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Department of Mathematics and Statistics
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia
Email: ajstat@gmail.com

The exact probability density function of a bivariate chi-square distribution with two correlated components is derived. Some moments of the product and ratio of two correlated chi-square random variables have been derived. The ratio of the two correlated chi-square variables is used to compare their variability. One such application is referred to. Another application is pinpointed in connection with the distribution of correlation coefficient based on a bivariate *t*-distribution.

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1. Introduction

Fisher (1915) derived the distribution of mean-centered sum of squares and sum of products in order to study the distribution of correlation coefficient from a bivariate normal sample. Let $X_1, X_2, \cdots, X_N \ (N > 2)$ be two-dimensional independent random vectors where $X_j = (X_{1j}, X_{2j})'$, $j = 1, 2, \cdots, N$ is distributed as a bivariate normal distribution denoted by $N_2(\theta, \Sigma)$ with $\theta = (\theta_1, \theta_2)'$ and a 2×2 covariance matrix $\Sigma = (\sigma_{ik})$, $i = 1, 2;\ k = 1, 2$. The sample mean-centered sums of squares and sum of products are given by

$$a_{ii} = \sum_{j=1}^{N} (X_{ij} - \bar{X}_i)^2 = mS_i^2,\ m = N - 1, (i = 1, 2) \text{ and}$$

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$a_{12} = \sum_{j=1}^{N} (X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2) = mRS_1S_2$ respectively. The quantity R is the sample product moment correlation coefficient. The distribution of a_{11}, a_{22} and a_{12} was derived by Fisher (1915) and may be called the bivariate Wishart distribution after Wishart (1928) who obtained the distribution of a p-variate Wishart matrix as the joint distribution of sample variances and covariances from the multivariate normal population. Obviously a_{11}/σ_{11} has a chi-square distribution with m degrees of freedom.

Let $U_1 = mS_1^2/\sigma_1^2$ and $U_2 = mS_2^2/\sigma_2^2$. The joint distribution of U_1 and U_2 is called the bivariate chi-square distribution after Krishnaiah, Hagis and Steinberg (1963). In this paper we derive its exact pdf (probability density function) in Theorem 2.1. Some properties of the distribution available in literature are reviewed in Kotz, Balakrishnan and Johnson (2000). We feel the properties need to be thoroughly investigated in the light of the joint pdf derived in this paper. There are also a number of other bivariate chi-square and gamma distributions excellently reviewed by Kotz, Balakrishnan and Johnson (2000).

We provide moments of the product two correlated chi-square random variables in Corollary 3.2 and Corollary 3.3. In case the correlation coefficient vanishes, the moments, as expected, coincide with that of the bivariate independent chi-squares random variables. An application is pinpointed in connection with the distribution of sample product moment correlation coefficient based on a bivariate *t*-distribution. The bivariate *t*-distribution has fatter tails compared to bivariate normal distribution and has found numerous applications to business data especially stock return data.

Ratios of two independent chi-squares are widely used in statistical tests of hypotheses. However, testing with the ratio of correlated chi-squares variables are rarely noticed. If $\sigma_1 = \sigma_2$, Finney (1938) derived the sampling distribution of the square root of the ratio of correlated chi-squares variables ($T = \sqrt{U/V}$) directly from the joint distribution of sample variances and correlation coefficient. He compared the variability of the measurements of standing height and stem length for different age groups of schoolboys by his method and by Hirschfeld (1937).

In corollary 3.7 and Corollary 3.8, we have provided moments of the ratio of two correlated chi-square variables. Further investigation is needed to utilize these moments to derive distributional properties of correlated chi-square variables by the inverse Mellin transform (Provost , 1986). It needs further investigation to derive the marginal and conditional distribution, and the distribution of the sum, product, ratio and similar quantities directly from the joint density function of the bivariate chi-square distribution with pdf in Theorem 2.1.

In what follows, for any nonnegative integer k , we will use the notation

$c_{(k)} = c(c+1)\cdots(c+k-1),\ c_{(0)} = 1,\ (c \neq 0)$.

2. The PDF of the Bivariate Chi-square Distribution

The pdf (probability density function) of the bivariate Wishart distribution originally derived by Fisher (1915) can be written as

$$f_1(a_{11},a_{22},a_{12}) = \frac{(1-\rho^2)^{-m/2}(\sigma_1\sigma_2)^{-m}}{2^m\sqrt{\pi}\,\Gamma(\frac{m}{2})\Gamma(\frac{m-1}{2})}\ (a_{11}a_{22}-a_{12}^2)^{(m-3)/2}$$
$$\times \exp\left(-\frac{a_{11}}{2(1-\rho^2)\sigma_1^2} - \frac{a_{22}}{2(1-\rho^2)\sigma_2^2} + \frac{\rho a_{12}}{(1-\rho^2)\sigma_1\sigma_2}\right)$$

(2.1)

where $a_{11} > 0,\ a_{22} > 0,\ -\sqrt{a_{11}a_{22}} < a_{12} < \sqrt{a_{22}a_{22}},\ m > 2,$ $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2,\ \ \sigma_{12} = \rho\sigma_1\sigma_2,\ \sigma_1 > 0, \sigma_2 > 0$ and ρ, $(-1<\rho<1)$, is the product moment correlation coefficient between X_{1j} and X_{2j} $(j = 1,2,\cdots,N)$ (Anderson, 2003, 123).

