COMMON FIXED POINTS ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce a general iteration scheme for a finite family of asymptotically quasi-nonexpansive mappings. The new iterative scheme includes the modified Mann and Ishikawa iterations, three-step iterative scheme of Xu and Noor and Khan and Takahashi scheme as special cases. Our results are generalizations as well as refinement of several known results in the current literature.

1. Introduction and Preliminaries

Let C be a nonempty subset of a real Banach space X and T a selfmapping of C. Denote by $F(T)$, the set of fixed points of T. Throughout this paper, we assume that $F(T) \neq \phi$. The mapping T is said to be (i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$; (ii) quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$; (iii) asymptotically nonexpansvie if there exists a sequence $\{u_n\}$ in $[0, +\infty)$ with $\lim_{n\to\infty} u_n = 0$ and $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, \ldots$; (iv) asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\}$ in $[0, +\infty)$ with $\lim_{n\to\infty} u_n = 0$ and $\|T^n x - p\| \leq (1 + u_n)\|x - p\|$, for all $x \in C,\ p \in F(T)$ and $n = 1, 2, \ldots$; (v) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$; (vi) uniformly Hölder continuous if there are constants $L > 0$ and $\gamma > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|^\gamma$ for all $x, y \in C$ and $n = 1, 2, 3, \ldots$ (cf. Zhou et. al [23], p.62); (vii) uniformly equi-continuous if and only if $\|T^n x_n - T^n y_n\| \to 0$ whenever $\|x_n - y_n\| \to 0$ as $n \to \infty$ (cf. Zhou et. al [23], p.62) and (viii) semi-compact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

From the above definitions, it follows that: (i) a nonexpansive mapping must be quasi-nonexpansive and asymptotically nonexpansive; (ii) an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive; (iii) uniformly L-Lipschitzian mapping is uniformly Hölder continuous; (iv) uniformly Hölder continuous is uniformly continuous. However, their converses are not true, in general due to the map $Tx = (1 - x^{3/2})^{2/3}$ for all $x \in [0, 1]$.

The map $T : C \to X$ is said to be demiclosed at 0 if for each sequence $\{x_n\}$ in C converging weakly to x and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$.

1991 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.

Key words and phrases. Modified Mann and Ishikawa iterations, asymptotically quasi-nonexpansive mapping, common fixed point, weak and strong convergence, Banach space.

A Banach space X is said to satisfy Opial's property if for each $x \in X$ and each sequence $\{x_n\}$ weakly convergent to x, the following condition holds for all $x \neq y$:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

It is well known that all Hilbert spaces and $\ell_p (1 < p < \infty)$ spaces are Opial spaces while L_p spaces $(p \neq 2)$ are not.

Schu [16], in 1991, considered the following modified Mann iteration process (cf. Mann [11]):

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1 \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ which is bounded away from 0 and 1, i.e., $a \leq \alpha_n \leq b$ for all n and some $0 < a \leq b < 1$. In 1994, Tan and Xu [20] studied the modified Ishikawa iteration process (cf. Ishikawa [7]):

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n((1-\beta_n)x_n + \beta_n T^n x_n), \quad n \geq 1 \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that $\{\alpha_n\}$ is bounded away from 0 and 1 and $\{\beta_n\}$ is bounded away from 1.

Xu and Noor [21], in 2002, introduced a three-step iterative scheme as follows:

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T^n x_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n z_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad n \geq 1 \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real numbers in $[0,1]$.

Finding common fixed points of a finite family $\{T_i : i = 1, 2, \ldots, k\}$ of mappings acting on a Hilbert space is a problem that often arises in applied mathematics. In fact, many algorithms have been introduced for different classes of mappings with a nonempty set of common fixed points. Unfortunately, the existence results of common fixed points of a family of mappings are not known in many situations. Therefore, it is natural to consider approximation results for these classes of mappings. Approximating common fixed points of a finite family of nonexpansive mappings by iteration has been studied by many authors (see, for example, Kuhfittig [10], Rhoades [15] and Takahashi and Shimoji [19]). Ghosh and Debnath [4] proved some convergence results for common fixed points of families of quasi-nonexpansive mappings.

