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# Computing Topological and Metrical Invariants

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**Abstract**

We study the dependence of the attractors on the contraction ratios of the iterated function systems, associated to expanding discontinuous maps with holes on the interval. For this class of maps, an extension of Milnor-Thurston theory is provided. Introducing weights on the formal power series, we establish a weighted kneading theory. We show that this method allows us to derive techniques to compute explicitly some topological and metrical invariants: the topological entropy, the Hausdorff dimension and the escape rate.

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## 1 Introduction

Let $I \subset \mathbb{R}$ be a compact interval and $f := \{f_i\}_{i=1}^{n}$ be an iterated function system (IFS), a collection of self-maps on $I$, defined by

$$f_i\,(x) := \rho_i\, x + \varrho_i, \text{ with } i = 1, \ldots, n,$$

where for all $i$, $0 < |\rho_i| < 1$ and $\varrho_i \in \mathbb{R}$. Let $E$ denote the attractor of the IFS, *i.e.*, the unique compact set $E \subset \mathbb{R}$ satisfying the equation

$$E = \bigcup_{i=1}^{n} f_i\,(E)\,.$$

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We consider IFS's where the open set condition is satisfied, [5] and [6]. On these circumstances the Hausdorff dimension of the attractor, which will be denoted by $\dim_H(E)$, can be rigorously calculated.

Observe that, if $f_i$ is monotone, then it is usual to see $E$ as the repeller of a linear expanding map $F : \bigcup_{i=1}^{n} f_i(I) \to I$, which will be denoted by $F = (F_1, \ldots, F_n)$, where

$$F_i(x) := f_i^{-1}(x) \text{ if } x \in f_i(I).$$

We consider the piecewise linear map $F$ with a single hole, *i.e.*, there is an open subinterval $I_h \subset I$ with $I_h \neq \emptyset$ such that $I$ is the disjoint union of $I_h$ and $\bigcup_{i=1}^{n} \mathrm{Im}(f_i)$, [9] and [11]. We denote the hole by $I_h = ]a_h, a_{h+1}[$. The points $x \in I_h$ will be mapped out of $I$ and the same will happen to all the points $x \in F^{-k}(I_h)$ for $k \geq 1$. The set $\bigcup_k F^{-k}(I_h)$ is open and dense in $I$ and has full Lebesgue measure [1]. We can obtain the same results for a finite union of disjoint holes $I_{h_j} \subset I$.

A brief overview of this paper is as follows. In Section 2, considering just the orbits of the turning points and discontinuity points of $F$, we define a Markov partition of $I$. We associate with the IFS a weighted subshift of finite type, which is described by a weighted transition matrix. This matrix allows us to compute explicitly the Hausdorff dimension of $E$. Section 3 contains an algorithm to define a fractal Markov measure, the maximum entropy measure to the fractal one-sided Markov subshift. Using this invariant probability measure, we characterize the metric entropy and the Lyapunov exponent. Thus, we show a relation between the Hausdorff dimension, the metric entropy and the Lyapunov exponent. In Section 4, to a three parameter family of IFS's, using a weighted kneading determinant, we relate the periodic, eventually periodic orbits and the orbits that lie in the hole with the topological entropy, the Hausdorff dimension and the escape rate.

## 2 Weighted transition matrix

The hole and the set of $n$ laps of $F$ determine a partition

$$\mathcal{P}_I := \{I_1, \ldots, I_h, \ldots, I_{n+1}\}$$

of the interval $I$. Let $a_i$, with $i = 1, \ldots, n+2$, be the discontinuity points and the turning points of the map $F$. Considering the orbits of these points, we define a Markov partition $\mathcal{P}'_I$ of $I$. The orbit of each point $a_i$ is defined by

$$o(a_i) := \left\{ x_k^{(i)} : x_k^{(i)} = F^k(a_i), \, k \in \mathbb{N}_0 \right\}.$$

Concerning the itinerary of each point $a_i$ we will have

$$F^k(a_i) = x_k^{(i)} \text{ with } x_k^{(i)} \notin I_h \text{ or } F^k(a_i) \in I_h.$$

In the first case, we have periodic, eventually periodic or aperiodic orbits [14]. While in the second one, after a finite number of iterates, the itinerary of the points lies in the hole.