Since the correlation coefficient between X_{1j} and X_{2j} $(j = 1,2,\cdots,N)$ is ρ, the correlation between U_1 and U_2 is ρ^2 (Joarder, 2006). In the following theorem we derive the pdf of the correlated bivariate chi-square distribution.

Theorem 2.1 The random variables U_1 and U_2 are said to have a correlated bivariate chi-square distribution each with m degrees of freedom, if its pdf is given by

$$f(u_1,u_2) = \frac{2^{-(m+1)}(u_1u_2)^{(m-2)/2}\ e^{\frac{-(u_1+u_2)}{2(1-\rho^2)}}}{\sqrt{\pi}\ \Gamma(\frac{m}{2})(1-\rho^2)^{m/2}} \sum_{k=0}^{\infty}[1+(-1)^k]\left(\frac{\rho\sqrt{u_1u_2}}{1-\rho^2}\right)^k \frac{\Gamma\left(\frac{k+1}{2}\right)}{k!\,\Gamma\left(\frac{k+m}{2}\right)},$$

$m = N-1 > 2,\ -1<\rho<1$.

Proof. Under the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2, a_{12} = mrs_1s_2$ in (2.1) with Jacobian $J((a_{11},a_{22},a_{12}) \to (s_1^2,s_2^2,r)) = m^3 s_1 s_2$, the pdf of S_1^2, S_2^2 and R is given by

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$$f_2(s_1^2,s_2^2,r)=\left(\frac{m}{2\sigma_1\sigma_2}\right)^m\frac{(1-\rho^2)^{-m/2}}{\sqrt{\pi}\Gamma(\frac{m}{2})\Gamma(\frac{m-1}{2})}(1-r^2)^{(m-3)/2}(s_1s_2)^{m-2}$$
$$\times\exp\left(-\frac{ms_1^2}{2(1-\rho^2)\sigma_1^2}-\frac{ms_2^2}{2(1-\rho^2)\sigma_2^2}+\frac{m\rho rs_1s_2}{(1-\rho^2)\sigma_1\sigma_2}\right).$$

By making the transformation $ms_1^{\ 2}=\sigma_1^{\ 2}u_1,\ ms_2^{\ 2}=\sigma_2^{\ 2}u_2,$ keeping r intact, with Jacobian $J\left(s_1^{\ 2},s_2^{\ 2}\rightarrow u_1u_2\right)=\left(\sigma_1\sigma_2/m\right)^2$, we have

$$f_3(u_1,u_2,r)=\frac{(1-\rho^2)^{-m/2}(u_1u_2)^{(m-2)/2}(1-r^2)^{(m-3)/2}}{2^m\sqrt{\pi}\ \Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}\exp\left(-\frac{u_1+u_2}{2(1-\rho^2)}+\frac{\rho r\sqrt{u_1u_2}}{1-\rho^2}\right).$$

Then integrating out r, the joint pdf of U_1 and U_2 is given by

$$f_3(u_1,u_2)=\frac{(1-\rho^2)^{-m/2}(u_1u_2)^{(m-2)/2}}{2^m\sqrt{\pi}\ \Gamma(\frac{m}{2})\Gamma(\frac{m-1}{2})}\exp\left(-\frac{u_1+u_2}{2(1-\rho^2)}\right)$$
$$\times\int_{-1}^{1}\left(1-r^2\right)^{(m-3)/2}\exp\left(\frac{\rho r\sqrt{u_1u_2}}{1-\rho^2}\right)dr.$$

By expanding the last exponential term of the above density function we have

$$f_3(u_1,u_2)=\frac{(1-\rho^2)^{-m/2}(u_1u_2)^{(m-2)/2}}{2^m\sqrt{\pi}\ \Gamma(\frac{m}{2})\Gamma(\frac{m-1}{2})}\exp\left(-\frac{u_1+u_2}{2(1-\rho^2)}\right)$$
$$\times\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\rho r\sqrt{u_1u_2}}{1-\rho^2}\right)^k\left[\int_{-1}^{0}r^k(1-r^2)^{(m-3)/2}+\int_{0}^{1}r^k(1-r^2)^{(m-3)/2}\right]dr$$

which can be written as

$$f_3(u_1,u_2)=\frac{(1-\rho^2)^{-m/2}(u_1u_2)^{(m-2)/2}}{2^m\sqrt{\pi}\ \Gamma(\frac{m}{2})\Gamma(\frac{m-1}{2})}\exp\left(-\frac{u_1+u_2}{2(1-\rho^2)}\right)$$
$$\times\sum_{k=0}^{\infty}[(-1)^k+1]\left(\frac{\rho\sqrt{u_1u_2y}}{1-\rho^2}\right)^k\frac{1}{k!}\int_0^1(1-y)^{(m-3)/2}\left(\frac{1}{2}y^{-1/2}\right)dy.$$

By completing the beta integral of the above density function, we have

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$$f_3(u_1,u_2)=\frac{(1-\rho^2)^{-m/2}(u_1u_2)^{(m-2)/2}}{2^{m+1}\sqrt{\pi}\ \Gamma(\frac{m}{2})\Gamma(\frac{m-1}{2})}\quad \exp\left(-\frac{u_1+u_2}{2(1-\rho^2)}\right)$$
$$\times\sum_{k=0}^{\infty}[1+(-1)^k]\left(\frac{\rho\sqrt{u_1u_2y}}{1-\rho^2}\right)^k\frac{1}{k!}\frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{k+m}{2}\right)}$$

which simplifies to what we have the theorem.

