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Comparison Results for Quasilinear Elliptic Equations via Picone-Type Identity

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Abstract

We find some comparison results between solutions of some quasilinear problems in regular bounded domains in $\mathbb{R}^n$, using a Picone-type multidimensional identity. Here we will not be investigating how to use the identity for super-sub-solutions methods purpose but to use it as to check how/if the concerned solutions "co-habit" or not in the concerned domains.

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1 Introduction

Let p and Q be two differential operators. To extend the comparison theorem of C. Sturm for solutions u and v of the Sturm-Liouville equations

$$-(p_1(x)u')' + p_0(x)u = 0; \quad p_1 > 0,$$

and

$$-(P_1(x)v')' + P_0(x)v = 0; \quad P_1 > 0,$$

Mauro Picone ([5]) used the fact that if u, v, pu' and Qv' are differentiable with $v \neq 0$, then

$$\begin{aligned} \frac{d}{dx}\left\{\frac{u}{v}[vpu' - uQv']\right\} &= u(pu')' - \frac{u^2}{v}(Qv')' \\ &+ (p-Q)(u')^2 + Q\left(u' - \frac{u}{v}v'\right)^2. \end{aligned} \tag{P}$$

This equation bears his name since. Many extensions and generalizations of that identity had been made (*e.g.*, [2, 3] and references therein). One of the interests of such identities is that integrating them over regular bounded domains provides crucial information about the functions u and v compared with each other in the required domain. This work is focused on the multidimensional type identity [3], namely, if G is a bounded domain in $\mathbb{R}^n \quad (n \geq 2)$ with a regular boundary (*e.g.*, $\partial G \in C^l,\ l \geq 1$), we define for $\alpha > 0$ and $f, F \in C(\overline{G} \times \mathbb{R};\ \mathbb{R})$ and $\mathbb{R}_+ := [0, \infty)$ the operators

$$\left.\begin{array}{l} Pu := P_f u = \nabla \cdot \{a\Phi(\nabla u)\} + c\phi(u) + f(x,u), \\ Qv := Q_F v = \nabla \cdot \{A\Phi(\nabla v)\} + C\phi(v) + F(x,v), \\ \text{where} \quad a, A \in C^1(\overline{G};\ \mathbb{R}_+), \qquad c, C \in C(\overline{G}; \mathbb{R}); \\ \phi(t) = |t|^{\alpha-1}t \quad \text{and} \quad \Phi(\xi) = |\xi|^{\alpha-1}\xi \text{ for } t \in \mathbb{R},\ \xi \in \mathbb{R}^n. \end{array}\right\} \tag{1}$$

Solutions will be supposed to be in

$$\mathcal{D}_P(G) := \{u \in C^1(\overline{G};\ \mathbb{R}) \mid a\Phi(\nabla u) \in C^1(G;\ \mathbb{R}) \bigcap C(\overline{G};\ \mathbb{R})\} \tag{2}$$

and respectively in $\mathcal{D}_Q(G)$ which is defined similarly. As in [3], we note that $s\phi'(s) = \alpha\phi(s)$; $\phi(s) \neq 0$ if $s \neq 0$; $\phi(s)\phi(t) = \phi(st)$; $\phi(s)\Phi(\xi) = \Phi(s\xi)$. So, similar to Theorem 1.1 of [3], we have the following result:

Lemma 1.1 *If* $u \in \mathcal{D}_P(G),\ v \in \mathcal{D}_Q(G)$ *with* $v \neq 0$ *in* G, *then from*

$$\nabla \cdot \left\{\frac{u}{\phi(v)}[\phi(v)a\Phi(\nabla u)]\right\} = a|\nabla u|^{\alpha+1} + uPu - c|u|^{\alpha+1} - uf(x,u) \quad \textit{and}$$

$$\nabla \cdot \left\{u\phi(u)\ \frac{A\Phi(\nabla v)}{\phi(v)}\right\} = (\alpha+1)A\phi(u/v)\ \nabla u \cdot \Phi(\nabla v)$$
$$-\alpha A\left|\frac{u}{v}\nabla v\right|^{\alpha+1} + \frac{u}{\phi(v)}\phi(u)Qv - C|u|^{\alpha+1} - \frac{u}{\phi(v)}\phi(u)F(x,v),$$