Goebel and Kirk [6], in 1972, introduced the notion of an asymptotically nonexpansive mapping and established that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space X and T is an asymptotically nonexpansive selfmapping of C, then T has a fixed point. Bose [1] initiated in 1978, the study of iterative construction for fixed points of asymptotically nonexpansive mappings. Xu and Ori [22], in 2001, introduced an implicit iteration process for a finite family of nonexpansive mappings. Sun [18], in 2003, modified this implicit iteration process for a finite family of asymptotically quasi-nonexpansvie mappings. Khan and Takahashi [8] have approximated common fixed points of two asymptotically nonexpansive mappings by the modified Ishikawa iteration. Recently, Shahzad and Udomene [17] established convergence theorems for the modified Ishikawa iteration process of two asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings.

For a finite family of mappings, it is desirable to devise a general iteration scheme which extends the modified Mann iteration (1.1), the modified Ishikawa iteration

(1.2), Khan and Takahashi scheme [8] and the three-step iteration by Xu and Noor [21], simultaneously. Thereby, to achieve this goal, we introduce a new iteration process for a finite family $\{T_i : i = 1, 2, \dots, k\}$ of asymptotically quasi-nonexpansive mappings as follows:

Let C be a convex subset of a Banach space X and $x_1 \in C$. Suppose that $\alpha_{in} \in [0, 1]$, $n = 1, 2, 3, \dots$ and $i = 1, 2, \dots, k$. Let $\{T_i : i = 1, 2, \dots, k\}$ be a family of selfmappings of C. The iteration scheme is defined as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_{kn})x_n + \alpha_{kn}T_k^n\, y_{(k-1)n}, \\ y_{(k-1)n} &= (1-\alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n\, y_{(k-2)n}, \\ y_{(k-2)n} &= (1-\alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n\, y_{(k-3)n}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_{2n} &= (1-\alpha_{2n})x_n + \alpha_{2n}T_2^n\, y_{1n}, \\ y_{1n} &= (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n\, y_{0n}, \end{aligned} \tag{1.3}$$

where $y_{0n} = x_n$ for all n.

Clearly, the iteration process (1.3) generalizes the modified Mann iteration (1.1), the modified Ishikawa iteration (1.2) and the three-step iteration scheme from one mapping to the finite family of mappings $\{T_i : i = 1, 2, \dots, k\}$.

In the sequel, we assume that $F = \bigcap_{i=1}^{k} F(T_i)$.

The main purpose of this paper is to:

(i) establish a necessary and sufficient condition for convergence of the iteration scheme (1.3) to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in a Banach space;
(ii) prove weak and strong convergence results of the iteration scheme (1.3) to a common fixed point of a finite family of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space.

Our work is a significant generalization of the corresponding results of Khan and Takahashi [8], Qihou [13], Schu [16], Shahzad and Udomene [17], Tan and Xu [20] and Xu and Noor [21]. Moreover, these results provide analogue of the results of Sun [18], for the iteration scheme (1.3) instead of the implicit iteration.

We need the following useful known lemmas for the development of our convergence results.

Lemma 1 [18, Lemma 2.2]. *Let the sequences $\{a_n\}$ and $\{u_n\}$ of real numbers satisfy:*

$$a_{n+1} \le (1+u_n)a_n,\ a_n \ge 0,\ u_n \ge 0, \sum_{n=1}^{\infty} u_n < +\infty.$$

Then (i) $\lim_{n\to\infty} a_n$ exists; (ii) if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 [16, Lemma 1.3]. *Let X be a uniformly convex Banach space. Assume that $0 < b \le t_n \le c < 1$, $n = 1, 2, 3, \dots$ Let the sequences $\{x_n\}$ and $\{y_n\}$ in X be such that $\limsup_{n\to\infty} \|x_n\| \le a$, $\limsup_{n\to\infty} \|y_n\| \le a$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = a$, where $a \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

2. Convergence Theorems in Banach Spaces

The aim of this section is to prove some results for the iterative process (1.3) to converge to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in a Banach space. We begin with the following:

Lemma 3. Let C be a nonempty closed convex subset of a Banach space, and $\{T_i : i = 1, 2, \dots, k\}$ a family of asymptotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - p_i\| \leq (1 + u_{in})\|x - p_i\|$ for all $x \in C$ and $p_i \in F(T_i),\ i = 1, 2, \dots, k$ where $\{u_{in}\}$ are sequences in $[0, +\infty)$ with $\lim_{n\to\infty} u_{in} = 0$ for each i. Assume that $F \neq \phi$ and $\sum_{n=1}^{\infty} u_{in} < +\infty$ for each i. Define the sequence $\{x_n\}$ as in (1.3). Then
(a) there exists a sequence $\{\nu_n\}$ in $[0, +\infty)$ such that $\sum_{n=1}^{\infty} \nu_n < +\infty$ and $\|x_{n+1} - p\| \leq (1 + \nu_n)^k \|x_n - p\|$, for all $p \in F$ and all n;
(b) there exists a constant $M > 0$ such that $\|x_{n+m} - p\| \leq M\|x_n - p\|$, for all $p \in F$ and $n, m = 1, 2, 3, \dots$.

Proof. (a) Let $p \in F$ and $\nu_n = \max_{1\leq i\leq k} u_{in}$, for all n. Since $\sum_{n=1}^{\infty} u_{in} < +\infty$ for each i, therefore $\sum_{n=1}^{\infty} \nu_n < +\infty$. Now we have

$$\begin{aligned}
\|y_{1n} - p\| &\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}\|T_1^n x_n - p\| \\
&\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}(1 + u_{1n})\|x_n - p\| \\
&= (1 + \alpha_{1n}u_{1n})\|x_n - p\| \\
&\leq (1 + \nu_n)\|x_n - p\|.
\end{aligned}$$

Assume that $\|y_{jn} - p\| \leq (1 + \nu_n)^j \|x_n - p\|$ holds for some $1 \leq j \leq k - 2$. Then

$$\begin{aligned}
\|y_{(j+1)n} - p\| &\leq (1 - \alpha_{(j+1)n})\|x_n - p\| + \alpha_{(j+1)n}\|T_{j+1}^n\, y_{jn} - p\| \\
&\leq (1 - \alpha_{(j+1)n})\|x_n - p\| + \alpha_{(j+1)n}(1 + u_{(j+1)n})\|y_{jn} - p\| \\
&\leq (1 - \alpha_{(j+1)n})\|x_n - p\| + \alpha_{(j+1)n}(1 + u_{(j+1)n})(1 + \nu_n)^j\|x_n - p\| \\
&\leq (1 - \alpha_{(j+1)n})\|x_n - p\| + \alpha_{(j+1)n}(1 + \nu_n)^{j+1}\|x_n - p\| \\
&= \left[1 - \alpha_{(j+1)n} + \alpha_{(j+1)n}\left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!}\nu_n^r\right)\right]\|x_n - p\| \\
&= \left[1 + \alpha_{(j+1)n}\sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!}\nu_n^r\right]\|x_n - p\| \\
&\leq (1 + \nu_n)^{j+1}\|x_n - p\|.
\end{aligned}$$

Thus, by induction, we have

$$\|y_{in} - p\| \leq (1 + \nu_n)^i\|x_n - p\|, \text{ for all } i = 1, 2, \dots, k - 1. \tag{2.1}$$

Now, by (2.1), we obtain

$$\begin{aligned}
\|x_{n+1}-p\| &\leq (1-\alpha_{kn})\|x_n-p\|+\alpha_{kn}\|T_k^n y_{(k-1)n}-p\| \\
&\leq (1-\alpha_{kn})\|x_n-p\|+\alpha_{kn}(1+u_{kn})\|y_{(k-1)n}-p\| \\
&\leq (1-\alpha_{kn})\|x_n-p\|+\alpha_{kn}(1+u_{kn})(1+\nu_n)^{k-1}\|x_n-p\| \\
&\leq (1-\alpha_{kn})\|x_n-p\|+\alpha_{kn}(1+\nu_n)^{k}\|x_n-p\| \\
&= \left[1-\alpha_{kn}+\alpha_{kn}\left(1+\sum_{r=1}^{k}\frac{k(k-1)\cdots(k-r+1)}{r!}\nu_n^r\right)\right]\|x_n-p\| \\
&= \left[1+\alpha_{kn}\sum_{r=1}^{k}\frac{k(k-1)\cdots(k-r+1)}{r!}\nu_n^r\right]\|x_n-p\| \\
&\leq (1+\nu_n)^k\|x_n-p\|.
\end{aligned}$$

This completes the proof of (a).

