Journal of Probability and Statistical Science
6(1), 115-123, Feb. 2008

On the Distributions of Norms of Spherical Distributions

Anwar H. Joarder, Walid S. Al-Sabah, and M. H. Omar
King Fahd University of Petroleum and Minerals

ABSTRACT This paper reviews some important results dealing with the norms of distributions of several members of spherical distributions in an accessible manner. Moments of the norms of some spherical distributions are discussed. Then they have been used to derive covariance matrices and higher order standardized moments.

(AMS classification: 60-01, 60E10)

Keywords Spherical distributions; Multivariate normal distribution; Uniform spherical distributions on or inside spheres; Multivariate t distribution; Multivariate Pearson type II distribution; Standardized moments; Covariance matrix.

1. Introduction

The distribution of the norm of a spherical distribution is known in its general form. We specialize it to several members of spherical distributions, namely, multivariate normal distribution, uniform spherical distributions on or inside $p(>2)$-dimensional spheres, multivariate t distribution and multivariate Pearson type II distribution. Some functions of norms are found to have standard distributions. Moments of norms of some spherical distributions are discussed. They are then used to derive covariance matrices and standardized moments. The standardized moments are moments of Mahalanobis distance. It may pointed out that direct derivation of the above quantities are sometimes intractable.

A p-dimensional random variable $\boldsymbol{Z}$ is said to have a spherical distribution if its probability density function (pdf) is given by

$$f(\mathbf{z}) = g(\mathbf{z}'\mathbf{z})\,. \tag{1.1}$$

□ Received March 2006, revised January/September 2007, in final form October 2007.

□ Authors of this article are affiliated with the Department of Mathematical Sciences at King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia. Corresponding author: Anwar H. Joarder, email: anwarj@kfupm.edu.sa.

 ISSN 1726-3328

We refer to Muirhead [14] for a decent introduction to the spherical and elliptical distributions. Much of the theoretical developments are available in Fang and Anderson [3] and Fang *et al.* [4]. For applications of such distributions we refer to Lange *et al.* [13], Billah and Saleh [1], Kibria and Saleh [9], Kibria [10], Kotz and Nadarajah [12] and Kibria [11] and the references therein.

Let $R = (\boldsymbol{Z}'\boldsymbol{Z})^{1/2} = \|\boldsymbol{Z}\|$ be the the norm of the distribution of $\boldsymbol{Z}$. The following theorem about the distribution of R is originally due to Goldman [5] and Goldman [6].

Theorem 1.1 *Let* $\boldsymbol{z}' = \left(z_1, z_2, \ldots, z_p\right)$, $p \geq 2$ *with p.d.f.* $f(\boldsymbol{z}) = g(\boldsymbol{z}'\boldsymbol{z})$. *Consider the transformation from rectangular coordinate to polar coordinate for* $\boldsymbol{z}$

$$
\begin{aligned}
z_1 &= r\cos\theta_1, \qquad (1.2)\\
z_i &= r\left(\prod_{j=1}^{i-1}\sin\theta_j\right)\cos\theta_i, \ \text{for } i = 2,3,\ldots,p-1,\\
z_p &= r\sin\theta_1\ \sin\theta_2\ \ldots\ \sin\theta_{p-2}\ \sin\theta_{p-1}
\end{aligned}
$$

where $0 \leq r < \infty$; $0 \leq \theta_i < \pi$, $i = 1,2,\ldots,p-2$; $0 \leq \theta_{p-1} < 2\pi$. *Then the random variables* $R, \Theta_1, \Theta_2, \ldots, \Theta_{p-1}$ *are independent and have the following probability density functions respectively:*

$$
\begin{aligned}
h(r) &= \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1}\, g(r^2),\ \ 0 \leq r < \infty \qquad (1.3)\\
u(\theta_i) &= \frac{1}{B\left(\frac{1}{2}, \frac{p-i}{2}\right)}\ \sin^{p-i-1}\theta_i,\ \ 0 \leq \theta_i < \pi,\quad i = 1,2,\ldots,p-2,\\
v\left(\theta_{p-1}\right) &= \frac{1}{2\pi},\qquad 0 \leq \theta_{p-1} < 2\pi
\end{aligned}
$$

where

$$B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \qquad (1.4)$$

is the usual beta function (Fang et al. [4], 37). Conversely, if $R, \Theta_1, \Theta_2, \ldots, \Theta_{p-1}$ *are independent and have probability density functions given by (1.3) and* $\boldsymbol{z}$ *is defined by (1.2), then* $\boldsymbol{Z}$ *has a spherical distribution.*