we get

$$\begin{aligned} &\nabla \cdot \left\{\frac{u}{\phi(v)}[\phi(v)a\Phi(\nabla u) - \phi(u)A\Phi(\nabla v)]\right\} \\ &= (a-A)|\nabla u|^{\alpha+1} + (C-c)|u|^{\alpha+1} \\ &+ A\left\{|\nabla u|^{\alpha+1} - (\alpha+1)\left|\frac{u}{v}\nabla v\right|^{\alpha-1}\nabla u \cdot \left(\frac{u}{v}\nabla v\right) + \alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1}\right\} \\ &+ \frac{u}{\phi(v)}\{[\phi(v)Pu - \phi(u)Qv] + [\phi(u)F(x,v) - \phi(v)f(x,u)]\}. \end{aligned} \tag{3}$$

Remarks 1.2
In the formulae (3) diverse values attributed to the coefficients $a,\ A,\ c,\ C$ and the functions f and F provide various identities.

1) a) For a general operator $Ku := \nabla \cdot \{a\Phi(\nabla u)\} + g(x,u)$, results related to P can apply after defining $f(x,u) = g(x,u) - c\phi(u)$.
b) Let a be a function as that in (1). For the operator

$$E_\lambda u := \nabla \cdot \{a(x)\Phi(\nabla u)\} + \lambda\phi(u),$$

the corresponding "first eigenvalue" in the domain G if it exists will be defined for $\nu := \alpha + 1$ as the number

$$\lambda_1 := \lambda_1(a; G) = \inf \left\{ \frac{\int_G a(x)|\nabla v|^\nu \, dx}{\int_G |v|^\nu \, dx}; \quad v \in W_0^{1,\nu}(G) \setminus \{0\} \right\}. \tag{λ}$$

Let G be a bounded and regular domain and a be bounded and bounded away from 0. Then λ_1 exists and the solution u_1, say, of

$$\nabla \cdot \{a\Phi(\nabla u)\} + \lambda_1\phi(u) = 0 \quad \text{in } G; \quad u|_{\partial G} = 0, \tag{$E_{\lambda_1}(G)$}$$

is unique modulo a constant multiplier and belongs to $C^{1+\theta}(\overline{G})$ for some $\theta \in (0,\ 1)$ [1, 4, 6]. The solution u_1 is called the eigenfunction corresponding to λ_1 and can be chosen positive.
2) a) For any $\mu > 0$,

$$\Psi(\mu u) := \nabla \cdot \{a(x)\Phi(\nabla \mu u)\} + c\phi(\mu u) = \mu^\alpha \Psi(u)$$

and we will say that a function $f(x,\cdot)$ is $\alpha-$*homogeneous* if

$$f(x, \mu t) = \mu^\alpha f(x,t) \qquad \forall \mu > 0.$$

b) We will denote $P_0 u$ and $Q_0 v$ for the operators in (1) where $f(x,u) \equiv 0$ and $F(x,v) \equiv 0$ respectively.
3) We have the following result from [3, Lemma 2.1]:
Given $\alpha > 0$,

$$\forall \xi,\ \eta \in \mathbb{R}^n \qquad |\xi|^{\alpha+1} + \alpha|\eta|^{\alpha+1} - (\alpha+1)|\eta|^{\alpha-1}\xi \cdot \eta \geq 0 \tag{4a}$$

and the equality holds if and only if $\xi = \eta$.

Some identities:

4) If $a = A,\ c = C,\ Pu = Qv = 0$ in G, then (3) becomes

$$\begin{aligned} &\nabla \cdot \left\{ \frac{u}{\phi(v)}\, a[\phi(v)\Phi(\nabla u) - \phi(u)\Phi(\nabla v)] \right\} \\ &= Z(u,v) + u\phi(u) \left[\frac{F(x,v)}{\phi(v)} - \frac{f(x,u)}{\phi(u)} \right] \\ &= a \left\{ |\nabla u|^{\alpha+1} - (\alpha+1) \left| \frac{u}{v}\nabla v \right|^{\alpha-1} \nabla u \cdot \left(\frac{u}{v}\nabla v \right) + \alpha \left| \frac{u}{v}\nabla v \right|^{\alpha+1} \right\} \\ &+ u\phi(u) \left[\frac{F(x,v)}{\phi(v)} - \frac{f(x,u)}{\phi(u)} \right], \end{aligned} \tag{4b}$$

where $Z_a(u,v)$ or $Z(u,v)$ (if there is no confusion) is defined by

$$Z(u,v) := a\left\{|\nabla u|^{\alpha+1} + \alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1} - (\alpha+1)\left|\frac{u}{v}\nabla v\right|^{\alpha-1}\nabla u\cdot\left(\frac{u}{v}\nabla v\right)\right\}.$$