(b) If $t\geq 0$, then $1+t\leq e^t$ and so, $(1+t)^k\leq e^{kt},\ k=1,2,\ldots.$ Thus, from part (a), we get

$$\begin{aligned}
\|x_{n+m}-p\| &\leq (1+\nu_{n+m-1})^k\|x_{n+m-1}-p\| \\
&\leq \exp\{k\nu_{n+m-1}\}\|x_{n+m-1}-p\|\leq\cdots\leq\exp\left\{k\sum_{i=1}^{n+m-1}\nu_i\right\}\|x_n-p\| \\
&\leq \exp\left\{k\sum_{i=1}^{\infty}\nu_i\right\}\|x_n-p\|.
\end{aligned}$$

Setting $M=\exp\{k\sum_{i=1}^{\infty}\nu_i\}$, completes the proof.

The above lemma generalizes Theorem 3.1 for two asymptotically quasi-nonexpansive mappings by Shahzad and Udomene [17] to the case of any finite family of such mappings.

The next result is the main result of this section. It deals with a necessary and sufficient condition for the convergence of $\{x_n\}$ generated by the iteration process (1.3) to a point of F; for this we follow the arguments of Qihou ([13, Theorem 1).

Theorem 1. Let C be a nonempty closed convex subset of a Banach space X, and $\{T_i : i=1,2,\ldots,k\}$ a family of asymptotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x-p_i\|\leq(1+u_{in})\|x-p_i\|$, for all $x\in C$ and $p_i\in F(T_i),\ i=1,2,\ldots,k$. Suppose that $F\neq\phi$, $x_1\in C$ and $\sum_{n=1}^{\infty}u_{in}<+\infty$ for all i. Then the iterative sequence $\{x_n\}$, defined by (1.3), converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty}d(x_n,F)=0$, where $d(x,F)=\inf_{p\in F}\|x-p\|$.

Proof. We will only prove the sufficiency; the necessity is obvious. From Lemma 3 (a), we have

$$\|x_{n+1}-p\|\leq(1+\nu_n)^k\|x_n-p\|,$$

for all $p\in F$ and all n. Therefore,

$$\begin{aligned}
d(x_{n+1},F) &\leq (1+\nu_n)^k d(x_n,F) \\
&= \left(1+\sum_{r=1}^{k}\frac{k(k-1)\cdots(k-r+1)}{r!}\nu_n^r\right)d(x_n,F).
\end{aligned}$$

As $\sum_{n=1}^{\infty}\nu_n < +\infty$, so $\sum_{n=1}^{\infty}\sum_{r=1}^{k}\frac{k(k-1)\cdots(k-r+1)}{r!}\nu_n^r < +\infty$. By Lemma 1 and $\liminf_{n\to\infty} d(x_n,F)=0$, we get that $\lim_{n\to\infty} d(x_n,F)=0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 3 (b), we have

$$\|x_{n+m}-p\| \leq M\|x_n-p\|, \tag{2.2}$$

Since $\lim_{n\to\infty} d(x_n,F)=0$, therefore for each $\epsilon>0$, there exists a natural number n_1 such that

$$d(x_n,F) \leq \frac{\epsilon}{3M}, n \geq n_1.$$

Hence, there exists $z_1 \in F$ such that

$$\|x_{n_1}-z_1\| \leq \frac{\epsilon}{2M}. \tag{2.3}$$

From (2.2) and (2.3), for all $n \geq n_1$, we have

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \|x_{n+m}-z_1\|+\|x_n-z_1\| \\ &\leq M\|x_{n_1}-z_1\|+M\|x_{n_1}-z_1\| \\ &\leq M\left(\frac{\epsilon}{2M}\right)+M\left(\frac{\epsilon}{2M}\right)=\epsilon.\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence and so converges to $q \in X$. Finally, we show that $q \in F$. For any $\overline{\epsilon}>0$, there exists a natural number n_2 such that