To simplify the presentation, we consider the points $a_1$ and $a_{n+2}$ as fixed points. Now, let

$$\{b_1, \ldots, b_{m+1}\} := \{o\,(a_i) : \ i = 1, \ldots, n+2\}$$

be the set of the points corresponding to the orbits of the discontinuity points and turning points, ordered on the interval $I$. This set allows us to define a subpartition $\mathcal{P}'_I$ of $\mathcal{P}_I$. The subpartition

$$\mathcal{P}'_I := \{J_1, \ldots, J_m\}$$

with $m \geq n$ determines a Markov partition of the interval $I$. Note that, the hole is an element of the Markov partition. Note also that, the map $F$ determines $\mathcal{P}'_I$ uniquely, but the converse is not true.

The IFS $f$ induces a subshift of finite type whose $m \times m$ transition matrix $A := [a_{ij}]$ is defined by

$$a_{ij} := \begin{cases} 1 & \text{if } F\,(\operatorname{int} J_j) \supseteq \operatorname{int} J_i \\ 0 & \text{otherwise.} \end{cases}$$

We note that, if there exists $k$ points $b_i$ such that $b_i \in I_h$, with $1 < i < m+1$, then the matrix $A$ has $k+1$ columns with all elements equal to zero, corresponding to the hole. We denote this subshift by $(\Sigma_A, \sigma)$, where $\sigma$ is the shift map on $\Sigma_m^{\mathbb{N}}$ defined by $\sigma\,(x_1 x_2 \ldots) := x_2 x_3 \ldots$, with $\Sigma_m := \{1, \ldots, m\}$ corresponding to the $m$ states of the subshift. The topological entropy of $(\Sigma_A, \sigma)$ is $\log \lambda_A$, where $\lambda_A$ is the spectral radius of the transition matrix $A$. See [9] and [11] for the topological entropy, which is related with the kneading theory.

Concerning the subshift $(\Sigma_A, \sigma)$ and the associated Markov partition $\mathcal{P}'_I$, we consider a Lipschitz function $\phi : I \to \mathbb{R}$, defined by

$$\phi := \{\phi_i : J_i \to \mathbb{R},\, 1 \leq i \leq m\}\,,$$

where

$$\phi_i\,(x) := -\beta\,\varphi_i\,(x) \ \text{ and } \varphi_i\,(x) := \log\left|F'_i\,(x)\right|, \text{ with } \beta \in \mathbb{R}.$$

This function is a weight for the dynamical system associated to the subshift, depending on the real parameter $\beta$, compare with [18].

Let $\mathcal{L}^1\,(I)$ be the set of all Lebesgue integrable functions on $I$. The transfer operator $L_\phi : \mathcal{L}^1\,(I) \to \mathcal{L}^1\,(I)$, associated with $F$ and $\mathcal{P}'_I$, is defined by

$$(L_\phi\, g)\,(x) := \sum_{j=1}^{m} \exp \phi_j \left(F_j^{-1}\,(x)\right)\, g\left(F_j^{-1}\,(x)\right)\, \chi_{F(\operatorname{int} I_j)}, \tag{2.1}$$

where $\chi_{I_j}$ is the characteristic function of $I_j$. We refer to [2, 3, 18, 19] and the references therein for other important spectral properties of the transfer operator, and [5] for this operator with respect to the cookie-cutter system.

Nevertheless, we consider a class of one-dimensional transformations that are piecewise linear Markov transformations. Consequently, the transfer operator has the following matrix representation

$$L_\phi\, g = Q_\beta\, \pi_g$$

with $g \in \mathcal{C}$, where $\mathcal{C}$ is the class of all functions that are piecewise constant on the partition $\mathcal{P}_I'$ and $\pi_g = (\pi_1, \ldots, \pi_m)^T$, [2] and [11]. If $D_\beta$ is the diagonal matrix defined by

$$D_\beta := (\exp \phi_1, \ldots, \exp \phi_m)$$

and $A$ is the transition matrix, then the matrix $Q_\beta$ is the $m \times m$ weighted transition matrix defined by

$$Q_\beta := A\, D_\beta = [q_{ij}], \text{ where } q_{ij} := \frac{a_{ij}}{\left|F_j'\right|^\beta}. \tag{2.2}$$

The matrix $Q_\beta$ allows us to consider a weighted subshift of finite type naturally generated by $(\Sigma_A, \sigma)$. See [4] for the similar weighted incidence matrix associated with a graph directed construction.

