + (\alpha_n^{(1)} + \alpha_n^{(2)}\beta_n^{(1)})\|x_n - p\| + (\beta_n^{(1)}\gamma_n^{(2)} + \gamma_n^{(1)})M \\
&\leq \left(\alpha_n^{(1)} + \alpha_n^{(2)}\beta_n^{(1)} + \alpha_n^{(3)}\beta_n^{(1)}\beta_n^{(2)} + \beta_n^{(1)}\beta_n^{(2)}\beta_n^{(3)}\right)\|x_n - p\| \\
&\quad + (\beta_n^{(1)}\beta_n^{(2)}\gamma_n^{(3)} + \beta_n^{(1)}\gamma_n^{(2)} + \gamma_n^{(1)})M \\
&\leq \|x_n - p\| + (\gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)})M.
\end{aligned}$$

By Lemma 2.1, $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F$.

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and$T_i (i = 1, 2, 3)$ be nonexpansive selfmaps on C. Let $\{x_n\}$ be the sequence defined in (1.2) with $F \neq \phi$ and $\sum_{n=1}^{\infty}\gamma_n^{(j)} < \infty$ for $j = 1, 2, 3$. Then, for any $p_1, p_2 \in F$, $lim_{n\to\infty}\|tx_n + (1 - t)p_1 - p_2\|$ exists for any $t \in [0, 1]$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F$ and therefore $\{x_n\}$ is bounded. Hence, there exists a ball $B_r(0) = \{x \in E : \|x\| \leq r\}$ for some $r > 0$ such that $\{x_n\} \subset K = B_r(0) \cap C$. Thus K is a nonempty bounded closed convex subset of E. Let $a_n(t) = \|tx_n + (1 - t)p_1 - p_2\|$. Then $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty}\|x_n - p_2\|$ exists as proved in Lemma 3.1. Define

$W_n : K \to K$ by:

$W_n x = \alpha_n^{(1)} T_1[\alpha_n^{(2)} T_2(\alpha_n^{(3)} x + \beta_n^{(3)} T_3 x + \gamma_n^{(3)} u_n^{(3)}) + \beta_n^{(2)} x + \gamma_n^{(2)} u_n^{(2)}] + \beta_n^{(1)} x + \gamma_n^{(1)} u_n^{(1)}.$

It is easy to verify that

$$\|W_n x - W_n y\| \le \|x - y\| \quad \text{for all } x, y \in K.$$

Set

$$R_{n,m} = W_{n+m-1} W_{n+m-2} ... W_n, \ m \ge 1 \text{ and}$$

$$b_{n,m} = \|R_{n,m}(tx_n + (1-t)p_1) - (tR_{n,m}x_n + (1-t)p_1)\| .$$

Then

$$\|R_{n,m}x - R_{n,m}y\| \le \|x - y\| \text{ and } R_{n,m}x_n = x_{n+m}.$$

We first show that for any $p \in F, \|R_{n,m}p - p\| \to 0$ as $n \to \infty$ and for all $m \ge 1$.

Consider

$$\begin{aligned}
\|R_{n,m}p - p\| &\le \|W_{n+m-1}W_{n+m-2}...W_n p - W_{n+m-1}W_{n+m-2}...W_{n+1}p\| \\
&\quad + \|W_{n+m-1}W_{n+m-2}...W_{n+1}p - W_{n+m-1}W_{n+m-2}...W_{n+2}p\| \\
&\quad + ... + \|W_{n+m-1}p - p\| \\
&\le \|W_n p - p\| + \|W_{n+1}p - p\| + ... + \|W_{n+m-1}p - p\| \\
&\le (\gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)})M + (\gamma_{n+1}^{(1)} + \gamma_{n+1}^{(2)} + \gamma_{n+1}^{(3)})M \\
&\quad + ... + (\gamma_{n+m-1}^{(1)} + \gamma_{n+m-1}^{(2)} + \gamma_{n+m-1}^{(3)})M \\
&= \sum_{k=0}^{m-1} (\gamma_{n+k}^{(1)} + \gamma_{n+k}^{(2)} + \gamma_{n+k}^{(3)})M \to 0 \text{ as } n \to \infty.
\end{aligned}$$