Note that the pdf of the correlated bivariate chi-square distribution each with m degrees of freedom can be written as

$$f(u_1,u_2)=\frac{(u_1u_2)^{(m-2)/2}\ e^{\frac{-(u_1+u_2)}{2(1-\rho^2)}}}{2^m\sqrt{\pi}\ \Gamma(\frac{m}{2})(1-\rho^2)^{m/2}}\sum_{k=0}^{\infty}\left(\frac{\rho\sqrt{u_1u_2}}{1-\rho^2}\right)^k\frac{\Gamma\left(\frac{k+1}{2}\right)}{k!\Gamma\left(\frac{k+m}{2}\right)},$$

$m=N-1>2$, $-1<\rho<1$, and $k=0,2,4,\cdots$

In case $\rho=0$, the pdf in Theorem 2.1 becomes the product of that of the two independent chi-square random variables each with m degrees of freedom. The surfaces of the correlated bivariate chi-square distribution for various parameters are provided in the Appendix.

3. Moments of the Product and Ratio of Two Correlated Chi-square Random Variables

Theorem 3.1 The (a,b) th product moment $\mu'(a,b;\rho)=E\left(U_1^aU_2^b\right)$ is given by

$$E\left(U_1^aU_2^b\right)=\frac{2^{a+b-1}(1-\rho^2)^{a+b}}{L(m,\rho)}$$
$$\times\sum_{k=0}^{\infty}[1+(-1)^k]\frac{(2\rho)^k}{k!\Gamma\left(\frac{k+m}{2}\right)}\Gamma\left(\frac{k+m}{2}+a\right)\Gamma\left(\frac{k+m}{2}+b\right)\Gamma\left(\frac{k+1}{2}\right)$$

where $m>2\max(a,b)$, $-1<\rho<1$ and

$$L(m,\rho)=\sqrt{\pi}\ \Gamma(m/2)\left(1-\rho^2\right)^{-m/2}. \tag{3.1}$$

Proof. It follows from Theorem 2.1 that the (a,b)-th th product moment of the distribution of U_1 and U_2 is given by

$$E\left(U_1^a U_2^b\right)=\frac{(1-\rho^2)^{-m/2}}{2^m\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)}$$

$$\times\sum_{k=0}^{\infty}[1+(-1)^k]\left(\frac{\rho}{1-\rho^2}\right)^k\frac{1}{k!}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+m}{2}\right)}\int_0^\infty\int_0^\infty u_1^{\frac{m+k}{2}+a-1}\,u_2^{\frac{m+k}{2}+b-1}\,e^{-\frac{(u_1+u_2)}{2(1-\rho^2)}}du_1du_2.$$

The theorem then follows by having evaluated the above integrals.

Obviously $\mu'(a,b;\rho)=\mu'(b,a;\rho)=E\,(U_1^aU_2^b)$. If $\rho=0$, the product moment in Theorem 3.1 simplifies to

$$E\,(U_1^aU_2^b)=\frac{2^{a+b}}{\Gamma^2(m/2)}\Gamma\left(\tfrac{m}{2}+a\right)\Gamma\left(\tfrac{m}{2}+b\right)$$

which is the product of the a-th and b-th moments of two independent variables $U_1\sim\chi_m^2$ and $U_2\sim\chi_m^2$. In case a is a nonnegative integer, then

$$\Gamma\left(\tfrac{k+m}{2}+a\right)=\Gamma\left(\tfrac{k+m}{2}\right)\left(\tfrac{k+m}{2}\right)_{(a)}. \tag{3.2}$$

Corollary 3.1 If the integers (i) a and b are nonnegative, or, (ii) $a>0, b<0$ with $m>2b$, then the product moment in Theorem 3.1 can be written as

$$E\left(U_1^aU_2^b\right)=\frac{2^{a+b}(1-\rho^2)^{a+b+m/2}}{\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\rho^k\gamma_{k,m+2b}\left(\tfrac{k+m}{2}\right)_{(a)} \tag{3.3}$$

where

$$\gamma_{k,m}=[1+(-1)^k]\,\frac{2^{k-1}}{k!}\Gamma\left(\tfrac{k+1}{2}\right)\Gamma\left(\tfrac{k+m}{2}\right). \tag{3.4}$$

Plugging in $a=b$ in Corollary 3.1, we have the following corollary.

Corollary 3.2 For $m>2, -1<\rho<1$, the a th moment of $V=U_1U_2$ is given by

$$E\,(V^a)=\frac{4^a(1-\rho^2)^{2a+m/2}}{\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{k+m}{2}+a\right)}{\Gamma\left(\frac{k+m}{2}\right)}\gamma_{k,m+2a}\,\rho^k \tag{3.5}$$

where $\gamma_{k,m}$ is defined by (3.4).