**Theorem 1** *Let $(\Sigma_A, \sigma)$ be the subshift of finite type associated with $F$. If $\beta$ is the unique positive real number for which the spectral radius of the matrix $Q_\beta$ is equal to one, then*

$$\dim_H(E) = \beta.$$

This result is equivalent to the Bowen's equation. See [11] for the proof and for the connection with a weighted kneading determinant. Now, we define the trace of the transfer operator by

$$\operatorname{Tr} L_\phi := \sum_{x \in \operatorname{Fix}(F)} \exp \phi(x),$$

where Fix$(F)$ denotes the set of fixed points of $F$. We consider the pressure function of $\phi(x) = \log |F'(x)|^{-\beta}$ as $\beta$ varies, $P_\beta(\phi)$, defined by

$$\begin{aligned} P_\beta(\phi) &:= \lim_{k\to\infty} \tfrac{1}{k} \log \sum_{x \in \operatorname{Fix}(F^k)} \left| \left(F^k\right)'(x) \right|^{-\beta} \\ &= \lim_{k\to\infty} \tfrac{1}{k} \log \left( Tr\, Q_\beta^k \right) \\ &= \log(\lambda_\beta), \end{aligned} \tag{2.3}$$

where $\mathrm{Fix}\,(F^k)$ denotes the set of fixed points of $F^k$, see [5, 18] and [19]. Thus, $\exp P_\beta\,(\phi)$ is the largest eigenvalue $\lambda_\beta$ of the transfer operator $L_\phi$, and is equal to the spectral radius of the matrix $Q_\beta$ [18].

## 3 Maximum entropy measure

It is well known that the subshift of finite type $(\Sigma_A, \sigma)$ has an invariant probability measure of maximal entropy, [3] and [17]. In this section, we will present an algorithm to define a fractal Markov measure, considering $\beta = \dim_H(E)$.

By the Ruelle-Perron-Frobenius Theorem there exist $\lambda_\beta > 0$ and $v_\beta \in \mathcal{C}$, with $v_\beta\,(J_i) > 0$ for all $1 \le i \le m$, such that $v_\beta$ is the eigenvector of $Q_\beta$ with largest eigenvalue $\lambda_\beta$, *i.e.*, $Q_\beta\, v_\beta = \lambda_\beta\, v_\beta$. This eigenvector is used to construct a transition probability matrix. Now we present a simple algorithm to compute this matrix. Let $\overline{\mu}$ be a measure with support in $\mathcal{P}'_I$. We denote the adjoint operator of $L_\phi$ by $L^*_\phi$, which is defined by a bounded linear map on measures, *i.e.*,

$$\left(L^*_\phi\,\overline{\mu}\right)(g) := \overline{\mu}\,(L_\phi\, g)\,.$$

Note that the adjoint operator $L^*_\phi$ is represented by the matrix $Q^T_\beta$ [12]. The eigenvalues of the matrices $Q_\beta$ and $Q^T_\beta$ are equal. If $v_\beta$ and $r_\beta$ are the right eigenvectors associated with $\lambda_\beta$ of the matrices $Q_\beta$ and $Q^T_\beta$, respectively, then we have

$$\sum_{i=1}^{m} q_{ij}\, v_j = \lambda_\beta\, v_i \text{ and } \sum_{i=1}^{m} q_{ij}\, r_i = \lambda_\beta\, r_j. \tag{3.1}$$

Let $u_\beta := (u_1, \ldots, u_m)$ be the left eigenvector and $v_\beta := (v_1, \ldots, v_m)$ be the right eigenvector, strictly positive, of the matrix $Q_\beta$. Thus, there exists a unique $\lambda_\beta > 0$ and a unique probability measure $u_\beta$ such that $Q^T_\beta\, u_\beta = \lambda_\beta\, u_\beta$. Furthermore, the measure given by $(u_1 v_1, \ldots, u_m v_m)$, up to a multiplicative constant, is $F$-invariant, ergodic, positive on non-empty open sets, see [3, 18] and [19].