The distribution of $Y = R^2$ is obviously

$$w(y) = \frac{\pi^{p/2}}{\Gamma(p/2)}\ y^{p/2-1}\, g(y). \qquad (1.5)$$

The k-th moment of R is given by

$$E(R^k) = \int_0^\infty r^k\ h(r)dr = \int_0^\infty \frac{2\pi^{p/2}}{\Gamma(p/2)}\ r^{p+k-1}\, g(r^2)dr. \qquad (1.6)$$

In Section 2, we provide the distribution of norm or some functions of norm and derive moments for norms of spherical distributions. In Section 3, we find product moments of some spherical distributions. In Section 4, we demonstrate how to derive covariance matrices of some elliptical distributions of $\boldsymbol{X} = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$ with the help of the moments of norms of the elliptical distributions. Finally in Section 5, we derive standardized moments of some elliptical distributions.

2. Distributions of the Norms of Spherical Distributions

In this section, we sketch distributions of some functions of norm of different spherical distributions. Most of these results are originally due to Fang *et al.* [4] who emphasized the so-called characteristic generator rather than the usual probability density function. The purpose of this section is to sketch accessible proofs of the well-known results on the distribution of norms.

Theorem 2.1 *Let $\boldsymbol{Z}$ have the spherical multivariate normal distribution given by*

$$f(\boldsymbol{z}) = g(\boldsymbol{z}'\boldsymbol{z}) = (2\pi)^{-p/2}\, exp\left(-\frac{1}{2}\, z'z\right). \tag{2.1}$$

Then $R^2 = (\boldsymbol{Z}'\boldsymbol{Z})$ has the usual chi-square distribution with p degrees of freedom and that the $k/2$-th moment of R^2 is given by

$$E\left(R^k\right) = 2^{k/2}\,\frac{\Gamma((p+k)/2)}{\Gamma(p/2)}. \tag{2.2}$$

Proof. The theorem follows by applying (2.1) to (1.5) and (1.6). □

In particular $E\left(R^2\right) = p$.

Theorem 2.2 *Let $\boldsymbol{Z}$ have the uniform distribution on a p-dimensional unit sphere with p.d.f.*

$$f(\boldsymbol{z}) = g(\boldsymbol{z}'\boldsymbol{z}) = \frac{I_T(\boldsymbol{z})}{S(p,1)} \tag{2.3}$$

where $S(p,1) = \dfrac{2\pi^{p/2}}{\Gamma(p/2)}$ is the surface area of a p-dimensional unit sphere and $I_T(\boldsymbol{z})$ is the indicator function of the set $T = \{\boldsymbol{z} : \boldsymbol{z}'\boldsymbol{z} = 1\}$. Then $P(R=1) = 1$ and $E\left(R^k\right) = 1$.

Proof. For unit sphere, $r = 1$, i.e. R is a degenerate random variable with all the mass at $r = 1$. Symbolically $P(R = 1) = 1$. Obviously $E(R^k) = 1$. Alternatively, by applying (2.3) to (1.3), we have

$$h(r) = \frac{2\pi^{p/2}}{\Gamma(p/2)} r^{p-1}\left(\frac{\Gamma(p/2)}{2\pi^{p/2}}\right) = r^{p-1} \tag{2.4}$$

i.e. $h(1) = 1$. □

Theorem 2.3 *Let $\boldsymbol{Z}$ have the uniform distribution inside a p-dimensional unit sphere with p.d.f.*

$$f(\boldsymbol{z}) = g(\boldsymbol{z}'\boldsymbol{z}) = \frac{I_T(z)}{V(p,1)} \tag{2.5}$$

where

$$V(p,1) = \frac{\pi^{p/2}}{\Gamma(p/2+1)}$$

is the volume of a p-dimensional unit sphere, and $I_T(\boldsymbol{z})$ is the indicator function of the set $T = \{\boldsymbol{z} : \boldsymbol{z}'\boldsymbol{z} \le 1\}$. Then R has a Beta distribution with parameter p and 1, i.e. $R \sim Beta(p,1)$, and that

$$E\left(R^k\right) = \frac{B(k+p,1)}{B(p,1)} = \frac{p}{p+k} \tag{2.6}$$

where $B(a,b)$ is the usual beta function defined in (1.4).

