Coincidence and Fixed Points of Nonself Contractive Maps with Applications

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Abstract

We prove some coincidence and common fixed point theorems for nonself maps (not necessarily continuous) satisfying different contractive conditions on an arbitrary nonempty subset of a metric space. As applications, we demonstrate the existence of: (i) common fixed points of the maps from the set of best approximations, (ii) solutions to nonlinear eigenvalue problems. Our work sets analogues, unifies and improves the earlier results of a number of authors.

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1 Introduction

In 1982, Sessa [15] introduced the concept of weakly commuting maps to generalize commutativity. Jungck [9], in 1986, generalized weak commutativity to the notion of compatible maps. In 1996, Jungck [11] further weakened compatibility to the concept of weak compatibility. Since then, many interesting fixed point theorems of compatible and weakly compatible maps under various contractive conditions have been obtained by a number of authors (see, for example, Aamri and El Moutawakil [1], Jachymski [7], Jungck [8-11], and Pant [13]). Aamri and El Moutawakil [1], in 2002, defined the property $(E \cdot A)$ for selfmaps on a metric space X which always holds for any two noncompatible maps (need not be continuous) on X and proved two common fixed point theorems for weakly compatible maps satisfying the property $(E \cdot A)$ and certain strict contractive conditions. Very recently, Ćirić [4] has established fixed point theorems for nonself maps satisfying certain contractive conditions on a

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nonempty closed subset of a metric space of hyperbolic type (Takahashi [17] uses the term "convex metric space").

In [9], Jungck generalized the Banach contraction principle to the case of two commuting selfmaps on a metric space. Baskaran and Subrahmanyam [3] noted that the commutativity of the maps in Jungck's theorem can be replaced by weak commutativity and then they obtained some common fixed point theorems for two maps on the closed ball of a Banach space. They also provided a solution to a nonlinear eigenvalue problem for operators on the closed ball of a Banach space. The existence of fixed points of maps defined on closed balls has been studied by several authors; for example, see Delbosco [5] and Liu [12].

In this paper, we establish new results related to coincidence and common fixed points of weakly compatible nonself maps satisfying the property $(E \cdot A)$ and strict contractive conditions on an arbitrary nonempty subset of a metric space. Applications of our results to best approximation and eigenvalue problems will also be given.

2 Preliminaries

Let f and g be selfmaps of a metric space (X, d). The maps f and g are

(1) weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$, for all $x \in X$,

(2) compatible if $\lim\limits_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$ for some $t \in X$,

(3) weakly compatible if they commute at their coincidence points; i.e., if $fu = gu$ for some u in X, then $fgu = gfu$,

(4) satisfying the property $(E \cdot A)$ if there exists a sequence $\{x_n\}$ such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$ for some $t \in X$.

The map f is nonexpansive if $d(fx, fy) \leq d(x, y)$, for all $x, y \in X$. We say f is g-nonexpansive if $d(fx, fy) \leq d(gx, gy)$, for all $x, y \in X$.

Note that weakly commuting maps are compatible and compatible maps are weakly compatible but the converse in each case does not hold (for examples and counter-examples, see [1], [9] and [11]). It is easy to see that two noncompatible maps satisfy the property $(E \cdot A)$ (see [1], Remark 1). Some fixed point results for noncompatible maps are obtained in [13].

Let M be a subset of X and $u \in X$. We denote by $P_M(u)$, the set of best approximations to u from M; that is,

$$P_M(u) = \{y \in M : d(y, u) = d(u, M)\},$$

where $d(u, M) = \inf\{d(u, m) : m \in M\}$. The existence of common fixed points in $P_M(u)$ has been studied by various authors; see Al-Thagafi [2], Hussain and Khan [6] and Shahzad [16].

3 Coincidence Point Results

Throughout this section, B denotes an arbitrary nonempty subset of a metric space X. We obtain some coincidence and common fixed point theorems for weakly compatible nonself maps (which need not be continuous) satisfying the property $(E \cdot A)$ and strict contractive conditions. We begin with an extension of Theorem 1 of Aamri and El Moutawakil [1] for nonself maps on $B \subseteq X$; our result is an improvement of Theorem 2.2 of Ćirić [4] in the sense that continuity of the map is removed.