For the $m$-dimensional vector space $\mathcal{P}'_I$, we consider two bases

$$\mathcal{B} := \{e_1, \ldots, e_m\} \text{ and } \mathcal{B}' := \left\{e'_1, \ldots, e'_m\right\}.$$

The set of vectors in $\mathcal{B}$ are defined by the column vector $e_j := (0, \ldots, 0, 1, 0, \ldots, 0)^T$, where 1 is in the $j-$th position. These vectors correspond to the intervals of the Markov partition. On the other hand, the set of vectors in $\mathcal{B}'$ are defined by $e'_j := (0, \ldots, 0, v_j, 0, \ldots, 0)^T$, which correspond to the coordinates of the vector $v_\beta$. If $M_\beta$ is the matrix which describes the change from the basis $\mathcal{B}'$ to the basis $\mathcal{B}$, then we define a new matrix, the $m \times m$ matrix

$$R_\beta := M^{-1}_\beta\, Q_\beta\, M_\beta = [r_{ij}], \text{ where } r_{ij} := q_{ij}\frac{v_j}{v_i} \text{ with } r_{ij} \ge 0.$$

The matrix $R_\beta$ is the matrix representation of $L_\phi$, with respect to the basis $\mathcal{B}'$. As the matrices $Q_\beta$ and $R_\beta$ are similar, the largest eigenvalue $\lambda_\beta$ of these matrices is the same.

We define an $m \times m$ stochastic matrix $S_\beta := [s_{ij}]$, where

$$s_{ij} := \frac{r_{ij}}{\lambda_\beta} \text{ with } s_{ij} \geq 0 \text{ and } \sum_{j=1}^{m} s_{ij} = 1. \tag{3.2}$$

We note that the transpose matrix $S_\beta^T$ corresponds to the modified or normalized transfer operator, with respect to the basis $\mathcal{B}'$ [12].

Let $u'_\beta := (u'_1, \ldots, u'_m)$ be the left eigenvector and $v'_\beta := (v'_1, \ldots, v'_m)$ be the right eigenvector, strictly positive, of the matrix $R_\beta$. The probability vector $p_\beta := (p_1, \ldots, p_m)$ is defined by

$$p_i := \frac{u'_i\, v'_i}{\sum\limits_{i=1}^{m} u'_i\, v'_i}, \text{ such that } \sum_{i=1}^{m} p_i\, s_{ij} = p_j \text{ and } \sum_{i=1}^{m} p_i = 1. \tag{3.3}$$

This vector defines the unique $F$-invariant equilibrium state for $\phi = -\beta \log |F'(x)|$. Note that, if we consider $\mu^* = (u_1 v_1, \ldots, u_m v_m)$, up to a multiplicative constant, then $\mu^* = p_\beta$, compare with [3, 17] and [19]. We will call the pair $(p_\beta, S_\beta)$ weighted one-sided Markov subshift, associated with the subshift of finite type $(\Sigma_A, \sigma)$.

The stochastic matrix $S_\beta$ and the probability vector $p_\beta$ allow us to define an invariant probability measure $\mu_\beta$ on the repeller, depending on the parameter $\beta$. Let $\Sigma_A$ and $\Sigma_m$ be as above. We define $\mu_\beta$ on the semi-algebra of measurable intervals by

$$\begin{aligned} &\mu_\beta \left( \left\{ (x_i)_{i\in\mathbb{N}} \in \Sigma_A : x_q = a_1, \ldots, x_{q+k-1} = a_k, \text{ with } a_k \in \Sigma_m \text{ and } k \in \mathbb{N} \right\} \right) \\ &\quad = p_{a_1} s_{a_1 a_2} s_{a_2 a_3} \ldots s_{a_{k-1} a_k}. \end{aligned} \tag{3.4}$$

We call this measure the weighted Markov measure, associated with the weighted one-sided $(p_\beta, S_\beta)$-Markov subshift, supported by the subshift of finite type $(\Sigma_A, \sigma)$, compare with [7] and [19]. This invariant measure gives nonvanishing probabilities only for the trajectories staying in the repeller. In particular, the measure of maximal entropy or Parry measure to the subshift of finite type $(\Sigma_A, \sigma)$ is obtained with $\beta = 0$. When $\beta = \dim_H(E)$, we call this measure fractal Markov measure. The fractal Markov measure is the measure of maximal entropy to the fractal one-sided $\left(p_{\dim_H(E)}, S_{\dim_H(E)}\right)$-Markov subshift.