By Lemma 2.2, there exists a strictly increasing continuous function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned}
g(b_{n,m}) &\le \|x_n - p_1\| - \|R_{n,m}x_n - R_{n,m}p_1\| \\
&= \|x_n - p_1\| - \|(R_{n,m}x_n - p_1) + (p_1 - R_{n,m}p_1)\| \\
&\le \|x_n - p_1\| + \|p_1 - R_{n,m}p_1\| - \|R_{n,m}x_n - p_1\| \\
&= \|x_n - p_1\| - \|x_{n+m} - p_1\| + \|p_1 - R_{n,m}p_1\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $b_{n,m} \to 0$ as $n \to \infty$ and for all $m \ge 1$.

Finally, from the inequality

$$\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\
&\le b_{n,m} + \|R_{n,m}(tx_n + (1-t)p_1) - p_2\| \\
&\le b_{n,m} + a_n(t) + \|R_{n,m}p_2 - p_2\| ,
\end{aligned}$$

it follows that

$$\limsup_{m\to\infty} a_{n+m}(t) \le \limsup_{m\to\infty} b_{n,m} + a_n(t) + \limsup_{m\to\infty} \|R_{n,m}p_2 - p_2\| .$$

That is,

$$\limsup_{m\to\infty} a_m(t) \le \liminf_{n\to\infty} a_n(t).$$

Hence, $lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for any $t \in [0, 1]$.

This completes the proof.

Lemma 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T_i (i = 1, 2, 3)$ be nonexpansive selfmaps on C. Let $\{x_n\}$ be the sequence defined in (1.2) with $F \neq \phi$ and $\sum_{n=1}^{\infty} \gamma_n^{(j)} < \infty$ for $j = 1, 2, 3$. Then, for any $p_1, p_2 \in F$, $\lim_{n\to\infty} \langle x_n, J(p_1 - p_2)\rangle$ exists; in particular, $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$.*

Proof. Take $x = p_1 - p_2$ with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in the inequality (2.1) to get:

$$\begin{aligned}
\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle &\leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\
&\leq \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle \\
&\quad + b(t\|x_n - p_1\|).
\end{aligned}$$

As $\sup_{n\geq 1}\|x_n - p_1\| \leq M'$ for some $M' > 0$, it follows that

$$\begin{aligned}
\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle &\leq \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|^2 \\
&\leq \frac{1}{2}\|p_1 - p_2\|^2 + b(tM') \\
&\quad + t\liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle.
\end{aligned}$$

That is,

$$\limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle \leq \liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle + \frac{b(tM')}{tM'}M'.$$

If $t \to 0$, then $\lim_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle$ exists for all $p_1, p_2 \in F$; in particular, we have $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$.

Lemma 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T_i (i = 1, 2, 3)$ be nonexpansive selfmaps on C. Let $\{x_n\}$ be the sequence defined in (1.2) with $F \neq \phi$. If, for each $j = 1, 2, 3; \beta_n^{(j)} \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty}\gamma_n^{(j)} < \infty$, then*

$$\lim_{n\to\infty}\|x_n - T_i x_n\| = 0 \text{ for } i = 1, 2, 3.$$

Proof. Let $p \in F$. As proved in Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists and let it be c. Let M be the real number introduced in the proof of Lemma 3.1. When $c = 0$, there is nothing to prove. Assume $c > 0$.

Observe that

$$\begin{aligned}
\|z_n - p\| &= \left\|\alpha_n^{(3)}x_n + \beta_n^{(3)}T_3x_n + \gamma_n^{(3)}u_n^{(3)} - p\right\| \\
&\leq \beta_n^{(3)}\|T_3x_n - p\| + (1-\beta_n^{(3)})\|x_n - p\| + \gamma_n^{(3)}\left\|u_n^{(3)} - x_n\right\| \\
&\leq \beta_n^{(3)}\|x_n - p\| + (1-\beta_n^{(3)})\|x_n - p\| + \gamma_n^{(3)}M \\
&= \|x_n - p\| + \gamma_n^{(3)}M \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}\|y_n - p\| &= \left\|\beta_n^{(2)}(T_2 z_n - p) + (1-\beta_n^{(2)})(x_n - p) + \gamma_n^{(2)}(u_n^{(2)} - x_n)\right\| \\ &\le \beta_n^{(2)} \|T_2 z_n - p\| + (1-\beta_n^{(2)}) \|x_n - p\| + \gamma_n^{(2)} \left\|u_n^{(2)} - x_n\right\| \\ &\le \beta_n^{(2)} \|z_n - p\| + (1-\beta_n^{(2)}) \|x_n - p\| + \gamma_n^{(2)} M. \qquad (3.2)\end{aligned}$$