Theorem 3.1 *Let $f, g : B \to X$ be such that:*

(i) f and g satisfy the property $(E \cdot A)$,

(ii) gB is complete or fB is complete with $fB \subseteq gB$,

(iii) for all $x \neq y$ in B, the following contractive condition holds:

$$d(fx, fy) < \max\Big\{d(gx, gy), rd(fx, gx) + \alpha d(fy, gy), \frac{1}{2}[d(fx, gy) + d(fy, gx)]\Big\} \quad (3.1)$$

where $r \in [0, +\infty)$ and $\alpha \in [0, 1)$.

Then f and g have a coincidence point in B. Further, if a is a coincidence point of f and g such that $fa \in B$ and f and g are weakly compatible, then f and g have a unique common fixed point in B.

Proof. By (i), there exists a sequence $\{x_n\}$ in B such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$. If gB is complete, then $\lim_{n\to\infty} gx_n = t = ga$, for some $a \in B$. Now, we show that $fa = ga$. By (iii), we have

$$d(fx_n, fa) < \max\left\{d(gx_n, ga), rd(fx_n, gx_n) + \alpha d(fa, ga), \frac{1}{2}[d(fx_n, ga) + d(fa, gx_n)]\right\}.$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} d(ga, fa) &\leq \max\left\{d(ga, ga), rd(ga, ga) + \alpha d(fa, ga), \frac{1}{2}[d(ga, ga) + d(fa, ga)]\right\} \\ &= \max\left\{\alpha d(fa, ga), \frac{1}{2}d(fa, ga)\right\}. \end{aligned}$$

This is possible only if $d(ga, fa) = 0$; that is, $fa = ga$.

Now, if $fa \in B$ and f and g are weakly compatible, then $ffa = fga = gfa = gga$. We prove that fa is a common fixed point. Suppose not; then

$$\begin{aligned} d(fa, ffa) &< \max\left\{d(ga, gfa), rd(fa, ga) + \alpha d(ffa, gfa), \frac{1}{2}[d(fa, gfa) + d(ffa, ga)]\right\} \\ &= d(fa, ffa) \end{aligned}$$

a contradiction. Thus $ffa = gfa = fa$. Similarly, we can prove that the case fB is complete and $fB \subseteq gB$. Finally, assume that $a \neq b$ are two common fixed points of f and g. Then by (iii), we get

$$\begin{aligned} d(a, b) = d(fa, fb) &< \max\Big\{d(ga, gb), rd(fa, ga) + \alpha d(fb, gb), \frac{1}{2}[d(fa, gb) + d(fb, ga)]\Big\} \\ &= d(a, b) \end{aligned}$$

a contradiction. Hence $a = b$. ■

The significance of the factor $rd(fx, gx) + \alpha d(fy, gy)$ in (3.1) is that it makes Theorem 1 in [1] a special case of Theorem 3.1 with $r = \alpha = 1/2$.

The following example shows that our theorem works where Theorem 1 of Aamri and El Moutawakil [1] is not applicable.

Example 3.2 Let X be the usual space of reals. Define $f(x) = x^2$ and $g(x) = x^4$. It is easy to verify that f and g satisfy the property $(E \cdot A)$ for the sequence $\left\{1 + \frac{1}{n}\right\}$, $n = 1, 2, 3, \ldots$ Note that the contractive condition of Theorem 1 in [1] is not satisfied (take $x = 1$ and $y = 0$). Now, if $f, g : B \to X$ where $B = [1, 2]$, then for all $x \neq y$ in B, (3.1) holds because $|f(x) - f(y)| = |x^2 - y^2| < |x^4 - y^4| = |g(x) - g(y)|$. Thus all the conditions of Theorem 3.1 are satisfied and 1 is the common fixed point of f and g in $[1, 2]$.

In the presence of a contractive condition such as (3.1), it is common to study results with a contractive condition in terms of a suitable function; we refer the reader to Amri and El Moutawakil ([1], Corollaries 2-5, Theorem 2), Ćirić ([4], p. 29) and Jachymski ([7], Lemma 2.2, Corollary 3.2, Theorem 3.3, Theorem 5.1).

In the following result, we replace the property $(E \cdot A)$ in Theorem 3.1 by a map ϕ satisfying a contractive condition. The proof is similar to that of Corollary 2 in [1] and hence is omitted.