**Lemma 1** *The weighted one-sided $(p_\beta, S_\beta)$-Markov subshift has metric entropy*

*$h_{\mu_\beta}(F)$ and Lyapunov exponent $\chi_{\mu_\beta}(F)$ with respect to the measure $\mu_\beta$, given by*

$$h_{\mu_\beta}(F) = -\sum_{i,j=1}^{m} p_i\, s_{ij} \log(s_{ij}), \tag{3.5}$$

$$\chi_{\mu_\beta}(F) = \sum_{i=1}^{m} p_i \log\left(\left|F_i'\right|\right),$$

*where the derivative $F_i'$ is evaluated on the interval $J_i$ of the partition $\mathcal{P}_I'$.*

**Proof.** The proof of (3.5) is similar to Theorem 4.27 of [19] (see also [13]) considering the next adaptations. Let $\Gamma = \{C_1, \ldots, C_m\}$ be a partition of $\Sigma_m^{\mathbb{N}}$, defined by

$$C_i = \{(x_k) : x_0 = a_i \text{ with } 1 \leq i \leq m,\, k \in \mathbb{N}\}.$$

Let $\mathcal{B}$ be a $\sigma$-algebra of $I$ and $\mathcal{A}$ be a finite subalgebra of $\mathcal{B}$. The elements of the partition $\Gamma\left(\bigvee_{i=1}^{m} \sigma^{-i}\mathcal{A}\right)$ are

$$C_{i_1} \cap \sigma^{-1}C_{i_2} \cap \cdots \cap \sigma^{-(m-1)}C_{i_m} = \{(x_k) : x_1 = a_1, \ldots, x_m = a_m,\, k \in \mathbb{N}\}$$

where the measure is given by (3.4). Thus, according to the conditions of (3.2) and (3.3), the definition of product $\sigma$-algebra and by Kolmogorov-Sinai Theorem, we have (3.5).

We consider the Lyapunov exponent of $F$ at $x$ defined by

$$\chi_{\mu_\beta}(F(x)) := \lim_{n\to\infty} \frac{1}{n} \log\left|(F^n)'(x)\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\left|F'\left(F^j(x)\right)\right|.$$

According to the strong law of large numbers and [16], we have

$$\chi_{\mu_\beta}(F(x)) = \lim_{n\to\infty} \frac{1}{n} \log \prod_{k=1}^{n} \left|F_{i_k}'(x)\right| = \sum_{i=1}^{m} p_i \log\left(\left|F_i'(x)\right|\right). \qquad \blacksquare$$

Next we will show that the Hausdorff dimension is related with the metric entropy and with the Lyapunov exponent, with respect to the fractal Markov measure $\mu_{\dim_H(E)}$.

**Theorem 2** *Let $\left(p_{\dim_H(E)}, S_{\dim_H(E)}\right)$ be the fractal one-sided Markov subshift above defined, then*

$$\dim_H(E) = \frac{h_{\mu_{\dim_H(E)}}(F)}{\chi_{\mu_{\dim_H(E)}}(F)}.$$

**Proof.** The statement follows from Lemma 1, definition (3.4) with $\beta = \dim_H(E)$, the variational principle for the topological pressure and by the expansiveness of the map $F$. Thus, there exists a probability $F$-invariant measure. This measure is an equilibrium state for $\phi$, [8] and [19]. According to the variational principle for the Hausdorff dimension, [5], we have that the Hausdorff dimension is the unique real number such that $h_{\mu_\beta} - \beta\, \chi_{\mu_\beta} = 0$. ■

If the IFS has the same contraction ratios in modulus, we verify the next result. See [7] to similar result with respect to the Parry measure.