Proof. By applying (2.5) to (1.3), the p.d.f. of R is given by

$$h(r) = \frac{2\pi^{p/2}}{\Gamma(p/2)}\ r^{p-1}\left(\frac{\Gamma(p/2+1)}{\pi^{p/2}}\right) = p\ r^{p-1},\quad p \ge 3.$$

Theorem 2.4 *Let $\boldsymbol{Z}$ have the multivariate t-distribution with ν degrees of freedom. Then the p.d.f. of Z is given by*

$$f(\boldsymbol{z}) = g(\boldsymbol{z}'\boldsymbol{z}) = \frac{1}{C(\nu,p)\pi^{p/2}}\left(1+\frac{\boldsymbol{z}'\boldsymbol{z}}{\nu}\right)^{-(\nu+p)/2} \tag{2.7}$$

where $C(\nu,p)$ is given by $C(\nu,p) = \nu^{p/2}\Gamma(\nu/2)/\Gamma((\nu+p)/2)$. Then $R^2/p \sim F(p,\nu)$ and that

$$E\left(R^k\right) = \nu^{k/2}\ \frac{\Gamma((p+k)/2)\Gamma((\nu-k)/2)}{\Gamma(p/2)\,\Gamma(\nu/2)}, \nu > k \tag{2.8}$$

(cf. Fang *et al.* [4], 22).

Proof. The results follow by applying (2.7) to (1.5) and (1.6). ☐

In particular,

$$E\left(R^2\right) = p\ \frac{\nu}{\nu-2},\quad \nu > 2 \tag{2.9}$$

$$V\left(R^2\right) = 2p\ \frac{\nu^2(p+\nu-2)}{(\nu-2)^2(\nu-4)},\ \nu > 4.$$

We remark that as $\nu \to \infty$, $E\left(R^2\right) = p$ and $V\left(R^2\right) = 2p$ which are the mean and variance respectively of $R^2 \sim \chi^2_p$.

Theorem 2.5 *Let $\boldsymbol{Z}$ have the Multivariate Pearson Type II distribution with p.d.f.*

$$f(\boldsymbol{z}) = g(\boldsymbol{z}'\boldsymbol{z}) = A(\alpha,p)(1-\boldsymbol{z}'\boldsymbol{z})^{(\alpha-p)/2},\ 0 < \boldsymbol{z}'\boldsymbol{z} < 1 \tag{2.10}$$

where

$$A(\alpha, p) = \frac{\Gamma(\alpha/2 + 1)}{\pi^{p/2}\,\Gamma((\alpha - p)/2 + 1)}.$$

Then

$$R^2 \sim Beta\left(\frac{p}{2}, \frac{\alpha - p}{2} + 1\right)$$

and

$$E\left(R^k\right) = \frac{B\left(\frac{k+p}{2}, \frac{\alpha-p}{2} + 1\right)}{B\left(\frac{p}{2}, \frac{\alpha-p}{2} + 1\right)} \tag{2.11}$$

(cf. Fang et al. [4], 89) where $B(a, b)$ is the usual beta function defined by (1.4).

Proof. Applying (2.10) to (1.5) and (1.6), we have the theorem. □

In particular $E\left(R^2\right) = p/(\alpha + 2)$.

3. Product Moments and Norms

In this section we discuss mixed moments of spherical distributions, most of which are discussed in Fang *et al.* [4].