$$\|x_n-q\| \leq \frac{\overline{\epsilon}}{2(2+\nu_1)}, n \geq n_2. \tag{2.4}$$

Again, $\lim_{n\to\infty} d(x_n,F)=0$ implies that there exists a natural number $n_3 \geq n_2$ such that

$$d(x_n,F) \leq \frac{\overline{\epsilon}}{3(4+3\nu_1)}, n \geq n_3.$$

Thus, there exists $z_2 \in F$ such that

$$\|x_{n_3}-z_2\| \leq \frac{\overline{\epsilon}}{2(4+3\nu_1)}. \tag{2.5}$$

From (2.4) and (2.5), for any $T_i,\ i=1,2,\ldots,k$, we get

$$\begin{aligned}\|T_iq-q\| &= \|T_iq-z_2+z_2-T_ix_{n_3}+T_ix_{n_3}-z_2+z_2-x_{n_3}+x_{n_3}-q\| \\ &\leq \|T_iq-z_2\|+2\|T_ix_{n_3}-z_2\|+\|x_{n_3}-z_2\|+\|x_{n_3}-q\| \\ &\leq (1+\nu_1)\|q-z_2\|+2(1+\nu_1)\|x_{n_3}-z_2\|+\|x_{n_3}-z_2\|+\|x_{n_3}-q\| \\ &\leq (1+\nu_1)\|x_{n_3}-q\|+(1+\nu_1)\|x_{n_3}-z_1\| \\ &\quad +2(1+\nu_1)\|x_{n_3}-z_2\|+\|x_{n_3}-z_2\|+\|x_{n_3}-q\| \\ &= (2+\nu_1)\|x_{n_3}-q\|+(4+3\nu_1)\|x_{n_3}-z_2\| \\ &\leq (2+\nu_1)\frac{\overline{\epsilon}}{2(2+\nu_1)}+(4+3\nu_1)\frac{\overline{\epsilon}}{2(4+3\nu_1)}=\overline{\epsilon}.\end{aligned}$$

Since $\overline{\epsilon}$ is arbitrary, therefore $\|T_iq-q\|=0$, for all i, i.e., $T_iq=q, i=1,2,\ldots,k$. Thus $q \in F$.

Remark 1. *Theorem 1 contains as special cases, Theorem 3.2 of Shahzad and Udomene [17] and Theorem 1 by Qihou [13] together with its Corollaries 1 and 2, which are themselves extensions of the results of Ghosh and Debnath [5] and Petryshyn and Williamson [12].*

An asymptotically nonexpansvie mapping is asymptotically quasi-nonexpansive, so we have:

Corollary 1. *Let C be a nonempty closed convex subset of a Banach space X, and $\{T_i : i = 1, 2, \dots, k\}$ a family of asymsptotically nonexpansive selfmappings of C, i.e., $\|T_i^n x - T_i^n y\| \leq (1 + u_{in})\|x - y\|$, for all $x, y \in C$ and $i = 1, 2, \dots, k$. Suppose that $F \neq \phi$, $x_1 \in C$ and $\sum_{n=1}^{\infty} u_{in} < +\infty$, for all i. Then the iterative sequence $\{x_n\}$, defined by (1.3), converges strongly to a point $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Corollary 2. *Let C, $\{T_i : i = 1, 2, \dots, k\}$, F and u_{in} be as in Theorem 1. Then the iterative sequence $\{x_n\}$, defined by (1.3), converges strongly to a point $p \in F$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.*