From (3.1) and (3.2), we get

$$\|y_n - p\| \le \|x_n - p\| + \left(\beta_n^{(2)}\gamma_n^{(3)} + \gamma_n^{(2)}\right) M.$$

Taking lim sup on both sides, we have

$$\limsup_{n\to\infty} \|y_n - p\| \quad \le \quad c. \qquad (3.3)$$

We note that:

$$\begin{aligned}\left\|T_1 y_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)\right\| &\le \|T_1 y_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - x_n\right\| \\ &\le \|y_n - p\| + \gamma_n^{(1)} M.\end{aligned}$$

By applying lim sup on both sides of this inequality and then using (3.3), we get

$$\limsup_{n\to\infty} \left\|T_1 y_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)\right\| \le c.$$

Also

$$\begin{aligned}\left\|x_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)\right\| &\le \|x_n - p\| + \gamma_n^{(1)} \left\|u_n^{(1)} - x_n\right\| \\ &\le \|x_n - p\| + \gamma_n^{(1)} M\end{aligned}$$

gives that

$$\limsup_{n\to\infty} \left\|x_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)\right\| \le c.$$

Further, $\lim_{n\to\infty} \|x_{n+1} - p\| = c$ means that

$$\lim_{n\to\infty} \left\|\beta_n^{(1)}(T_1 y_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n)) + (1-\beta_n^{(1)})(x_n - p + \gamma_n^{(1)}(u_n^{(1)} - x_n))\right\| = c.$$

Now by Lemma 2.5, we obtain

$$\lim_{n\to\infty} \|x_n - T_1 y_n\| = 0.$$

Since

$$\begin{aligned}\|x_n - p\| &\le \|x_n - T_1 y_n\| + \|T_1 y_n - p\| \\ &\le \|x_n - T_1 y_n\| + \|y_n - p\|,\end{aligned}$$

therefore we obtain

$$c \ \le \liminf_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|y_n - p\| \ \le c.$$

That is,

$$\lim_{n\to\infty} \|y_n - p\| = c.$$

Now $\lim_{n\to\infty} \|y_n - p\| = c$ means that

$$\lim_{n\to\infty} \left\|\beta_n^{(2)}(T_2 z_n - p + \gamma_n^{(2)}(u_n^{(2)} - x_n)) + (1-\beta_n^{(2)})(x_n - p + \gamma_n^{(2)}(u_n^{(2)} - x_n))\right\| = c.$$

Moreover,

$$\limsup_{n\to\infty} \left\|T_2 z_n - p + \gamma_n^{(2)}(u_n^{(2)} - x_n)\right\| \ \le c$$

and

$$\limsup_{n\to\infty} \left\|x_n - p + \gamma_n^{(2)}(u_n^{(2)} - x_n)\right\| \le c.$$

So again by Lemma 2.5, we have

$$\lim_{n\to\infty} \|T_2 z_n - x_n\| = 0.$$

Now

$$\begin{aligned} \|x_n - p\| &\le \|x_n - T_2 z_n\| + \|T_2 z_n - p\| \\ &\le \|x_n - T_2 z_n\| + \|z_n - p\|, \end{aligned}$$

yields:

$$c \ \le \liminf_{n\to\infty} \|z_n - p\| \le \limsup_{n\to\infty} \|z_n - p\| \ \le c.$$