**Proposition 1** *If $\varphi_i(x) = \log |F_i'(x)|$ is constant on each interval $J_i$ of the partition $\mathcal{P}_I'$, then the metric entropy of the weighted one-sided $(p_\beta, S_\beta)$-Markov subshift, with respect to the measure $\mu_\beta$, is equal to the topological entropy of the subshift of finite type $(\Sigma_A, \sigma)$.*

**Proof.** If $\varphi_i(x) = \log |F_i'(x)|$ is constant on each $J_i$ of $\mathcal{P}_I'$, then the entries of $Q_\beta$ are $q_{ij} = a_{ij}\, |F'|^{-\beta}$. The largest eigenvalue of $Q_\beta$ satisfies $\lambda_\beta = \lambda_A \left|F_j'\right|^{-\beta}$, where $\lambda_A$ is the spectral radius of the transition matrix $A$. Substituting (3.2) and (3.3) into the formula (3.5) for the metric entropy, we have

$$\begin{aligned} h_{\mu_\beta}(F) &= -\sum_{i,j=1}^{m} u_i\, \tfrac{q_{ij}}{\lambda_\beta}\, v_j \,\log\left(\frac{a_{ij}\, v_j}{\left|F_j'\right|^{\beta} \lambda_\beta\, v_i}\right) \\ &= \sum_{i,j=1}^{m} u_i\, \tfrac{q_{ij}}{\lambda_\beta}\, v_j \,\log\left(\left|F_j'\right|^{\beta}\, \lambda_\beta\right) + \sum_{i,j=1}^{m} u_i\, \tfrac{q_{ij}}{\lambda_\beta}\, v_j\, \left(\log v_i - \log\left(a_{ij}\, v_j\right)\right). \end{aligned}$$

It follows from (3.1) and (3.3) that the first sum is equal to $\log \lambda_A$, *i.e.*, the topological entropy, [9]. On the other hand, according to (3.1) and to the definition of the transition matrix, the second sum is equal to zero. ■

# 4 Family of fractal sets

In this section, we consider a three-parameter family of expanding discontinuous maps with holes on the unit interval parametrized by $a$, $b$ and $c$, defined by

$$F_{a,b,c}(x) = \begin{cases} \frac{x}{a} & \text{if } x \in [0, y_1]\,, \\ \frac{x-y_1}{b} & \text{if } x \in\, ]y_1, y_2]\,, \\ \frac{x-y_3}{c} & \text{if } x \in [y_3, 1]\,, \end{cases}$$

where $\frac{y_2-y_1}{b}=\frac{1-y_3}{c}=1$, $\frac{z-y_1}{b}=z$ with $y_1<z<y_2$ and $z\,y_1^{-1}\leq a^{-1}\leq y_1^{-1}$. Note that, if $1<a^{-1}\leq z\,y_1^{-1}$, then the transition matrices are reducibles with topological entropy $\log 2$ . According to the value of the parameter $a^{-1}$, the orbit $o\left(y_1^-\right)$ can be periodic, eventually periodic, aperiodic or lie in the hole. To the point $y_1$, we consider the orbits of the lateral limit points $o\left(y_1^-\right)$ and $o\left(y_1^+\right)$. Set

$$\left\{y_1^-,y_1^+,y_2,y_3\right\}:=\left\{b_1,b_2,b_3,b_4\right\}.$$

We associate with the orbit of each point $b_i$, with $1\leq i\leq 4$, a sequence of symbols $S^{(i)}$ given by

$$S^{(i)}:=S_0^{(i)}S_1^{(i)}\dots S_k^{(i)}\dots,$$

where

$$S_k^{(i)}:=\begin{cases} L & \text{if } F_{a,b,c}^k\left(b_i\right)\in\left[0,y_1\right], \\ M & \text{if } F_{a,b,c}^k\left(b_i\right)\in\left]y_1,y_2\right], \\ H & \text{if } F_{a,b,c}^k\left(b_i\right)\in\left]y_2,y_3\right[, \\ R & \text{if } F_{a,b,c}^k\left(b_i\right)\in\left[y_3,1\right]. \end{cases}$$