Corollary 3.3 *Let $f, g : B \to X$ be such that:*

(i) there exists a map $\phi : B \to \mathbb{R}^+$ (the set of all nonnegative reals) such that $d(fx, gx) < \phi(gx) - \phi(fx)$, for all x in B,

(ii) gB is complete or fB is complete with $fB \subseteq gB$,

(iii) for all $x \neq y$ in B, (3.1) holds.

Then f and g have a coincidence point in B. Further, if a is a coincidence point of f and g such that $fa \in B$ and f and g are weakly compatible, then f and g have a unique common fixed point.

Suppose that $F : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and $0 < F(t) < t$, for all $t \in (0, \infty)$. The next theorem deals with four nonself maps under a contractive condition in terms of the function F; this theorem is a considerable improvement of Theorem 2 of Aamri and El Moutawakil [1] for nonself maps on an arbitrary subset of a metric space (compare our result also with Theorem 2.3 in [4]).

Theorem 3.4 *Let $f, g, p, q : B \to X$ be such that:*

(i) the pair (f, p) or (g, q) satisfies the property $(E \cdot A)$,

(ii) the range of one of the maps f, g, p or q is complete, $fB \subseteq qB$ and $gB \subseteq pB$,

(iii) for all x, y in B, the following condition holds:

$$d(fx, gy) \leq F(\max\{d(px, qy), d(px, gy), d(qy, gy)\}). \tag{3.2}$$

Then:

(a) f and p have a coincidence point, and g and q have a coincidence point,

(b) if a is a coincidence point of f and p such that $fa \in B$ and f and p are weakly compatible, then they have a common fixed point,

(c) if b is a coincidence point of g and q such that $gb \in B$ and g and q are weakly compatible, then they have a common fixed point,

(d) f, g, p and q have a unique common fixed point provided (b) and (c) hold.

Proof. (a) Assume that g and q satisfy the property $(E \cdot A)$; that is, there exists a sequence $\{x_n\}$ in B such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} qx_n = t$, for some $t \in X$. Since $gB \subseteq pB$, there exists a sequence $\{y_n\}$ in B with $gx_n = py_n$, for all n. So, $\lim_{n\to\infty} py_n = t$. By (iii), we have

$$\begin{aligned} d(fy_n, gx_n) &\leq F(\max\{d(py_n, qx_n), d(py_n, gx_n), d(qx_n, gx_n)\}) \\ &\leq F(d(gx_n, qx_n)) \\ &< d(gx_n, qx_n). \end{aligned}$$

Thus $\lim_{n\to\infty} d(fy_n, t) = 0$ and so, $\lim_{n\to\infty} fy_n = t$. Let pB be complete. Then $t = pa$, for some $a \in B$. By (iii), we get

$$d(fa, gx_n) \leq F(\max\{d(pa, qx_n), d(pa, gx_n), d(qx_n, gx_n)\}).$$

Taking the limit as $n \to \infty$, it follows that $fa = pa$. Also $fB \subseteq qB$ implies that $fa = qb$, for some $b \in B$. We show that $fa = gb$. Suppose not; then

$$\begin{aligned} d(fa, gb) &\leq F(\max\{d(pa, qb), d(pa, gb), d(qb, gb)\}) \leq F(d(fa, gb)) \\ &< d(fa, gb) \end{aligned}$$

a contradiction. Thus $fa = pa = gb = qb$.

(b) If $fa \in B$ and f and p are weakly compatible, then $ffa = fpa = pfa = ppa$. Hence fa is a common fixed point of f and p.

(c) Similar to the case (b).

(d) We show that fa is a common fixed point of f, g, p and q. If not; then

$$\begin{aligned} d(ffa, fa) &= d(ffa, gb) \leq F(\max\{d(pfa, qb), d(pfa, gb), d(qb, gb)\}) \\ &\leq F(d(ffa, fa)) \\ &< d(ffa, fa) \end{aligned}$$

sets a contradiction. Thus $fa = ffa = pfa$. Similarly, $gb = ggb = qgb$. Since $fa = gb$, therefore fa is a common fixed point of f, g, p and q. The proof is similar if qB, fB or gB is complete. Finally, if $u \neq v$ are two common fixed points of f, g, p and q, then

$$d(u, v) = d(fu, gv) \leq F(\max\{d(pu, qv), d(pu, gv), d(qv, gv)\}) \leq F(d(u, v)) < d(u, v)$$

gives a contradiction. Thus $u = v$ proves the uniqueness of the common fixed point. ■

4 Invariant Approximation

In this section, we obtain common fixed points of best approximation. Our work provides analogues of most of the well-known results for the class of weakly compatible maps on a metric space.