That is,

$$\lim_{n\to\infty} \|z_n - p\| = c,$$

or

$$\lim_{n\to\infty} \left\|\beta_n^{(3)}(T_3 x_n - p + \gamma_n^{(3)}(u_n^{(3)} - x_n)) + (1-\beta_n^{(3)})(x_n - p + \gamma_n^{(3)}(u_n^{(3)} - x_n))\right\| = c$$

and hence again by Lemma 2.5,

$$\lim_{n\to\infty} \|T_3 x_n - x_n\| = 0.$$

Next

$$\begin{aligned} \|T_2 x_n - x_n\| &\le \|T_2 x_n - T_2 z_n\| + \|T_2 z_n - x_n\| \\ &\le \|x_n - z_n\| + \|T_2 z_n - x_n\| \\ &\le \beta_n^{(3)} \|x_n - T_3 x_n\| + \|T_2 z_n - x_n\| + \gamma_n^{(3)} M \end{aligned}$$

gives that

$$\lim_{n\to\infty} \|T_2 x_n - x_n\| = 0.$$

Finally,

$$\begin{aligned} \|T_1 x_n - x_n\| &\le \|T_1 x_n - T_1 y_n\| + \|T_1 y_n - x_n\| \\ &\le \|x_n - y_n\| + \|T_1 y_n - x_n\| \\ &\le \beta_n^{(2)} \|x_n - T_2 z_n\| + \|T_1 y_n - x_n\| + \gamma_n^{(2)} M \end{aligned}$$

implies that

$$\lim_{n\to\infty}\|T_1x_n-x_n\|=0.$$

From the above conclusions, we have

$$\lim_{n\to\infty}\|T_ix_n-x_n\|=0 \text{ for } i=1,2,3.$$

Lemma 3.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T_i: C\to C(i=1,2)$ be nonexpansive maps with $F\neq\phi$ and $\sum_{n=1}^{\infty}\gamma_n^{(j)}<\infty$ for each $j=1,2,3$. Then, for the sequence $\{x_n\}$ given by* (1.3), *where $\beta_n^{(j)}\in[\delta,1-\delta]$ for some $\delta\in(0,1)$ and $j=1,2$, we have*

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=0 \text{ for } i=1,2.$$

Proof. Let $p\in F$. If we choose $T_3=T_1$, then as in Lemma 3.4, we can show that

$$\lim_{n\to\infty}\|x_n-T_1y_n\|=0$$

and

$$\lim_{n\to\infty}\|T_2z_n-x_n\|=0.$$

Since

$$\|x_n-y_n\|\leq\beta_n^{(2)}\|x_n-T_2z_n\|+\gamma_n^{(2)}\left\|u_n^{(2)}-x_n\right\|\to 0 \text{ as } n\to\infty,$$

therefore we have

$$\begin{aligned}\|T_1x_n-x_n\| &\leq \|T_1x_n-T_1y_n\|+\|x_n-T_1y_n\|\\ &\leq \|x_n-y_n\|+\|x_n-T_1y_n\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

On the other hand

$$\|x_n-z_n\|\leq\beta_n^{(3)}\|x_n-T_1x_n\|+\gamma_n^{(3)}\left\|u_n^{(3)}-x_n\right\|\to 0 \text{ as } n\to\infty,$$

gives that

$$\begin{aligned}\|T_2x_n-x_n\| &\leq \|T_2x_n-T_2z_n\|+\|T_2z_n-x_n\|\\ &\leq \|x_n-z_n\|+\|T_2z_n-x_n\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

This completes the proof.

Remark 3.1. *A comparision of the statements of Lemma* 3.4 *and Lemma* 3.5 *reveals that replacement of the scheme* (1.2) *of three maps by the scheme* (1.3) *of two maps, makes the condition $0<\beta_n^{(3)}<1$ superfluous so that the scheme* (1.3) *can be used to approximate the common fixed points under a free parameter . Moreover, all the above Lemmas* 3.1-3.5 *which hold for the scheme* (1.2)*, also hold for the scheme* (1.3)*.*

4. Weak and Strong Convergence Theorems

In this section, we prove our weak and strong convergence theorems.