Theorem 3.1 *Let $\mathbf{Z} = (z_1, z_2, \ldots z_p)$ have a spherical distribution given by (1.1). Then for any integers $k_1, k_2, \ldots k_p$ where $k = \sum_{i=1}^{p} k_i$, the product moment is given by*

$$E\left(\prod_{i=1}^{p} Z_i^{k_i}\right) = \begin{cases} 0 \text{ if at least one } k_i (i = 1, 2, \ldots p) \text{ is odd} \\ E\left(R^k\right) \dfrac{\Gamma(p/2)}{2^k \Gamma((k+p)/2)} \displaystyle\prod_{i=1}^{p} \frac{k_i!}{(k_i/2)!} \text{ if all } k_i's (i = 1, 2, \ldots p) \text{ are even.} \end{cases}$$

1. Spherical normal distribution (see equation (2.1))

$E\left(\prod_{i=1}^{p} Z_i^{k_i}\right) = 0$ if any $k_i (i = 1, 2, \ldots p)$ is odd. However if all $k_i's$ are even, then applying (2.2) to Theorem 3.1 we have

$$E\left(\prod_{i=1}^{p} Z_i^{k_i}\right) = 2^{-k/2} \prod_{i=1}^{p} \frac{k_i!}{(k_i/2)!}.$$

2. Uniform distribution on a p-dimensional unit sphere (see equation (2.3)).

If all $k_i's$ are even then applying Theorem 2.2 to Theorem 3.1 we have

$$E\left(\prod_{i=1}^{p} Z_i^{k_i}\right) = \frac{\Gamma(p/2)}{2^k \Gamma((k+p)/2)} \prod_{i=1}^{p} \frac{k_i!}{(k_i/2)!},$$

where $k = \sum_{i=1}^{p} k_i$(cf. Fang *et al.* [4], 72).

3. Uniform distribution inside a p-dimensional unit sphere (see equation (2.5))

If all k_i's are even then by applying (2.6) to Theorem 3.1 we have

$$E\left(\prod_{i=1}^{p} \boldsymbol{Z}_i^{k_i}\right) = \frac{p}{p+k}\ \frac{\Gamma(p/2)}{2^k\Gamma((k+p)/2)} \prod_{i=1}^{p} \frac{k_i!}{(k_i/2)!},$$

where $k = \sum_{i=1}^{p} k_i$ (cf. Fang *et al.* [4], 75).

4. Multivariate t-distribution (see equation (2.7)).

If all $k_i's$ are even then, by applying (2.8) to Theorem 3.1 we have

$$E\left(\prod_{i=1}^{p} \boldsymbol{Z}_i^{k_i}\right) = \frac{\nu^{k/2}}{2^k} \frac{\Gamma((\nu-k)/2)}{\Gamma(\nu/2)} \prod_{i=1}^{p} \frac{k_i!}{(k_i/2)!},$$

where $k = \sum_{i=1}^{p} k_i$(Fang *et al.* [4], 88). The above was also derived by Joarder [7] laboriously by differentiating the characteristic function of the multivariate t-distribution.

5. Multivariate Pearson type II distribution (see equation (2.10))

If all $k_i's$ are even, then by applying (2.11) to Theorem 3.1 we have

$$E\left(\prod_{i=1}^{p} \boldsymbol{Z}_i^{k_i}\right) = \frac{\Gamma(\alpha/2+1)}{2^k\Gamma((k+\alpha)/2+1)} \prod_{i=1}^{p} \frac{k_i!}{(k_i/2)!}$$

where $k = \sum_{i=1}^{p} k_i$.

4. Covariance Matrices of Some Elliptical Distributions

Consider the elliptical random variable $\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$ where $\boldsymbol{Z}$ has the p.d.f. given by (1.1). It is well known (Cambanis *et al.* [2]) that the covariance matrix of $\boldsymbol{X}$ is given by $Cov(\boldsymbol{X}) = -2\psi'_{\boldsymbol{X}}(\boldsymbol{0})\boldsymbol{\Sigma}$ where $\psi_{\boldsymbol{X}}(t) = \exp(i\boldsymbol{t}'\boldsymbol{\theta})\psi(||\boldsymbol{\Sigma}^{1/2}t||)$ is the characteristic function of $\boldsymbol{X}$.

Since most elliptical distributions do not have closed form for characteristic functions, an easy way out is to exploit their stochastic decomposition, that is $\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{X}-\boldsymbol{\theta}) = \boldsymbol{Z} = R\boldsymbol{U}$ where $R = (\boldsymbol{Z}'\boldsymbol{Z})^{1/2}$ is independent of $\boldsymbol{U}$ and the random variable $\boldsymbol{U}$ is uniformly distributed on the surface of unit sphere in $\mathcal{R}^p$.