We denote by $\mathcal{A}$ the ordered set of symbols corresponding to the laps and the hole of $F_{a,b,c}$ and according to the real line order, *i.e.*,

$$\mathcal{A}=\{L,\,M,\,H,\,R\}\ \text{ and } L\prec M\prec H\prec R.$$

If we consider values of $a^{-1}$ for which the orbits are periodic, and denote these orbits by $((L)\,S_1^{(1)}\dots\,S_{p-1}^{(1)})^\infty$ with period $p$, then the kneading data to $F_{a,b,c}$ are

$$(((L)\,S_1^{(1)}\dots\,S_{p-1}^{(1)})^\infty,\,(M)L^\infty,\,(M)R^\infty,\,(R)L^\infty).$$

The weighted kneading increment for the point $a_1$ is

$$\nu_{y_1}\left(t,\beta\right)=\theta_{y_1^+}\left(t,\beta\right)-\theta_{y_1^-}\left(t,\beta\right)=\theta_{b_2}\left(t,\beta\right)-\theta_{b_1}\left(t,\beta\right),$$

where

$$\theta_{y_1^\pm}\left(t,\beta\right):=\lim_{x\to y_1^\pm}\ \theta_x\left(t,\beta\right)=\sum_{k=0}^{\infty}\tau_k\left(y_1^\pm\right)\ S_k^{(i)}\,t^k$$

with $\tau_0\left(y_1^\pm\right):=1$ and for $k>0$

$$\tau_k\left(y_1^\pm\right):=\prod_{l=0}^{k-1}\varepsilon\left(S_l^{(i)}\right)\ \left|F'_{a,b,c}\left(F^l_{a,b,c}\left(y_1^\pm\right)\right)\right|^{-\beta}.$$

For more details see [11], where we develop a weighted kneading theory for expanding discontinuous maps with holes. Thus, we have

$$\begin{aligned}\theta_{y_1^-}(t,\beta) &= \tfrac{L+\tau_1\left(y_1^-\right)S_1^{(1)}\,t+\ldots+\tau_{p-1}\left(y_1^-\right)S_{p-1}^{(1)}\,t^{p-1}}{1-\tau_p\left(y_1^-\right)t^p},\\ \theta_{y_1^+}(t,\beta) &= \tfrac{b^\beta\,t}{1-a^\beta\,t}L+M.\end{aligned}$$

After separating the terms associated with the different symbols $\{L,\ M,\ R\}$, if we write

$$L_p := \sum_{\substack{i=1\\ S_i^{(1)}=L}}^{p-1} \tau_i\left(y_1^-\right)\ t^i$$

and analogously for $M_p$ and $R_p$, then we have

$$\theta_{y_1^-}(t,\beta) = \frac{1+L_p}{1-\tau_p\left(y_1^-\right)\ t^p}\,L + \frac{M_p}{1-\tau_p\left(y_1^-\right)\ t^p}\,M + \frac{R_p}{1-\tau_p\left(y_1^-\right)\ t^p}\,R.$$

Consequently, the weighted kneading increment for the point $y_1$ is

$$\begin{aligned}\nu_{y_1}(t,\beta) &= \left(\tfrac{b^\beta\,t}{1-a^\beta\,t} - \tfrac{1+L_p}{1-\tau_p\left(y_1^-\right)t^p}\right)L\\ &\quad + \left(1-\tfrac{M_p}{1-\tau_p\left(y_1^-\right)t^p}\right)M - \tfrac{R_p}{1-\tau_p\left(y_1^-\right)t^p}\,R.\end{aligned}$$

The weighted kneading increments for the points $y_2$ and $y_3$ are

$$\nu_{y_2}(t,\beta) = M + \frac{b^\beta\,t}{1-c^\beta\,t}\,R \text{ and } \nu_{y_3}(t,\beta) = \frac{c^\beta\,t}{1-a^\beta\,t}\,L + R.$$