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T_i: C\to C(i=1,2,3)$ be nonexpansive maps. Let $\{x_n\}$ be the sequence defined in* (1.2) *with $F\neq\phi$ and for each $j=1,2,3;\beta_n^{(j)}\in[\delta,1-\delta]$ for some $\delta\in(0,\frac{1}{2})$ and $\sum_{n=1}^{\infty}\gamma_n^{(j)}<\infty$. Assume that one of the following conditions holds:* (1) *E satisfies the Opial property;* (2) *E has a Fréchet differentiable norm;*

(3) *E^* has the Kadec-Klee property. Then $\{x_n\}$ converges weakly to some $p \in F$.*

Proof. Let $p \in F$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 3.1. Since E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to some $z_1 \in C$. By Lemma 3.4 and Lemma 2.3, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ and $I - T_i$ is demiclosed at 0 for each $i = 1, 2, 3$, respectively. Therefore, we obtain $T_i z_1 = z_1$ for each $i = 1, 2, 3$. That is, $z_1 \in F$. In order to show that $\{x_n\}$ converges weakly to z_1, take an other subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to some $z_2 \in C$. Again, as before, we can prove that $z_2 \in F$. Next, we prove that $z_1 = z_2$. Assume (1) is given and suppose that $z_1 \neq z_2$. Then by the Opial property, we obtain:

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This contradiction implies that $z_1 = z_2$. Next suppose that (2) is satisfied. From Lemma 3.3, we have that $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$. Now $\|z_1 - z_2\|^2 = \langle z_1 - z_2, J(z_1 - z_2)\rangle = 0$ gives that $z_1 = z_2$. Finally, let (3) be given. As $\lim_{n\to\infty} \|tx_n + (1-t)z_1 - z_2\|$ exists, therefore by Lemma 2.4, we obtain $z_1 = z_2$. Hence $x_n \rightharpoonup p \in F$. This completes the proof.

The following results are immediate consequences of our weak convergence theorem.

Corollary 4.1 [7, Theorem 3.2]. *Let E be a uniformly convex Banach space satisfying the Opial property or whose norm is Fréchet differentiable. Let C be a nonempty closed convex subset of E and $T_1, T_2 : C \to C$ be nonexpansive maps with $F \neq \phi$. For an arbitrary $x_1 \in C$, define $\{x_n\}$ by (1.1), where $\alpha_n, \beta_n \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. Then $\{x_n\}$ converges weakly to some $p \in F$.*

Corollary 4.2 [8, Theorem 4.1]. *Let E be a uniformly convex Banach space and E^* has the Kadec-Klee property. Let C be a nonempty closed convex subset of E and $T_1, T_2 : C \to C$ be nonexpansive maps with $F \neq \phi$. For an arbitrary $x_1 \in C$, define $\{x_n\}$ by (1.1), where $\alpha_n, \beta_n \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. Then $\{x_n\}$ converges weakly to some $p \in F$.*

Corollary 4.3 [17, Theorem 1]. *Let E be a uniformly convex Banach space satisfying the Opial property. Let C be a nonempty closed convex subset of E and $T_1, T_2 : C \to C$ be nonexpansive maps with $F \neq \phi$. For an arbitrary $x_1 \in C$, define $\{x_n\}$ by (1.4), where $\beta_n^{(1)}, \beta_n^{(2)} \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. Then $\{x_n\}$ converges weakly to some $p \in F$.*

To prove our strong convergence theorem, we need the following:

Definition 4.1. A family $\{T_i : i = 1, 2, 3, ..., n\}$ of maps is said to satisfy condition(A) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with

$f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\frac{1}{n}(\sum_{i=1}^{n} \|x - T_i x\|) \geq f(d(x, F^{\prime}))$ for all $x \in C$, where $d(x, F^{\prime}) = \inf\{\|x - p\| : p \in F^{\prime} = \cap_{i=1}^{n} F(T_i)\}$.

It is remarked that the condition(A) reduces to the condition(I) in ([24], p.375) when $T_i = T$ for $i = 1, 2, 3, ..., n$.

By using the condition(A), we obtain a strong convergence theorem; a generalization of Theorem 2.4 in [6].