If we interchange the functions u and v in (4b) assuming that none of them takes zero value inside G then as in (4b) we have

$$\begin{aligned}
&\nabla\cdot\left\{\frac{v}{\phi(u)}\,a[\phi(u)\Phi(\nabla v) - \phi(v)\Phi(\nabla u)]\right\}\\
&= a\left\{|\nabla v|^{\alpha+1} - (\alpha+1)\left|\frac{v}{u}\nabla u\right|^{\alpha-1}\nabla v\cdot\left(\frac{v}{u}\nabla u\right) + \alpha\left|\frac{v}{u}\nabla u\right|^{\alpha+1}\right\}\\
&+ v\phi(v)\left[\frac{F(x,u)}{\phi(u)} - \frac{f(x,v)}{\phi(v)}\right]\\
&= Z(v,u) + v\phi(v)\left[\frac{F(x,u)}{\phi(u)} - \frac{f(x,v)}{\phi(v)}\right].
\end{aligned} \tag{4c}$$

5) For the functions u and v above, if $\Omega\subset G$ has a nonempty interior and $f(x,t)\equiv F(x,t)$, then after integrating (4b) over Ω we get

$$\begin{aligned}
&\int_{\partial\Omega} au\left\{|\nabla u|^{\alpha-1}\frac{\partial u}{\partial\nu_\Omega} - |\nabla v|^{\alpha-1}\frac{\partial v}{\partial\nu_\Omega}\right\}ds\\
=\ &\int_\Omega \left[Z(u,v) + |u|^{\alpha+1}\left\{\chi(x,v) - \chi(x,u)\right\}\right]dx,
\end{aligned} \tag{4d}$$

and from (4c)

$$\begin{aligned}
&\int_{\partial\Omega} av\left\{|\nabla v|^{\alpha-1}\frac{\partial v}{\partial\nu_\Omega} - |\nabla u|^{\alpha-1}\frac{\partial u}{\partial\nu_\Omega}\right\}ds\\
=\ &\int_\Omega \left[Z(v,u) + |v|^{\alpha+1}\left\{\chi(x,u) - \chi(x,v)\right\}\right]dx,
\end{aligned} \tag{4e}$$

where ν_Ω denotes the outward normal unit vector to $\partial\Omega$ and

$$\chi(x,t) := f(x,t)/\phi(t). \tag{5}$$

In the sequel G and any subset of its are assumed to be of the class $C^l,\ l\geq 1$.

2 Main results

When the perturbations are homogeneous only for $\alpha = 1$, obviously the possibility that a solution of the problem coincides with a multiple of that of a half-linear equation cannot hold.

Theorem A *Let* $f \in C(\overline{G} \times \mathbb{R}; \mathbb{R})$ *and let* $u,\ v \in \mathcal{D}_P(G)$ *be two solutions of*

$$Pw := \nabla \cdot \{a\Phi(\nabla w)\} + c\phi(w) + f(x,w) = 0 \ in\ G; \quad w|_{\partial G} = 0$$

such that each of them remains nonzero inside G *and they have the same sign.*
1) If $t \mapsto f(x,t)/\phi(t) := \chi(x,t)$ *is monotone in* $\mathbb{R}$ *uniformly for* $x \in G$*, then the two solutions have at least one common point.*
If Ω *is a subset of* G *and* $\chi(x,t)$ *is increasing in* $t > 0$*, then if* $v \geq u$ *in* Ω*,*

$$\int_{\partial\Omega} au \left\{ |\nabla u|^{\alpha-1} \frac{\partial u}{\partial \nu_\Omega} - |\nabla v|^{\alpha-1} \frac{\partial v}{\partial \nu_\Omega} \right\} ds \geq 0 \tag{$a1$}$$

and if, in addition, $u = v$ *on* $\partial\Omega$*, then*

$$\begin{aligned} 0 &\geq \int_\Omega \left\{ Z(v,u) + |v|^{\alpha+1}[\chi(x,u) - \chi(x,v)] \right\} dx \\ &= -\int_\Omega \left\{ Z(u,v) + |u|^{\alpha+1}[\chi(x,v) - \chi(x,u)] \right\} dx. \end{aligned} \tag{$a2$}$$

2) If $\chi(x,t)$ *is decreasing in* t *and nonnegative for any* $x \in \Omega$*, then the two solutions coincide.*

In the sequel, $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^1$; $a \in C^1(\overline{\Omega}; \mathbb{R}_+)$ and $K \in C(\overline{\Omega} \times \mathbb{R}_+; \mathbb{R}_+)$.