Theorem 2. *Let C be a nonempty closed convex subset of a Banach space X, and $\{T_i : i = 1, 2, \dots, k\}$ a family of asymptotically nonexpansive selfmappings of C. Suppose that $F \neq \phi$, $x_1 \in C$ and $\sum_{n=1}^{\infty} u_{in} < +\infty$ for all i. Let $\{x_n\}$ be the sequence defined by (1.3). If $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, $i = 1, 2, \dots, k$ and one of the mappings is semi-compact, then $\{x_n\}$ converges strongly to $p \in F$.*

Proof. Let T_ℓ be semi-compact for some $1 \leq \ell \leq k$. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$. Hence

$$\|p - T_i p\| = \lim_{n_j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0, .$$

Thus, $p \in F$ and by Corollary 2, $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

Theorem 3. *Let C, $\{T_i : i = 1, 2, \dots, k\}$, F and u_{in} be as in Theorem 1. Suppose that there exists a map T_j which satisfies the following conditions:*
(i) $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$;
(ii) there exists a constant M such that $\|x_n - T_j x_n\| \geq M d(x_n, F)$, for all n.

Then the sequence $\{x_n\}$, defined by (1.3), converges strongly to a point $p \in F$.

Proof. From (i) and (ii), it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 1, $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

3. Results in Uniformly Convex Banach Spaces

In this section, we establish some weak and strong convergence results for the iterative scheme (1.3) by removing the condition $\liminf_{n\to+\infty} d(x_n, F) = 0$ from the results obtained in Section 2; for this we have to consider the class of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings on a uniformly convex Banach space.

Lemma 4. Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $\{T_i : i = 1, 2, 3, \dots, k\}$ a family of uniformly equi-continuous and asymptotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - p_i\| \leq (1 + u_{in})\|x - p_i\|$ for all $x \in C$ and $p_i \in F(T_i)$, where $\{u_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$, for each $i \in \{1, 2, 3, \dots, k\}$. Assume that $F \neq \phi$ and the sequence $\{x_n\}$ is as in (1.3) with $\alpha_{in} \in [\delta, 1 - \delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$. Then
(i) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$;
(ii) $\lim_{n\to\infty} \|x_n - T_j^n y_{(j-1)n}\| = 0$, for each $j = 1, 2, , \dots, k$;
(iii) $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$, for each $j = 1, 2, \dots, k$.

Proof. Let $p \in F$ and $\nu_n = \max_{1\leq i\leq k} u_{in}$, for all n.
(i) By Lemma 1 (i) and Lemma 3 (a), it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists for

all $p \in F$. Assume that

$$\lim_{n\to\infty} \|x_n - p\| = c. \tag{3.1}$$

(ii) The inequality (2.1) and (3.1) give that

$$\limsup_{n\to\infty} \|y_{jn} - p\| \le c, 1 \le j \le k-1. \tag{3.2}$$

We also note that:

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_{kn})(x_n - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| \\ &\le (1-\alpha_{kn})\|x_n - p\| + \alpha_{kn}(1+v_n)\|y_{(k-1)n} - p\| \\ &\quad \cdots\cdots\cdots\cdots \\ &\le (1-\alpha_{kn}\alpha_{(k-1)n}\cdots\alpha_{(j+1)n})(1+v_n)^{k-j}\|x_n - p\| \\ &\quad +\alpha_{kn}\alpha_{(k-1)n}\cdots\alpha_{(j+1)n}(1+v_n)^{k-j}\|y_{jn} - p\|. \end{aligned}$$

Therefore,

$$\|x_n - p\| \le \frac{\|x_n - p\|}{\delta^{k-j}} - \frac{\|x_{n+1} - p\|}{\delta^{k-j}(1+v_n)^{k-j}} + \|y_{jn} - p\|$$

and hence

$$c \le \liminf_{n\to\infty} \|y_{in} - p\|, \ 1 \le j \le k-1. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\lim_{n\to\infty} \|y_{jn} - p\| = c, j = 1, 2, 3, \ldots, k-1.$$

That is,

$$\lim_{n\to\infty} \|(1-\alpha_{jn})(x_n - p) + \alpha_{jn}(T_j^n y_{(j-1)n} - p)\| = c,$$

for each $j = 1, 2, 3, \ldots, k-1$.