Recently, Hussain and Khan [6] have obtained in Theorem 3.1, a generalization of Theorem 3 by Sahab et al. [14] for a class of noncommuting selfmaps on a Hausdorff locally convex space. An analogue of Theorem 3.1 in [6] is given below in the setup of an arbitrary metric space.

Theorem 4.1 *Let M be a subset of a metric space X and f and g be selfmaps of X. Assume that u is a common fixed point of f and g, and $D = P_M(u)$ is nonempty. Suppose that:*

(i) f and g are weakly compatible and satisfy the property $(E \cdot A)$ on D,

(ii) $gD = D$, $f(\partial M) \subseteq M$ (∂M denotes the boundary of M), and fD or D is complete,

(iii) f is g-nonexpansive on $D \cup \{u\}$,

(iv) for all $x \neq y$ in D, (3.1) holds.

Then f and g have a unique common fixed point in $P_M(u)$.

Proof. Let $y \in D$. Then $gy \in D$. By the definition of $P_M(u)$, $y \in \partial M$ and since $f(\partial M) \subseteq M$, it follows that $fy \in M$. As f is g-nonexpansive on $D \cup \{u\}$, so

$$d(fy, u) = d(fy, fu) \leq d(gy, gu) = d(gy, u).$$

Now, $fy \in M$ and $gy \in D$ imply that $fy \in D$; consequently, f and g are selfmaps of D. By Theorem 3.1, there exists a unique $b \in D$ such that $b = fb = gb$. ■

The following example illustrates our theorem.

Example 4.2 *Let $X = \mathbb{R}$ and $M = [1, 4]$. Define $f(x) = \frac{1}{3}(x + 2)$ and $g(x) = \frac{1}{2}(x + 1)$. The maps f and g being commuting are weakly compatible and satisfy the property $(E \cdot A)$ for the sequence $\left\{1 + \frac{1}{n}\right\}$, $n = 1, 2, \ldots$. Also, $|fx - fy| < |gx - gy|$. All the conditions of Theorem 4.1 are satisfied. Clearly, $P_M(0) = \{1\}$ and 1 is the common fixed point of f and g.*

The existence of a unique common fixed point from the set of best approximations for four weakly compatible maps is established in the next result. It is remarked that the study of best approximations in the context of four maps is a new one in the literature.

Theorem 4.3 *Let f, g, p and q be selfmaps of a metric space X and M be a subset of X. Assume that u is a common fixed point of f, g, p and q, and $D = P_M(u)$ is nonempty. Suppose that:*

(i) the pairs (f, p) and (g, q) are weakly compatible, and the pair (f, p) or (g, q) satisfies the property $(E \cdot A)$ on D,

(ii) $pD = D$, $qD = D$, $f(\partial M) \subseteq M$, $g(\partial M) \subseteq M$, and D, fD, or gD is complete,

(iii) f is p-nonexpansive and g is q-nonexpansive on $D \cup \{u\}$,

(iv) for all $x, y \in D$, (3.2) holds.

Then f, g, p and q have a unique common fixed point in $P_M(u)$.

Proof. As in the proof of Theorem 4.1, we can prove that $fy \in D$ and $gy \in D$. Thus f, g, p and q are selfmaps of D. Therefore, by Theorem 3.4, there exists a unique $b \in D$ such that b is a common fixed point of f, g, p and q. ■

Following Al-Thagafi [2], we define for $g : M \to X$, $C_M^g(u) = \{x \in M : gx \in P_M(u)\}$ and $D_M^g(u) = P_M(u) \cap C_M^g(u)$. Note that $D_M^g(u) = P_M(u) = C_M^g(u)$ whenever g is the identity map on M.

We establish the analogues of Theorem 3.2 by Al-Thagafi [2] and Theorem 3.3 due to Hussain and Khan [6] in the following result.