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In case a is a nonnegative integer, then by the use of (3.2) in (3.5) we have the following:

$$E\left(V^{a}\right)=\frac{4^{a}(1-\rho^{2})^{2a+m/2}}{\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\left(\tfrac{k+m}{2}\right)_{(a)}\gamma_{k,m+2a}\,\rho^{k} \tag{3.6}$$

where $\gamma_{k,m}$ is defined by (3.4). The moment generating function of V at h is given by

$$M_{V}(h)=E\left(e^{hV}\right)=E\sum_{a=0}^{\infty}\frac{(hV)^{a}}{a!}=\sum_{a=0}^{\infty}\frac{h^{a}}{a!}E(V^{a})\,.$$

The moments of V are evaluated from Corollary 3.2 and given in the following corollary.

Corollary 3.3 For $m>2,-1<\rho<1$, the first four raw moments of V are given by

$(i)\ E(V)=m\left(m+2\rho^{2}\right),$

$(ii)\ E\left(V^{2}\right)=m(m+2)\left[8\rho^{4}+8(m+2)\rho^{2}+m(m+2)\right],$

$(iii)\ E(V^{3})=16\left(\frac{m}{2}\right)_{(3)}\left[24\rho^{6}+36(m+4)\rho^{4}+36\left(\frac{m}{2}+1\right)_{(2)}\rho^{2}+4\left(\frac{m}{2}\right)_{(3)}\right],$

$$(iv)\ E(V^{4})=256\left(\frac{m}{2}\right)_{(4)}\left[24\rho^{8}+48(m+6)\rho^{6}+72\left(\frac{m}{2}+2\right)_{(2)}\rho^{4}\right.$$
$$\left.+16\left(\frac{m}{2}+1\right)_{(3)}\rho^{2}+\left(\frac{m}{2}\right)_{(4)}\right].$$

Proof. Define $L^{(r)}(m,\rho)=\sum_{k=0}^{\infty}k^{(r)}\gamma_{k,m}\,\rho^{k}$, with

$L^{(0)}(m,\rho)=\sum_{k=0}^{\infty}\gamma_{k,m}\,\rho^{k}$ which is denoted by $L(m,\rho)$ (Joarder, 2006), and given in (3.1) in this paper. Then it can be checked that

$L^{(1)}(m+2,\rho)=(m+2)\rho^{2}(1-\rho^{2})L(m+2,\rho)$, and that

$$\sum_{k=0}^{\infty}\rho^k\gamma_{k,m+2}\ ((k+m)/2)$$
$$=\frac{1}{2}\left[L^{(1)}(m+2,\rho)+mL(m+2,\rho)\ \right]$$
$$=\frac{1}{2}\left[(m+2)\rho^2(1-\rho^2)^{-1}+m\right]L(m+2,\rho)$$
$$=\frac{\sqrt{\pi}}{2}\Gamma\left(\tfrac{m}{2}+1\right)\left[(m+2)\rho^2(1-\rho^2)^{-1}+m\right](1-\rho^2)^{-(m+2)/2}.$$

By plugging the above in to (3.6) with $a=1$, we have $E(V)=(1-\rho^2)[m+(m+2)\rho^2\ \ (1-\rho^2)]$ which simplifies to (i).

(ii) Using the following identities

$L^{(1)}(m+4,\rho)=(m+4)\rho^2(1-\rho^2)L(m+4,\rho)$ and

$L^{(2)}(m+4,\rho)=[(m+4)(m+5)\rho^4+(m+4)\rho^2](1-\rho^2)^{-2}L(m+4,\rho)$ in the expression

$$\sum_{k=0}^{\infty}\rho^k\gamma_{k,m+4}\ ((k+m)/2)_{(2)}$$
$$=\frac{1}{4}\left[L^{(2)}(m+4,\rho)\ +(2m+3)L^{(1)}(m+4,\rho)+m(m+2)L(m+4,\rho)\right]$$

and plugging in the value of $L(m+4,\rho)$ in (3.6) with $a=2$, we get (ii).

Similarly (iii) and (iv) can be proved.

In has been mentioned at the end of Theorem 2.1 that if $\rho=0$, then $U_1\sim\chi^2_m$ and $U_2\sim\chi^2_m$ will be independent. That is V will be the product of two independent chi-square random variables each with m degrees of freedom and evidently the resulting moments are in agreement with that situation. For example if $\rho=0$, from Corollary 3.3 (iv) we have $E(V^4)=[m(m+2)(m+4)(m+6)]^2$ which is the product of the fourth moments of two independent chi-squares each having m degrees of freedom , i.e., $E(V^4)=E(U_1^4)E(U_2^4)$.

Since the geometric mean of the chi-square variables are important in many applications, we define $G=\left(U_1U_2\right)^{1/2}$, the geometric mean of U_1 and U_2, and provide its moments below.