Theorem B *Consider in* Ω *the problems*

$$\begin{cases} Eu := \nabla \cdot \{a(x)|\nabla u|^{\alpha-1}\nabla u\} + K(x,u) = 0; \\ u|_{\partial\Omega} = 0; \quad u \in \mathcal{D}_E(\Omega). \end{cases} \tag{E}$$

If there are $\beta \in C(\overline{\Omega}; \mathbb{R}_+)$ *and* $W \in C^1(\overline{\Omega}; \mathbb{R}_+)$ *such that a.e. in* Ω

$$\begin{aligned} &W|_{\partial\Omega} = 0, \quad K(x,W) - \beta(x)\phi(W) \geq 0 \quad \text{and} \\ &\int_\Omega \left\{ a|\nabla W|^{\alpha+1} - c|W|^{\alpha+1} \right\} dx \leq 0, \end{aligned} \tag{Ea}$$

then any solution $u \in \mathcal{D}_E(\Omega)$ *of (E) has a zero inside* Ω *unless there is* $\lambda \in \mathbb{R}$ *such that* $u = \lambda W$ *and*

$$\int_\Omega W \left\{ \lambda^\alpha K(x,W) - K(x,\lambda W) \right\} dx = 0.$$

Theorem C *Suppose that* $a \in C^1(\overline{\Omega}; (0,\ \infty)\ )$ *and let* $\lambda_1 := \lambda_1(a, \Omega)$ *as in Remarks 1.2. If* K *is* $\alpha-$*homogeneous only for* $\alpha = 1$*, then*
1) if

$$K(x,t) - \lambda_1 \phi(t) \geq 0 \quad in\ \Omega \times \mathbb{R}_+, \tag{Eb}$$

(E) has no solution which is strictly positive in Ω*;*
2) assume that $K(x,t) := \mu\phi(t) + h(x,t)$ *where* $h(x,t) \leq 0$ *and* $\mu > 0$ *; then if* $\lambda_1 \geq \mu$ *, (E) has no solution which is strictly positive in* Ω*.*

The solution u_1 corresponding to λ_1 can be taken to satisfy $|u_1|_{C(\overline{\Omega})} \leq 1$ and replace W in (Ea). Thus the inequality in (Eb) to hold in $\Omega \times [0,\ 1]$ will be sufficient.
An application of these results is

Theorem D *Assume that* K *is* $\alpha-$*homogeneous only for* $\alpha = 1$*. Then the problem*

$$\begin{cases} Eu := \nabla \cdot \left\{ a(x) |\nabla u|^{\alpha-1} \nabla u \right\} + K(x,u) = 0, \quad u > 0 \quad in \quad \Omega; \\ u|_{\partial\Omega} = 0; \quad u \in \mathcal{D}_E(\Omega). \end{cases} \tag{$E+$}$$

has at most one solution satisfying for some $\beta \in C(\overline{\Omega}; \mathbb{R}_+)$

$$K(x,u) - \beta(x)\phi(u) \geq 0 \quad in \quad \Omega.$$

In particular, for any $\beta \in C(\overline{\Omega}; \mathbb{R}_+)$ *and* $f \in C(\overline{\Omega} \times \mathbb{R}_+; \mathbb{R}_+)$

$$\begin{aligned} &\nabla \cdot \left\{ a(x) |\nabla u|^{\alpha-1} \nabla u \right\} + \beta(x)\Phi(u) + f(x,u) = 0, \qquad u > 0 \ \ in \ \ \Omega; \\ &u|_{\partial\Omega} = 0; \qquad u \in \mathcal{D}_E(\Omega) \end{aligned}$$

has at most one solution.