For any elliptical random variable $\boldsymbol{X}$ where $\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$ with $\boldsymbol{Z}$ having the p.d.f. (1.1), it is well known that

$$\boldsymbol{\Lambda} = Cov(\boldsymbol{X}) = \frac{1}{p}E\left(R^2\right)\boldsymbol{\Sigma} \tag{4.1}$$

(Cambanis *et al.* [2] or Joarder [7]). In this section, we outline how the covariance matrix of elliptical distributions can be derived by the above result.

(i) Multivariate normal distribution

Let $\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$ where $\boldsymbol{Z}$ has the p.d.f. given by (2.1). Then by applying (2.2) to (4.1), we have $Cov(\boldsymbol{X}) = \frac{1}{p}(p\boldsymbol{\Sigma}) = \boldsymbol{\Sigma}$.

(ii) Uniform distribution on a p-dimensional unit ellipsoid.

Let $\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$ where $\boldsymbol{Z}$ has the p.d.f. given by (2.3). Then applying Theorem 2.2 to (4.1), we have $Cov(X) = \frac{1}{p}(1)\boldsymbol{\Sigma} = \frac{1}{p}\boldsymbol{\Sigma}$.

(iii) Uniform distribution inside a p-dimensional unit ellipsoid.

Let $\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$ where $\boldsymbol{Z}$ has the p.d.f. given by (2.5). Then applying (2.6) to (4.1), it follows that $Cov(\boldsymbol{X}) = \frac{1}{p}\left(\frac{p}{p+2}\right)\boldsymbol{\Sigma} = \frac{1}{p+2}\boldsymbol{\Sigma}$.

(iv) Multivariate t-distribution

Let $\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$ where $\boldsymbol{Z}$ has the p.d.f. given by (2.7). Then applying (2.8) to (4.1), we have $Cov(\boldsymbol{X}) = \frac{1}{p}\left(\frac{\nu p}{\nu-2}\right)\boldsymbol{\Sigma} = \frac{\nu}{\nu-2}\boldsymbol{\Sigma}$.

(v) Multivariate Pearson type II distribution

Let $\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$ where $\boldsymbol{Z}$ has the p.d.f. given by (2.10). Then applying (2.11) to (4.1), we have $Cov(\boldsymbol{X}) = \frac{1}{p}\left(\frac{p}{\alpha+2}\right)\boldsymbol{\Sigma} = \frac{1}{\alpha+2}\boldsymbol{\Sigma}$.

5. Standardized Moments of Some Elliptical Distributions

Consider the elliptical random variable $\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{Z}$ where $\boldsymbol{Z}$ has the p.d.f. given by (1.1). The Mahalanobis distance is defined by $Q = (\boldsymbol{X}-\boldsymbol{\theta})'\boldsymbol{\Lambda}^{-1}(\boldsymbol{X}-\boldsymbol{\theta})$ where $\boldsymbol{\Lambda} = E(\boldsymbol{X}-\boldsymbol{\theta})(\boldsymbol{X}-\boldsymbol{\theta})'$ is the covariance of the elliptical distribution of X. Then it follows from (4.1) that

$$Q = (\boldsymbol{X}-\boldsymbol{\theta})'\left(\frac{1}{p}E\left(R^2\right)\boldsymbol{\Sigma}\right)^{-1}(\boldsymbol{X}-\boldsymbol{\theta}) = \frac{pR^2}{E(R^2)} \tag{5.1}$$

where $R = (\boldsymbol{Z}'\boldsymbol{Z})^{1/2}$ is the norm of the spherical distribution.

The standardized moments or Mahalanobis moments (Joarder [8]) of the elliptical distribution of $\boldsymbol{X}$ is given by

$$\beta_a = E(Q^a) = p^a\frac{E(R^{2a})}{(E(R^2))^a}. \tag{5.2}$$

In particular, the first three standardized moments of the elliptical distribution of X is given by

$$\beta_1 = p,\ \beta_2 = p^2\frac{E(R^4)}{E^2(R^2)},\ \beta_3 = p^3\frac{E(R^6)}{E^3(R^2)}.$$

It is worth noting that β_2 is the kurtosis parameter of the related elliptical distribution. In this section, by the results of Section 2, we outline how the standardized moments of some elliptical distributions can be derived by (5.2).