Corollary 3.4 For $m>2, -1<\rho<1$, the first four raw moments of G are given by

$$(i)\ E\left(G\right)=\frac{2(1-\rho^2)^{(m+2)/2}}{\sqrt{\pi}\ \Gamma\left(\frac{m}{2}\right)}\ \sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{k+m+1}{2}\right)}{\Gamma\left(\frac{k+m}{2}\right)}\gamma_{k,m+1}\ \rho^k,$$

$$(ii)\ E\left(G^2\right)=m\left(m+2\rho^2\right),$$

$$(iii)\ E\left(G^3\right)=\frac{8(1-\rho^2)^{(m+6)/2}}{\sqrt{\pi}\ \Gamma\left(\frac{m}{2}\right)}\ \sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{k+m+3}{2}\right)}{\Gamma\left(\frac{k+m}{2}\right)}\gamma_{k,m+3}\ \rho^k,$$

$$(iv)\ E\left(G^4\right)=m(m+2)\left[8\rho^4+8(m+2)\rho^2+m(m+2)\right].$$

Since $G=\sqrt{U_1U_2}=\sqrt{V}$, the moment generating function of G at h is given by $E(e^{hG})=\sum_{a=0}^{\infty}\frac{h^a}{a!}E(G^a)=\sum_{a=0}^{\infty}\frac{h^a}{a!}E(V^{a/2})$. Then by (3.4) we have the following corollary.

Corollary 3.5 If a is nonnegative with $m>2a$, then for $-1<\rho<1$, the moment generating function of G at h is given by

$$M_G(h)=\frac{(1-\rho^2)^{m/2}}{\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)}\sum_{a=0}^{\infty}\frac{(2h)^a(1-\rho^2)^a}{a!}\ \sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{k+m+a}{2}\right)}{\Gamma\left(\frac{k+m}{2}\right)}\gamma_{k,m+a}\ \rho^k,$$

where $\gamma_{k,m}$ is defined by (3.4). The following results from Corollary 3.2 and Corollary 3.5 by putting $\rho=0$.

Corollary 3.6 Let $G=\sqrt{U_1U_2}=\sqrt{V}$ be the geometric mean of two independent chi-square random variables U_1 and U_2 each with m degrees of freedom. Then the a-th moment and the moment generating function of G at h are given by

$$E(G^a)=\frac{2^a\ \Gamma^2\left(\frac{m+a}{2}\right)}{\Gamma^2\left(\frac{m}{2}\right)}\ \text{and}$$

$$M_G(h)=\sum_{a=0}^{\infty}\frac{(2h)^a}{a!}\frac{\Gamma^2\left(\frac{m+a}{2}\right)}{\Gamma^2\left(\frac{m}{2}\right)}$$

respectively (Joarder, 2007).

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Let $W = U_1/U_2$, the ratio of two correlated chi-square variables U_1 and U_2 that have probability density function in Theorem 2.1. Then the following result follows from Theorem 3.1.

Corollary 3.7 If a is nonnegative with $m > 2a$, then for $-1<\rho<1$, the a-th moment of W is given by

$$E(W^a) = \frac{(1-\rho^2)^{m/2}}{\sqrt{\pi}\,\Gamma(m/2)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{k+m}{2}+a\right)}{\Gamma\left(\frac{k+m}{2}\right)} \gamma_{k,m-2a}\, \rho^k ,$$

where $\gamma_{k,m-2a}$ is defined by (3.4). In case a is a nonnegative integer, we have

$$E(W^a) = \frac{(1-\rho^2)^{m/2}}{\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)} \sum_{k=0}^{\infty} \left(\tfrac{k+m}{2}\right)_{(a)} \gamma_{k,m-2a}\, \rho^k ,\ m > 2a,\ -1<\rho<1.$$

The moments of W up to 4th order are calculated below from Corollary 3.7. The derivation is similar to that of Corollary 3.3.

Corollary 3.8 For $m > 8, -1<\rho<1$, the first four moments of W are given below:

$$(i)\ E(W) = \frac{m-2\rho^2}{m-2},\quad m>2,$$

$$(ii)\ E\left(W^2\right) = \frac{1}{(m-2)(m-4)}\left(24\rho^4 - 8(m+2)\rho^2 + m(m+2)\right),\ m>4,$$

$$(iii)\ E(W^3) = \frac{\Gamma\left(\frac{m}{2}-3\right)}{8\Gamma\left(\frac{m}{2}\right)}\left[-480\rho^6 + 144(m+4)\rho^4 - 72\left(\tfrac{m}{2}+1\right)_{(2)}\rho^2 + 8\left(\tfrac{m}{2}\right)_{(3)}\right],$$

$$m>6,$$

$$(iv)\ E(W^4) = \frac{\Gamma\left(\frac{m}{2}-4\right)}{16\Gamma\left(\frac{m}{2}\right)}\Big[13440\rho^8 - 3840(m+6)\rho^6 + 1920\left(\tfrac{m}{2}+2\right)_{(2)}\rho^4$$

$$-256\left(\tfrac{m}{2}+1\right)_{(3)}\rho^2 + 8\left(\tfrac{m}{2}\right)_{(3)}\Big],\ m>8.$$

In case $\rho = 0$, then W will be the ratio of two independent chi-square variables each with the same degrees of freedom, and the above moments will be simply

$$E(W^a) = \frac{\left(\frac{m}{2}\right)_{(a)}}{\left(\frac{m}{2}-a\right)_{(a)}},\quad m>2a$$

which are evidently in agreement with the resulting situation of independence.