3 Some comparison results

Theorem 3.1 *Assume that there are* $u, v \in \mathcal{D}_P(G)$ *, respectively nontrivial solutions of*

$$P_0 u := \nabla \cdot \{a\Phi(\nabla u)\} + c\phi(u) = 0 \quad in \quad G$$

and

$$P_g v := \nabla \cdot \{a\Phi(\nabla v)\} + g(x,v) = 0 \quad in \quad G, \quad v|_{\partial G} = 0$$

for some $g \in C(\overline{G} \times \mathbb{R}; \mathbb{R})$ *.*
a) If

$$\int_G v(x)\{c\phi(v) - g(x,v)\}\, dx \geq 0, \tag{6}$$

then u has a zero inside G. If equality prevails in (6)*, then there is $k \in \mathbb{R}$ such that $u \equiv kv$ in G.*
b) If

$$\begin{aligned} & g(x,v) - c\phi(v) \geq 0 \quad in \quad G \quad or \\ & \int_G \frac{|u|^{\alpha+1}}{\phi(v)} [g(x,v) - c\phi(v)]\, dx \geq 0, \end{aligned} \tag{6'}$$

then v has a zero inside G. If equality prevails in (6′)*, then there is $k \in \mathbb{R}$ such that $u \equiv kv$ in G.*

Proof.

a) Assume that $u \neq 0$ in G.
Applied to u and v, (4c) where $f(x,v) := g(x,v) - c\phi(v)$ and $F(x,u) \equiv 0$ reads

$$\begin{aligned} & \nabla \cdot \left\{ \frac{v}{\phi(u)}\ a[\phi(u)\Phi(\nabla v) - \phi(v)\Phi(\nabla u)] \right\} \\ & = Z(v,u) + v\{c\phi(v) - g(x,v)\}. \end{aligned} \tag{6a}$$

The integration of (6a) over G and using (4a) leads to

$$0 = \int_G Z(v,u) + \int_G v(x)\{c\phi(v) - g(x,v)\}\, dx. \tag{7}$$

Thus if (6) holds, $v \neq 0$ cannot hold throughout G as otherwise (7) would not hold. From (7), if equality holds in (6), then by (4a) $\nabla u = (u/v)\nabla v$ whence $v\nabla u - u\nabla v := v^2\ \nabla(u/v) = 0$ and the conclusion follows.

b) If we suppose that $v \neq 0$ in G, then from (4b) we have

$$\begin{aligned} & \nabla \cdot \left\{ \frac{u}{\phi(v)}\ a[\phi(v)\Phi(\nabla u) - \phi(u)\Phi(\nabla v)] \right\} \\ & = Z(u,v) + \frac{|u|^{\alpha+1}}{\phi(v)}\{g(x,v) - c\phi(v)\} \end{aligned} \tag{6b}$$

and the proof is completed as in the a)-case. □

Theorem 3.2 *Let Ω be a regular bounded domain in $\mathbb{R}^n$ as G in* (1) *and Ω_1 be a domain containing Ω. Let $a, A \in C^1(\overline{\Omega}_1; \mathbb{R}_+)$ and $c, C \in C(\overline{\Omega}_1; \mathbb{R})$ with $a \equiv c \equiv 0$ in Ω. Assume that there is $u \in C^1(\overline{\Omega})$ such that $u|_{\partial\Omega} = 0$ and*

$$\int_\Omega \{A|\nabla u|^{\alpha+1} - C|u|^{\alpha+1}\}\, dx \leq 0. \tag{8}$$

Then any solution $v \in \mathcal{D}_Q(\Omega)$ *of*

$$\begin{cases} Qv := Q_f v = \nabla \cdot \{A\Phi(\nabla v)\} + C\phi(v) + f(x,v) = 0 \quad in \quad \Omega \\ satisfying \qquad \phi(v)f(x,v) \geq 0 \quad in \quad \Omega \\ or \quad \int_\Omega \left\{ Z(v,u) + |u|^{\alpha+1}\frac{f(x,v)}{\phi(v)} \right\} dx \geq 0 \end{cases} \tag{9}$$

must vanish somewhere in Ω *unless there is* $k \in \mathbb{R}$ *such that* $u = kv$ *and either* f *is* α*-homogeneous or* $\int_\Omega v\,\{k^\alpha f(x,v) - f(x,kv)\}\,dx = 0$.

Note that the (heavy) introduction of Ω_1 can be avoided by defining the operator P as $Pv \equiv f(x,v)$.