Theorem 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T_i : C \to C (i = 1, 2, 3)$ be nonexpansive maps. Let $\{x_n\}$ be the sequence in* (1.2) *with $F \neq \phi$ and for each $j = 1, 2, 3; \beta_n^{(j)} \in [\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty} \gamma_n^{(j)} < \infty$. Assume that T_1, T_2, T_3 satisfy the condition (A). Then $\{x_n\}$ converges strongly to some $p \in F$.*

Proof. As in Lemma 3.1, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + (\gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)})M. \tag{4.1}$$

This gives that

$$\inf_{p \in F} \|x_{n+1} - p\| \leq \inf_{p \in F} \|x_n - p\| + (\gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)})M.$$

That is,

$$d(x_{n+1}, F) \leq d(x_n, F) + (\gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)})M. \tag{4.2}$$

Comparing Lemma 2.1 and the inequality (4.2), we deduce that $\lim_{n\to\infty} d(x_n, F)$ exists. From Lemma 3.4, we have

$$\lim_{n\to\infty} \|T_i x_n - x_n\| = 0 \text{ for each } i = 1, 2, 3.$$

Hence, by the condition(A), $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing and $f(0) = 0$, therefore, we get $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we prove that $\{x_n\}$ is a Cauchy sequence. Let $h_n = (\gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)})M$. Let $\epsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} h_n < \infty$, there exists an integer n_0 such that for all $n \geq n_0$,

$$d(x_n, F) < \frac{\epsilon}{4} \text{ and } \sum_{j=n_0}^{\infty} h_j < \frac{\epsilon}{6}.$$

In particular,

$$d(x_{n_0}, F) < \frac{\epsilon}{4}.$$

That is,

$$\inf_{p \in F} \|x_{n_0} - p\| < \frac{\epsilon}{4}.$$

Thus there must exist $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{3}.$$

Now, for $n \geq n_0$, we have from the inequality (4.1) that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2\left[\|x_{n_0} - p^*\| + \sum\nolimits_{j=n_0}^{n_0+m-1} h_j\right] \\ &< 2\left(\frac{\epsilon}{3} + \frac{\epsilon}{6}\right) = \epsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in C and it must converge to a point of C. Let $\lim_{n\to\infty} x_n = q$ (say). Since $\lim_{n\to\infty} d(x_n, F) = 0$ and F is closed, therefore $q \in F$. This completes the proof of the theorem.

Corollary 4.4 [17, Theorem 2]. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $T_1, T_2 : C \to C$ be nonexpansive maps satisfying the condition(A). Let $\{x_n\}$ given by* (1.4) *be such that for each $j = 1, 2, \{u_n^{(j)}\}$ is a sequence in C and $\{\alpha_n^{(j)}\}, \{\beta_n^{(j)}\}$ and $\{\gamma_n^{(j)}\}$ are sequences in* $[0,1]$ *with $0 < \delta \leq \beta_n^{(1)}, \beta_n^{(2)} \leq 1 - \delta < 1$, $\alpha_n^{(j)} + \beta_n^{(j)} + \gamma_n^{(j)} = 1$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \gamma_n^{(j)} < \infty$. If $F \neq \phi$, then $\{x_n\}$ converges strongly to some $p \in F$.*

Corollary 4.5 [24, Theorem 1]. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T : C \to C$ a nonexpansive map satisfying the condition(I). Generate the sequence $\{x_n\}$ by:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n, \qquad \textit{for all } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ *with $0 < \delta \leq \alpha_n \leq 1 - \delta < 1$. Then $\{x_n\}$* converges strongly to some $p \in F(T)$.

Definition 4.2. We say a family $\{T_i : i = 1, 2, 3, ..., n\}$ of maps satisfies condition(B) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\max_{1\leq i\leq n} \|x - T_i x\| \geq f(d(x, \acute{F}))$ for all $x \in C$.

Remark 4.1. *From the procedures of proof of the above results, it is obvious that:*
(i) Condition(A) in Theorem 4.2 *and Corollary* 4.4 *can be replaced by the condition(B).*
(ii) Weak and strong convergence results for the scheme (1.3)*, similar to Theorems* 4.1- 4.2*, with the help of Lemma 3.5 can be established. These new restricted results will still generalize all the above corollaries* (*Corollaries* 4.1-4.4).
(iii) By modifying the schemes (1.2) *and* (1.3)*, we can prove all the above theorems and corollaries for asymptotically nonexpansive maps with suitable changes. We leave the details to the reader.*

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