Corollary 3.9 If a is an integer with $m > 2a$, then for $-1 < \rho < 1$, the moment generating function of W at h is given by

$$M_W(h) = \frac{(1-\rho^2)^{m/2}}{\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)} \sum_{a=0}^{\infty} \frac{h^a}{a!} \sum_{k=0}^{\infty} \left(\tfrac{k+m}{2}\right)_{(a)} \gamma_{k,m-2a}\, \rho^k$$

where $L(m,\rho)$ is defined in Theorem 3.1**.**

4. The Distribution of the Correlation Coefficient Based on Bivariate T-Population

The lengthy proof of the distribution of sample correlation coefficient based on a bivariate normal distribution by Fisher (1915) has been made shorter and elegant by Joarder (2007). Further he derived the distribution of correlation coefficient for bivariate *t*-distribution along Fisher (1915). It is overviewed here to show the application of the moments of Section 3.

It is known that the distribution of correlation coefficient is robust under violation of normality at least in the bivariate elliptical class of distributions (Fand and Anderson, 1990, 10, Fang and Zhang, 1990, 137 or Ali and Joarder, 1991). However a direct derivation by Joarder (2007) provides more insight for those who have been recently using *t*-distributions as parent populations.

Theorem 4.1 The probability density function of the correlation coefficient R based on a bivariate *t*-population is given by

$$(i)\, h(r) = \frac{2^{m-2}\,\Gamma^2\left(\frac{m}{2}\right)(1-\rho^2)^{m/2}}{\pi\,\Gamma(m-1)} (1-r^2)^{(m-3)/2} M_G(\rho r)$$

$$(ii)\, h(r) = \frac{2^{m-2}(1-\rho^2)^{m/2}}{\pi\,\Gamma(m-1)} (1-r^2)^{(m-3)/2} \sum_{k=0}^{\infty} \frac{(2\rho r)^k}{k!}\,\Gamma^2\left(\tfrac{m+k}{2}\right),$$

$-1 < r < 1$ (cf. Muirhead, 1982, 154) where $m > 2, -1 < \rho < 1$ and $M_G(\rho r)$ is defined in Corollary 3.6.

Since part (ii) is well known to be the distribution of the correlation coefficient for bivariate normal population, it proves the robustness of the distribution or of tests on correlation coefficient. Thus the assumption of bivariate normality under which many tests on correlation coefficient are developed can be relaxed to a broader class of bivariate *t*-distribution. Interested readers may go through Muddapur (1988) for some exciting exact tests on correlation coefficient. Further the readers are referred to Joarder and Ahmed (1996), Billah and Saleh (2000), Kibria (2004), Kibria and Saleh (2004), Kotz and Nadarajah (2004) and the references therein for applications of *t*-distribution.

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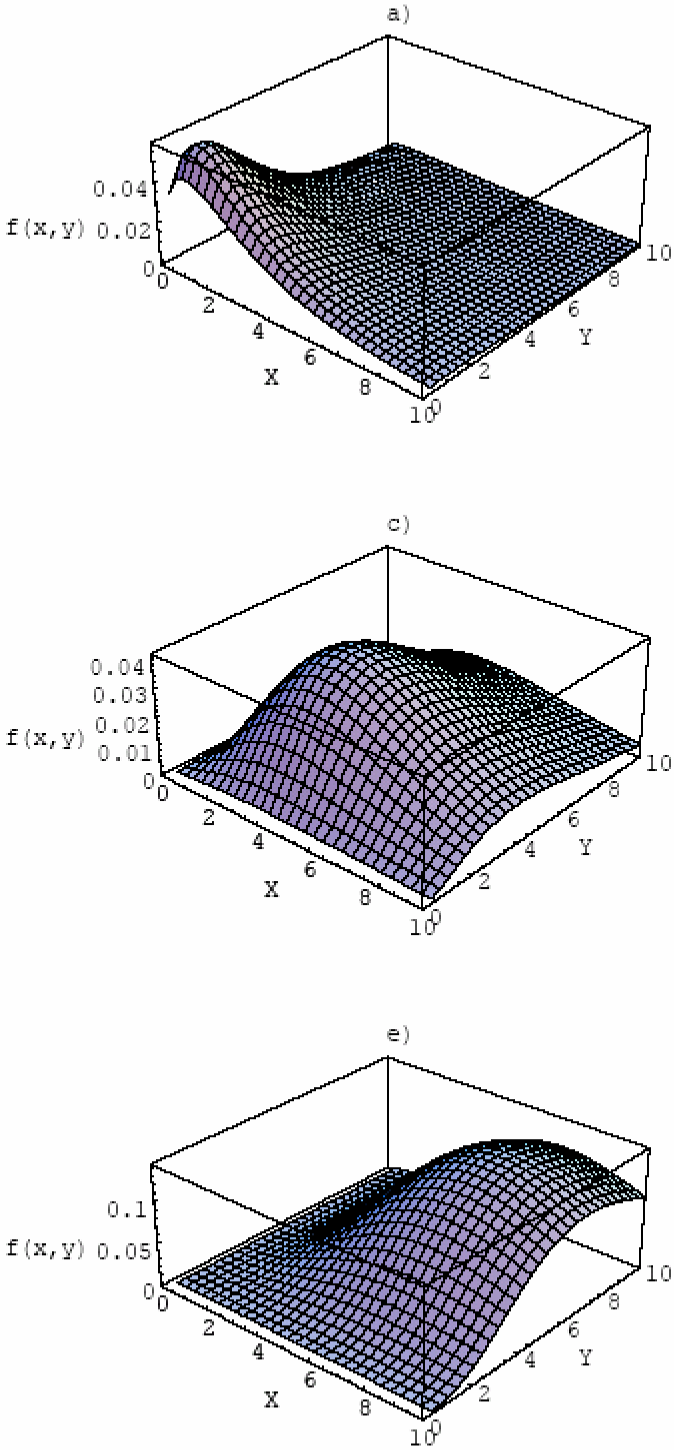


a)
f(x,y)
0.04
0.02
0
X
Y
c)
f(x,y)
0.04
0.03
0.02
0.01
0
X
Y
e)
f(x,y)
0.1
0.05
0
X
Y