Proof. Assume that such v is nonzero throughout Ω. With the operator $P := P_f$ as in (1), in Ω, $\quad Pu = f(x,u)$ and (3) reads

$$\nabla \cdot \{\phi(u/v)A\Phi(\nabla v)\} = A|\nabla u|^{\alpha+1} - C|u|^{\alpha+1} - Z(u,v) - u\phi(u/v)f(x,v). \tag{9a}$$

Integrating this over Ω we get

$$\int_\Omega \left\{A|\nabla u|^{\alpha+1} - C|u|^{\alpha+1}\right\}\,dx = \int_\Omega \left\{Z(u,v) + u\phi\left(\frac{u}{v}\right)f(x,v)\right\}\,dx \tag{10}$$

and the conclusion follows.

If $u = kv$, we have equality in (8) and (10) is reduced to

$$k\phi(k)\int_\Omega v\,\{k^\alpha f(x,v) - f(x,kv)\}\,dx = 0. \qquad \square$$

4 Proofs of Theorems

4.1 Proof of Theorem A

The statement (a1) follows from (4d). Adding (4d) and (4e), we get

$$\begin{aligned} &\int_{\partial\Omega} a(u-v)\{\Phi(u) - \Phi(v)\}\cdot \nu_\Omega\,ds \\ &= \int_\Omega \left\{Z(u,v) + Z(v,u) + \left[|u|^{\alpha+1} - |v|^{\alpha+1}\right](\ \chi(x,v) - \chi(x,u)\ )\right\}\,dx \end{aligned}$$

leading to (a2). For the two solutions, (4b) and (4c) lead (after an integration over G) to

$$\begin{aligned} 0 \leq \int_G Z(u,v)\,dx &= -\int_G u\phi(u)\left\{\frac{f(x,v)}{\phi(v)} - \frac{f(x,u)}{\phi(u)}\right\}\,dx \\ &= -\int_G |u|^{\alpha+1}\{\chi(x,v) - \chi(x,u)\}\,dx \end{aligned} \tag{A1}$$

and

$$0 \leq \int_G Z(v,u)\,dx = -\int_G v\phi(v)\left\{\frac{f(x,u)}{\phi(u)} - \frac{f(x,v)}{\phi(v)}\right\}dx \\ = \int_G |v|^{\alpha+1}\{\chi(x,v) - \chi(x,u)\}\,dx. \tag{A2}$$

Assume that $\chi(x,t)$ is increasing. If we suppose that $v > u$ in G, then (A1) provides a contradiction and if we suppose that $u > v$, (A2) would lead to a contradiction. Assume that $\chi(x,t)$ is decreasing and define

$$G_+ \ (G_-) \ := \{x \in G \mid X(x) := \chi(x,v) - \chi(x,u) > \ (<)\ 0\}.$$

Then (without loss of generality) $0 < v \leq u$ in G_+ and $v > u \geq 0$ in G_- whence

$$\int_{G_+} |v|^{\alpha+1} X(x)\,dx \leq \int_{G_+} |u|^{\alpha+1} X(x)\,dx \quad \text{and} \\ \int_{G_-} |v|^{\alpha+1} X(x)\,dx \leq \int_{G_-} |u|^{\alpha+1} X(x)\,dx. \tag{A3}$$

This implies from (A1) and (A2) that

$$0 \leq \int_G |v|^{\alpha+1} X(x)\,dx \leq \int_G |u|^{\alpha+1} X(x)\,dx \leq 0,$$

whence $\int_G Z(u,v)dx = 0$, leading to $v \equiv u$ in G by (4a).

If f is nonnegative and decreasing in t, χ is decreasing in t and the same conclusion is reached. □

4.2 Proof of Theorem B and Theorem C

B) This is an application of Theorem 3.2.

We just need to take $f(x,z) := K(x,z) - \beta\Phi(z)$.

C) 1) Follows easily from Theorem B when we take for W the eigenfunction u_1 corresponding to λ_1 and $\beta = \lambda_1$.