Also, from (3.2), we obtain

$$\limsup_{n\to\infty} \|T_j^n y_{(j-1)n} - p\| \le c, j = 1, 2, 3, \ldots, k-1.$$

By Lemma 2, we get

$$\lim_{n\to\infty} \|T_j^n y_{(j-1)n} - x_n\| = 0, j = 1, 2, 3, \ldots, k-1. \tag{3.4}$$

For the case $j = k$, by (2.1), we have

$$\|T_k^n y_{(k-1)n} - p\| \le (1+u_{kn})\|y_{(k-1)n} - p\| \le (1+u_{kn})(1+\nu_n)^{k-1}\|x_n - p\|.$$

But $\lim_{n\to\infty} \|x_n - p\| = c$, by part (i). So,

$$\limsup_{n\to\infty} \|T_k^n y_{(k-1)n} - p\| \le c.$$

Moreover,

$$\lim_{n\to\infty} \|(1-\alpha_{kn})(x_n - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = \lim_{n\to\infty} \|x_{n+1} - p\| = c.$$

Again by Lemma 2, we get

$$\lim_{n\to\infty} \|x_n - T_k^n y_{(k-1)n}\| = 0. \tag{3.5}$$

Thus, (3.4) and (3.5) imply that

$$\lim_{n\to\infty} \|T_j^n y_{(j-1)n} - x_n\| = 0, j = 1, 2, 3, \dots, k. \tag{3.6}$$

(iii) For $j = 1$, from part (ii), we have

$$\lim_{n\to\infty} \|T_1^n x_n - x_n\| = 0.$$

If $j = 2, 3, 4, \dots, k$, then we have

$$\begin{aligned} \|T_j^n x_n - x_n\| &= \|(T_j^n x_n - T_j^n y_{(j-1)n}) + (T_j^n y_{(j-1)n} - x_n)\| \\ &= \|T_j^n x_n - T_j^n y_{(j-1)n}\| + \|T_j^n y_{(j-1)n} - x_n\| \to 0 \end{aligned}$$

as $\|x_n - y_{(j-1)n}\| \to 0$ and T_j is uniformly equi-continuous.

Hence,

$$||T_j^n x_n - x_n\| \to 0, n \to \infty, 1 \le j \le k. \tag{3.7}$$

Let us observe that:

$$\begin{aligned} \|x_n - T_j x_n\| \le\ & \|x_n - x_{n+1}\| + \|x_{n+1} - T_j^{n+1} x_{n+1}\| \\ & + \|T_j^{n+1} x_{n+1} - T_j^{n+1} x_n\| + \|T_j^{n+1} x_n - T_j x_n\|. \end{aligned}$$

Using (3.5), (3.7) together with uniform equi- continuity of T_j, we get

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0, 1 \le j \le k.$$

This completes the proof.

Theorem 4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X satisfying the Opial property and let $\{T_i : i = 1, 2, 3, \dots, k\}$ be a family of uniformly equi- continuous and asymsptoticaly quasi-nonexpansive self-mappings of C, i.e., $\|T_i^n x - p_i\| \le (1 + u_{in})\|x - p_i\|$ for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, 3, \dots, k$ where $\{u_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$ for each $i = 1, 2, 3 \dots, k$. Let the sequence $\{x_n\}$ be as in (1.3) with $\alpha_{in} \in [\delta, 1 - \delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$. If $F \ne \phi$ and each $I - T_i, i = 1, 2, 3, \dots, k$, is demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, 3, \dots, k\}$.*

Proof. Let $p \in F$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma 4 (i) and hence $\{x_n\}$ is bounded. Since a uniformly convex Banach space is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to some $z_1 \in C$. By Lemma 4, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ and $I - T_i$ is demiclosed at 0 for $i = 1, 2, 3, \dots, k$, so we obtain $T_i z_1 = z_1$. That is, $z_1 \in F$. In order to show that $\{x_n\}$ converges weakly to z_1, take another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $z_2 \in C$. Again, as above, we can prove that $z_2 \in F$. Next, we show that $z_1 = z_2$. Assume

$z_1 \neq z_2$. Then by the Opial property

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{n_k\to\infty} \|x_{n_k} - z_2\| \\
&< \lim_{n_k\to\infty} \|x_{n_k} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This contradiction proves that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, 3, \ldots, k\}$.