(i) Multivariate normal distribution

Since $(\boldsymbol{Z}'\boldsymbol{Z}) \sim \chi^2_p$, from (5.1) we have

$$\beta_a = E(\boldsymbol{Z}'\boldsymbol{Z})^a = 2^a\frac{\Gamma(p/2+a)}{\Gamma(p/2)}.$$

Alternatively by applying $E(R^2) = p$ in (5.2) we have $\beta_a = E(R^{2a})$ which is the same as above.

(ii) Uniform distribution on a p-dimensional ellipse.

Applying Theorem 2.2 to (5.2) we have $\beta_a = p^a$.

(iii) Uniform distribution inside a p-dimensional ellipse.

Applying (2.6) to (5.2) we have

$$\beta_a = p\frac{(p+2)^a}{p+2a}.$$

(iv) Multivariate t-distribution

Applying (2.8) to (5.2) we have

$$\beta_a = 2^a\frac{\Gamma(p/2+a)}{\Gamma(p/2)}\left[\left(\frac{\Gamma(\nu/2)}{\Gamma(\nu/2-a)}\right)^{a-1}\frac{\Gamma(\nu/2-a)}{\Gamma(\nu/2-1)}\right],\ \nu > 2a.$$

Note as $\nu \to \infty$, the terms in the square bracket would converge to 1 and β_a would, as expected, match with that of multivariate normal distribution.

(v) Multivariate Pearson type II distribution

Applying (2.11) to (5.2) we have

$$\beta_a = (p+2)^a B\left(\frac{p}{2}+a, \frac{\alpha-p}{2}+1\right)/B\left(\frac{p}{2}+1, \frac{\alpha-p}{2}+1\right)$$

where $B(a,b)$ is the usual beta function.

Acknowledgements

The authors gratefully acknowledge the excellent research facilities provided by the King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

References

[1] Billah, M. B. and Saleh, A. K. M. E. (2000). Performance of the large sample tests in the actual forecasting of the pretest estimators for regression model with t-error, *Journal of Applied Statistical Science*, **9**(3), 237-251.

[2] Cambanis, S., Huang, S., and Simons, G. (1981). On the theory of elliptically contoured distributions, *Journal of Multivariate Analysis*, **11**, 368-385.

[3] Fang, K. T. and Anderson, T. W. (1990). Statistical Inference in Elliptically Contoured and Related Distributions, Allerton Press, New York.

[4] Fang, K. T., Kotz, S., and Ng, K. W. (1990). Symmetric Multivariate and Related Distributions, Chapman and Hall, London.

[5] Goldman, J. (1974). Statistical properties of a sum of sinusoidal and gaussian noise and its generalization to higher dimensions, *Bell System Technology Journal*, **53**, 557-580.

[6] Goldman, J. (1976). Detection in the presence of spherically symmetric random vectors, *IEEE Transactions on Information Theory*, **22**, 52-59.

[7] Joarder, A. H. (1992). Estimation of the scale matrix of a multivariate t-model, Unpublished Ph.D. thesis, The University of Western Ontario, Canada.

[8] Joarder, A. H. (2007). Mahalanobis moments of bivariate distributions. Festschrift in honor of George and Frances Ball Distinguished Professor of Statistics 'Mir Masoom Ali', (Ball State University, Muncie, Indiana, USA), May 18-19, 201-206.

[9] Kibria, B. M. G. and Saleh, A. K. M. E. (2000). Conflict among the shrinkage estimators induced by W, LR and LM tests under a Student's t regression model, *Journal of the Korean Statistical Society*, **33**(4), 411-433.

[10] Kibria, B. M. G. (2003). Robust Predictive Inference for the Multivariate Linear Models with Elliptically Contoured Distribution, *Far East Journal of Theoretical Statistics*, **10**(1), 11-24.

[11] Kibria, B. M. G. (2006). The Matric t Distribution and its Applications in Predictive Inference, *Journal of Multivariate Analysis,* **97**(3), 785-795.

[12] Kotz, S. and Nadarajah, S. (2004). Multivariate t-Distributions and Their Applications, Cambridge University Press.

[13] Lange, K. L., Little, R. J. A., and Taylor, J. M. G. (1989). Robust statistical modeling using the t distribution, *Journal of the American Statistical Association*, **84**, 881 –896.

[14] Muirhead, R. J. (1982). Aspects of Multivariate Statistical Theory, John Wiley, New York.