Theorem 4.4 *Let f and g be selfmaps of a metric space X and M be a subset of X. Assume that u is a common fixed point of f and g, and $D^* = D_M^g(u)$ is nonempty. Suppose that:*

(i) f and g are weakly compatible and satisfy the property $(E \cdot A)$ on D^*,

(ii) g is nonexpansive on $P_M(u) \cup \{u\}$ and f is g-nonexpansive on $D^* \cup \{u\}$,

(iii) $gD^* = D^*, f(\partial M) \subseteq M$, and fD^* or D^* is complete,

(iv) for all $x \neq y$ in D^*, (3.1) holds.

Then f and g have a unique common fixed point in D^*.

Proof. Let $y \in D^*$. Then $gy \in D^*$. By the definition of D^*, $y \in \partial M$ and so $fy \in M$. As f is g-nonexpansive on $D^* \cup \{u\}, d(fy, u) = d(fy, fu) \leq d(gy, u)$. Therefore, $fy \in P_M(u)$. Since g is nonexpansive on $P_M(u) \cup \{u\}$, therefore

$$d(gfy, u) = d(gfy, gu) \leq d(fy, u) = d(fy, fu) \leq d(gy, gu) = d(gy, u).$$

Thus, $gfy \in P_M(u)$ and so $fy \in C_M^g(u)$. Therefore, $fy \in D^*$. Hence f and g are selfmaps of D^*. Thus, by Theorem 3.1, there exists a unique $b \in D^* \subset P_M(u)$ such that $b = fb = gb$. ■

5 Eigenvalue Problems

The aim of this section is to seek solutions of certain nonlinear eigenvalue problems for operators defined on a normed space and closed balls of a reflexive Banach space.

We now apply Theorem 3.1 to solve an eigenvalue problem as follows:

Theorem 5.1 *Let X be a normed space and f be a selfmap of X with $f(0) \neq 0$. Suppose that:*

(i) there exists a sequence $\{x_m\}$ such that $\lim_{m\to\infty} f_n x_m = \lim_{m\to\infty} x_m = t$ for some $t \in X$, where $f_n = \left(1 - \frac{1}{n}\right) f$, $n = 2, 3, 4, \ldots,$

(ii) X or fX is complete,

(iii) for all $x \neq y$ in X, the following condition holds:

$$\|fx - fy\| \leq \max\left\{\|x - y\|, r\|f_n x - x\| + \alpha\|f_n y - y\|, \frac{1}{2}(\|f_n y - x\| + \|f_n x - y\|)\right\} \tag{5.1}$$

where $r \in [0, +\infty)$ and $\alpha \in [0, 1)$.

Then $M_n = 1/\left(1 - \frac{1}{n}\right)$ is an eigenvalue of f for each $n > 1$.

Proof. Clearly, f_n and I (the identity map on X) are commuting and satisfy the property $(E \cdot A)$. Note that $\|f_n x - f_n y\| < \|fx - fy\|$ for each $n > 1$. By this and (iii), for all $x \neq y$ in X and each $n > 1$, (3.1) is satisfied for the maps f_n and I. By Theorem 3.1, there exists $x_n \in X$ such that $x_n = f_n x_n$ for each $n > 1$; that is, $fx_n = M_n x_n$ for each $n > 1$. This and $f(0) \neq 0$ imply that $x_n \neq 0$ for each $n > 1$. Thus, for each $n > 1$, x_n is an eigenvector and M_n is an eigenvalue for f. ■

Example 5.2 *Let $X = \mathbb{R}^2$ and f be defined by $f(x, y) = (x-1, y+1)$. Clearly, $f(0,0) \neq (0,0)$ and (5.1) holds in view of $|f(x_1, y_1) - f(x_2, y_2)| = |(x_1, y_1) - (x_2, y_2)|$. Now, for the sequence $(x_n, y_n) = \left(\frac{1}{n} - 1, \frac{1}{n} + 1\right), \quad n = 1, 2, \ldots,$*

$$\lim_{n\to\infty} f_2(x_n, y_n) = \lim_{n\to\infty} \frac{1}{2} f(x_n, y_n) = (-1, 1) = \lim_{n\to\infty} (x_n, y_n).$$

Thus, by Theorem 5.1, $M_2 = 2$ is an eigenvalue of f. The corresponding eigenvector is $(-1, 1)$.