Theorem 5. Under the hypotheses of Lemma 4, assume that, for some $1 \leq i \leq k$, T_i^m is semi-compact for some positive integer m. Then $\{x_n\}$ converges strongly to some common fixed point of the family $\{T_j : j = 1, 2, 3, \ldots, k\}$.

Proof. By Lemma 4 (iii), we have

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0, 1 \leq j \leq k. \tag{3.8}$$

Fix $i \in \{1, 2, 3, \ldots, k\}$ and suppose T_i^m to be semi-compact for some $m \geq 1$. >From (3.8), we obtain

$$\begin{aligned}
\|T_i^m x_n - x_n\| &\leq \|T_i^m x_n - T_i^{m-1} x_n\| + \|T_i^{m-1} x_n - T_i^{m-2} x_n\| \\
&\quad + \cdots + \|T_i^2 x_n - T_i x_n\| + \|T_i x_n - x_n\| \\
&\leq \|T_i x_n - x_n\| + (m-1)L\|T_i x_n - x_n\| \to 0.
\end{aligned}$$

Since $\{x_n\}$ is bounded and T_i^m is semi-compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$ such that $x_{n_j} \to q \in C$. Hence, from (3.8), we have

$$\|q - T_i q\| = \lim_{n\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0, i = 1, 2, 3, \ldots, k.$$

Thus $q \in F$ and by Corollary 2.5, $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i = 1, 2, 3, \ldots, k\}$.

4. Concluding Remarks

1. The family of uniformly equi-continuous and asymptotically quasi-nonexpansive mappings in Lemma 4 can be replaced by a family of asymptotically nonexpansive mappings. We state this result as follows; the proof is similar to that of Lemma 4.

   Lemma 5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of asymptotically nonexpansive selfmappings of C. Assume that $F \neq \phi$ and $\sum_{i=1}^{\infty} u_{in} < \infty$ for each $i = 1, 2, \ldots, k$. Let $\{x_n\}$ be as in (1.3) with $\alpha_{in} \in [\delta, 1-\delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$. Then (i), (ii) and (iii) of Lemma 4 hold.*
2. Lemma 4 (i) extends Lemma 2.1 of Tan and Xu [20]. Lemma 4 (ii) extends Theorem 3.3 of Shahzad and Udomene [17] for two uniformly continuous asymptotically quasi-nonexpansive mappings to any finite family of uniformly equi- continuous and asymptotically quasi-nonexpansive mappings.
3. Lemma 5 (ii) and Lemma 5 (iii) contain as special cases, Lemma 2.2 of Xu and Noor [21] and Lemma 1.5 of Schu [16], respectively.

4. On the lines of the proof of Theorem 3.2 and using Lemma 1.6 of Schu [16] and Lemma 4.1, the following result can be easily proved.
   Theorem 6. *Under the hypotheses of Lemma 5, assume that the space X satisfies the Opial property. Then the sequence $\{x_n\}$ converges weakly to a common fixed point of the family of mappings.*
5. The special cases of Theorem 6 are Theorems 3.1-3.2 of Tan and Xu [20] and Theorem 2.1 due to Schu [16].
6. Following the arguments of the proof of Theorem 4, it is now easy to prove:
   Theorem 7. *Under the assumptions of Lemma 5, suppose that, for some $1 \leq i \leq k$, and a positive integer m, T_i^m is semi-compact. Then $\{x_n\}$ converges strongly to some common fixed point of the family of mappings.*
   The above theorem contains as a special case, Theorem 2.2 of Schu [16].
7. Theorem 3, Corollary 1, Theorem 5 and Theorem 7 about the iteration scheme (1.3) are analogues of Theorem 3.1, Corollary 3.2, Theorem 3.3 and Theorem 3.4 in the context of implcit iteration process by Sun [18], respectively.

Acknowledgement. The authors A.R. Khan and A.A. Domlo gratefully acknowledge support provided by King Fahd University of Petroleum and Minerals during this research.

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