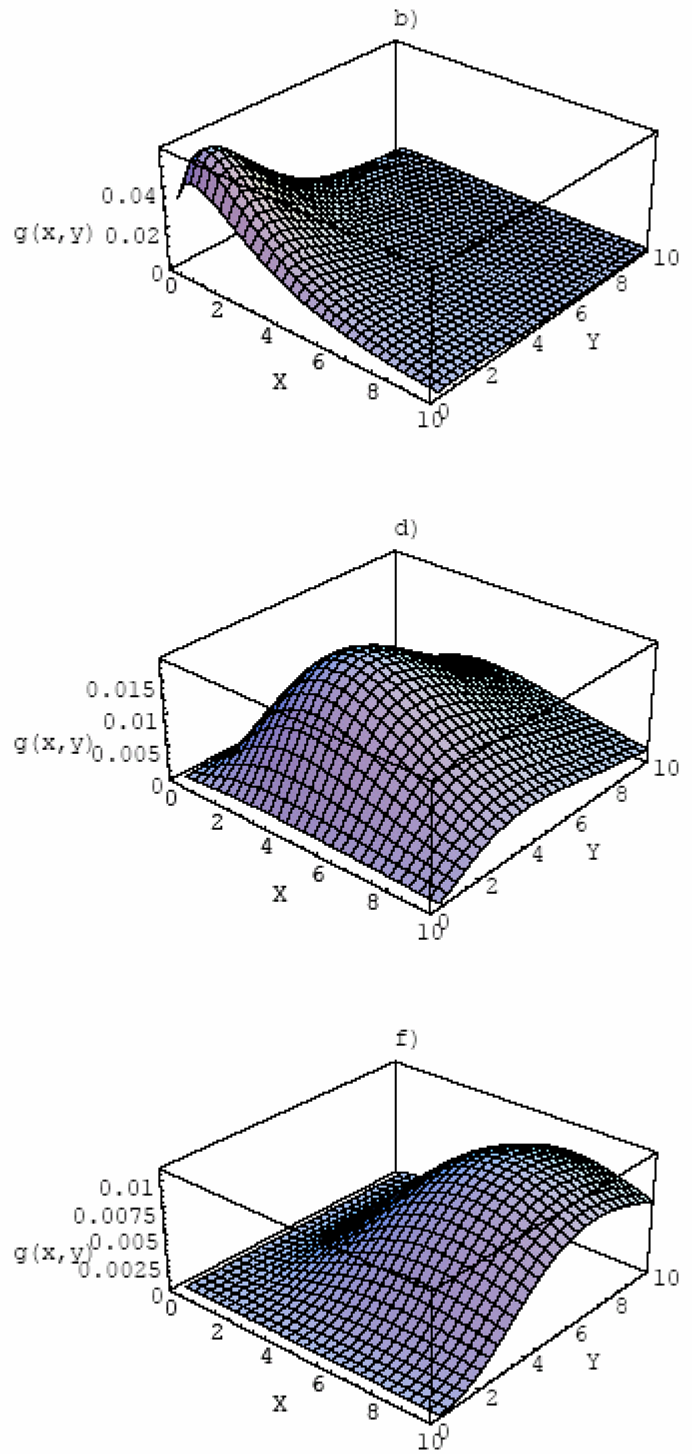
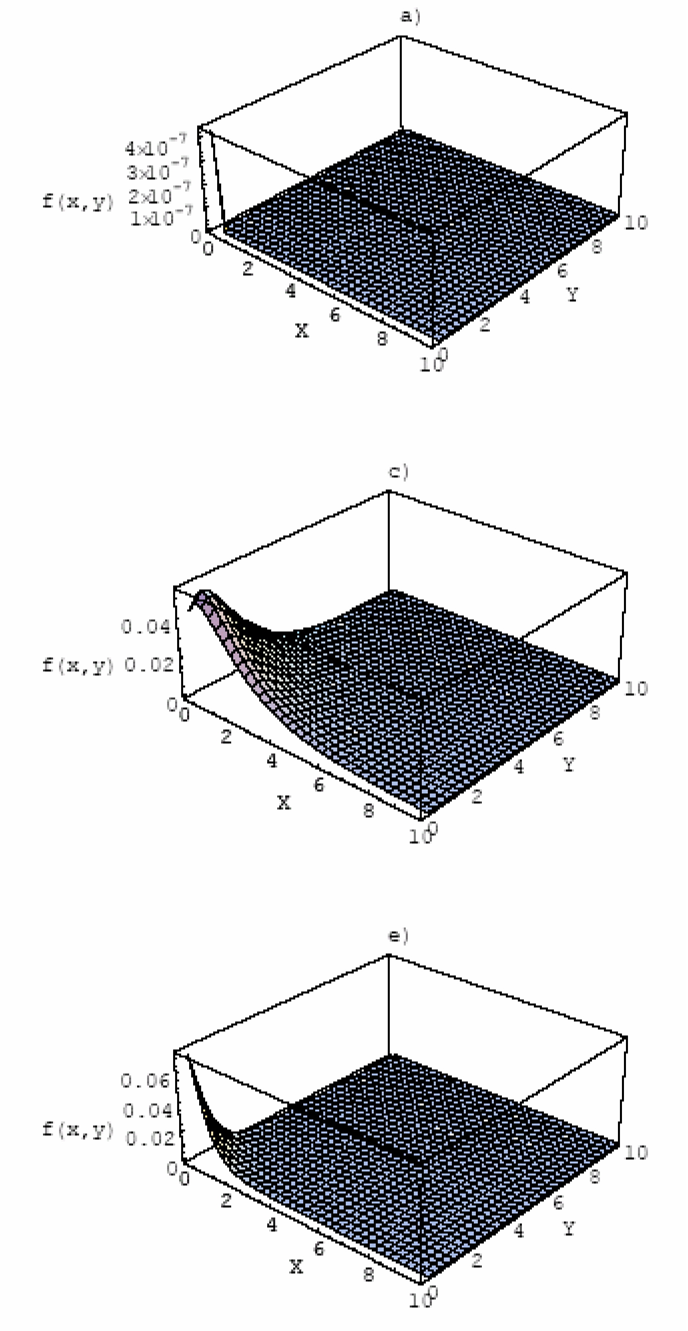