In the sequel, B denotes the closed ball $B = \{x \in X : \|x\| \leq r\}$.

For the solution of eigenvalue problems of nonself maps on closed balls, we need the following result.

Theorem A ([5], p. 92). *Let X be a reflexive Banach space and $f : B \to X$ be a weakly continuous map. Suppose that for each $x \in \partial B$, one of the following conditions holds:*

(i) $\|fx\| \leq \max\{\|fx - x\|, \|x\|\}$,

(ii) there exists $p > 1$ such that $\|fx\|^p \leq \|fx - x\|^p + \|x\|^p$.

Then f has a fixed point in B.

Theorem 5.3 *Let X be a reflexive Banach space and $f : B \to X$ be weakly continuous with $f(0) \neq 0$. Suppose that for each $x \in \partial B$ and for $k \in (0, 1]$, one of the following conditions holds:*

(i) $\|fx\| \leq \max\{\|kfx - x\|, \|x\|\}$,

(ii) there exists $p > 1$ such that $\|fx\|^p \leq \|kfx - x\|^p + \|x\|^p$.

Then $M = \dfrac{1}{k}$ is an eigenvalue for f.

Proof. Let $M = \dfrac{1}{k}, \quad k \in (0, 1]$. Define, $f_k = kf$. Assume that (i) or (ii) is satisfied; then we get one of the following:

(a) $\|f_k x\| \leq \|fx\| \leq \max\{\|f_k x - x\|, \|x\|\}$,

(b) $\|f_k x\|^p \leq \|fx\|^p \leq \{\|f_k x - x\|^p, \|x\|^p\}$.

By Theorem A, there exists $u \in B$ such that $f_k u = u$. So $fu = Mu$. This and $f(0) \neq 0$ imply that $u \neq 0$. Thus u is an eigenvector for f and so M is an eigenvalue for f as desired. ■

As an application of Theorem 5.3, we obtain the following analogue of Theorem 3.2 in [3].

Theorem 5.4 *Let C be a closed and bounded subset of $\mathbb{R}^n$ and $T : L^p(C) \to L^p(C)$. Suppose that:*

(i) $H = H(t, s) : C \times \mathbb{R} \to \mathbb{R}$ is weakly continuous with respect to s uniformly in t,

(ii) $x(t) \in L^p(C)$ implies $H(t, Tx(t)) \in L^p(C)$,

(iii) for $x(t) \in L^p(C)$ with $\|x(t)\|_p = 1$, $\|H(t, Tx(t))\|_p \leq \max\{1, \|kH(t, Tx(t)) - x(t)\|_p\}$, where $k \in (0, 1]$,

(iv) $H(t, T(0)) \neq 0$.

Then the operator equation

$$H(t, Tx(t)) = ux(t) \tag{5.2}$$

has a solution in B_1, the closed unit ball of $L^p(C)$, for each $u = \dfrac{1}{k}, \quad k \in (0, 1]$.

Proof. We know that $L^p \quad (1 < p < \infty)$ is a reflexive Banach space. The operator S defined by $Sx(t) = H(t, Tx(t))$ maps $L^p(C)$ into itself by (ii). In view of (iii), for the operator $S : B_1 \to L^p(C)$,

$$\|Sx(t)\|_p \leq \max\{\|x(t)\|_p, \|kSx(t) - x(t)\|_p\}$$

for each $x(t) \in \partial B_1$ and $k \in (0, 1]$. Now all the conditions of Theorem 5.3 are satisfied and hence for each $u = \frac{1}{k}, \quad k \in (0, 1]$, we get, $Sx(t) = ux(t)$; that is, the operator equation (5.2) has a solution in B_1 for each $u = \frac{1}{k}, \quad k \in (0, 1]$. ■

The following example supports the above theorem.

Example 5.5 *The eigenvalue problem*

$$e^t - \|x(t)\| = ux(t)$$

has a nontrivial solution in the closed unit ball B_1 of $L^2([0, 1])$.

Solution *Set $H(t, s) = e^t - s$, $Tx(t) = \|x(t)\|$, $C = [0, 1]$ and $p = 2$ in Theorem 5.4.*

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