2) This follows from Theorem 3.1, where $c = \lambda_1$ and $g(x,t) := K(x,t)$. □

4.3 Proof of Theorem D

To prove this it is enough to notice that if we suppose to have two such solutions, with one of them playing the role of W, the conclusion follows from Theorem B. When K is $\alpha-$homogeneous only for $\alpha = 1$, the last part of the conclusion of Theorem B can apply only for that value of α. □

4.4 Concluding remarks

Remark 4.1 Concerning the problems (E), the hypothesis for the results in Theorem B through Theorem D is more or less $K(x,u) - \beta(x)\phi(u) \geq 0$ in Ω; $\phi(t) = O(t^\alpha)$ for small $t > 0$.
As the datum on $\partial\Omega$ is 0, using the change $U(x) := u(x)/\{\max_{\overline{\Omega}} u(x)\}$ the results more likely hold for sublinear perturbations, (*e.g.*, $K(x,t) = O(t^q)$; $q \in (0,\ \alpha]$) .
But if we consider strictly positive solutions of the Neumann problem

$$\begin{cases} \nabla \cdot \{a(x)|\nabla v|^{\alpha-1}\nabla v\} + K(x,v) = 0 & \text{in } \Omega; \\ \nabla v|_{\partial\Omega} = 0; \quad v \in \mathcal{D}_E(\Omega), \end{cases} \tag{N}$$

using this time the change $V(x) := v(x)/\{\min_{\overline{\Omega}} v(x)\}$ the results apply for superlinear perturbations $(K(x,t) = O(t^p);\ p > \alpha)$.

Remark 4.2 In [3], some oscillation theorems are established for half-linear problems. Following similar approaches (hopefully with the help of Theorem 3.1), similar results could be obtained for some cases with perturbations.

Remark 4.3 A Wirtinger-type inequality states that
For a regular domain G, if there is a solution $v \in \mathcal{D}_{P_0}(G)$ of the half-linear equation $P_0 v = \nabla \cdot \{a(x)|\nabla v|^{\alpha-1}\nabla v\} + c(x)\phi(v) = 0$ such that $v \neq 0$ in G, then $\int_G [a(x)|\nabla u|^{\alpha+1} - c(x)|u|^{\alpha+1}]\,dx \geq 0$ holds for any nontrivial function $u \in C^1(\overline{G};\mathbb{R})$ such that $u|_{\partial G} = 0$ where the equality holds if and only if u is a constant multiple of v. (see [3])
For the perturbations cases, Theorem 3.2 provides some corresponding (alternative) version of the inequality, namely, for any domain G as above and $f \in C(\overline{G}\times\mathbb{R};\ \mathbb{R}_+)$

Proposition 4.1 *If $\int_G v\{k^\alpha f(x,v) - f(x,kv)\}\,dx \neq 0 \quad \forall k \notin \{0,\ 1\}$ where $v \in \mathcal{D}_P(G)$ is nonzero in G and solves $\nabla\cdot\{A\Phi(\nabla v)\} + C\phi(v) + f(x,v) = 0$ in G, then $\int_G \{\ A(x)|\nabla u|^{\alpha+1} - C(x)|u|^{\alpha+1}\ \}dx > 0$ holds for any nontrivial $u \in C^1(\overline{G};\mathbb{R})$ such that $u|_{\partial G} = 0$.*

Remark 4.4 For a nonhomogeneous perturbation $g(x,v)$, Theorem 3.1 implies that the respective solutions $u, v \in \mathcal{D}_P(G)$ of $\nabla \cdot \{a\Phi(\nabla u)\} + c\phi(u) = 0$ in G and $\nabla \cdot \{a\Phi(\nabla v)\} + g(x,v) = 0$ in G, $v|_{\partial G} = 0$ cannot be both nonzero inside G if $g(x,v) - c\phi(v)$ is nonzero (whence keeps the same sign) in G.

Remark 4.5 The uniqueness result in Theorem A can be deduced from (a1) and (4d) or (4e).
In fact, if there are two solutions u, v, say, such that in some $D \Subset G$ $v > u > 0$ and

$u = v$ on ∂D , then by (4.e) and (a1) $0 \geq \int_{\partial D} av\{|\nabla v|^{\alpha-1}\frac{\partial v}{\partial \nu_D} - |\nabla u|^{\alpha-1}\frac{\partial u}{\partial \nu_D}\}\, ds$ $= \int_D [Z(v,u) + |v|^{\alpha+1}\, \{\chi(x,u) - \chi(x,v)\}]\, dx$ and the last member is strictly positive if $\chi(x,t)$ is decreasing in t.

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