Figure 1. Correlated bivariate Chi-square density *f(x,y)*surface with $\rho = 0$ versus independent bivariate Chi-square density *g(x,y)* surfaces at 3 (Graph a and Graph b), 6 (Graph c and Graph d), and 9 (Graph e and Graph f) degrees of freedom.


a)
f(x,y)
4×10^{-7}
3×10^{-7}
2×10^{-7}
1×10^{-7}
0
X
Y
0 2 4 6 8 10
c)
f(x,y)
0.04
0.02
0
X
Y
0 2 4 6 8 10
e)
f(x,y)
0.06
0.04
0.02
0
X
Y
0 2 4 6 8 10

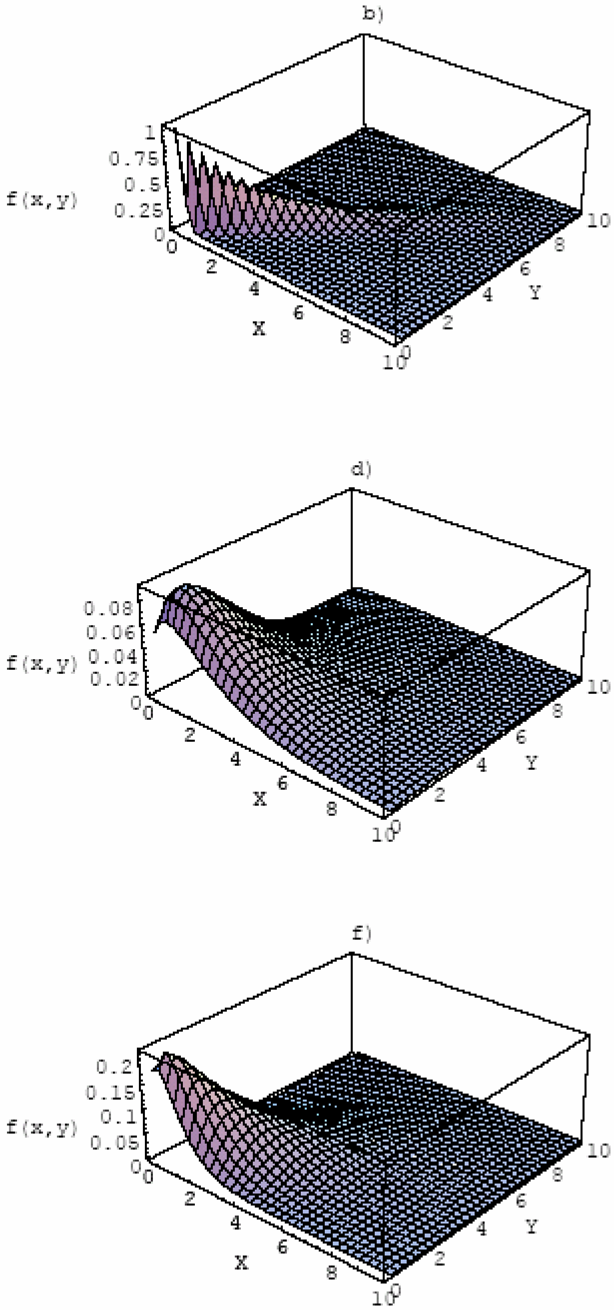


Figure 2. Correlated bivariate Chi-square density *f(x,y)* surface with 3 degrees of freedom at different values of ρ : Graph a (ρ = -0.99), Graph b (ρ = 0.99), Graph c (ρ = -0.5), Graph d (ρ = 0.5), Graph e (ρ = -0.85), and Graph f (ρ = 0.85).