The weighted kneading matrix of these kneading data is

$$N(t,\beta) = \begin{bmatrix} \tfrac{b^\beta\,t}{1-a^\beta\,t} - \tfrac{1+L_p}{1-\tau_p\left(y_1^-\right)t^p} & 1-\tfrac{M_p}{1-\tau_p\left(y_1^-\right)t^p} & \tfrac{-R_p}{1-\tau_p\left(y_1^-\right)t^p}\\ 0 & 1 & \tfrac{b^\beta\,t}{1-c^\beta\,t}\\ \tfrac{c^\beta\,t}{1-a^\beta\,t} & 0 & 1\end{bmatrix}.$$

The weighted kneading determinant is given by

$$\begin{aligned}D(t,\beta) &= \tfrac{1}{\left(1-a^\beta\,t\right)\left(1-c^\beta\,t\right)\left(1-\tau_p\left(y_1^-\right)t^p\right)}\left[1-\left(a^\beta+b^\beta+c^\beta\right)t + a^\beta c^\beta t^2\right.\\ &\quad + \left(1-a^\beta t - c^\beta t + a^\beta c^\beta t^2\right)L_p + b^\beta c^\beta t^2 M_p \\ &\quad \left. - \left(c^\beta t - c^{2\beta}t^2\right)R_p + b^\beta \tau_p\left(y_1^-\right)t^{p+1}\right].\end{aligned} \tag{4.1}$$

Note that, the weighted kneading determinant associated with an orbit that lies in the hole is the same, by removing the weighted cyclotomic polynomial $1 - \tau_p \left(y_1^-\right) t^p$. The next theorem relates the weighted kneading determinant with the characteristic polynomial of the matrix $Q_\beta$, which we will be denoted by $P_{Q_\beta}(t)$ [11].

**Theorem 3** *If the kneading data associated with an expanding discontinuous map with holes $F$ corresponds to periodic, eventually periodic orbits or to orbits that lie in the hole, then the weighted kneading determinant is given by*

$$D(t,\beta) = \frac{P_{Q_\beta}(t)}{R(t)},$$

*where $R(t)$ is a product of weighted cyclotomic polynomials corresponding to those periodic or eventually periodic orbits.*

The following statement will allows us to compute explicitly the Hausdorff dimension, the escape rate and the topological entropy.

**Theorem 4** *Let $D(t,\beta)$ be the weighted kneading determinant, under the conditions of the previous theorem.*

*(i) If $\beta$ is the unique solution of $D(1,\beta) = 0$, then $\beta$ is the Hausdorff dimension of the attractor $E$.*

*(ii) If $t_1$ is the least real positive solution of $D(t,1) = 0$, then $\log(t_1)$ is the escape rate of the pair $(E, F)$.*

*(iii) If $t_0$ is the least real positive solution of $D(t,0) = 0$, then $\log\left(t_0^{-1}\right)$ is the topological entropy of the map $F$.*

**Proof.** Considering the transfer operator given in (2.1), we have

$$\left(L_{\phi_j}\, g\right)(x) = \sum_{j=1}^{m} \left|F_j'(x)\right|^{-\beta} g\left(f_j(x)\right) \chi_{F(\operatorname{int} J_j)}.$$

Let $a_{ij}$ be the entries of the transition matrix $A$. For each $J_i \in \mathcal{P}_I'$, with $1 \le i \le m$, and $\beta \in \mathbb{R}$ the eigenvalue equation corresponding to an eigenvalue $\lambda_\beta$ is

$$\sum_{j=1}^{m} \frac{a_{ij}}{\left|F_j'(x)\right|^{\beta}} v_j = \lambda_\beta v_i$$

for the operator $L_\phi$ characterized by the matrix $Q_\beta$. According to [18] and using (2.3), the largest eigenvalue of the transfer operator is $\exp P_\beta(\phi)$. Hence, $\exp P_\beta(\phi)$ is the spectral radius $\lambda_\beta$ of the matrix $Q_\beta$.

If $\beta$ is the unique solution of $D(1,\beta)=0$, then by Theorem 1 and (2.3), we get $P_\beta(\phi)=0$. By [5] and [10], we can conclude that $\beta=\dim_H(E)$.

On the other hand, considering the parameter $\beta=1$, we have that $\lambda_1=\exp P_1(\phi)$ is the largest eigenvalue of the matrix $Q_1$. The second statement follows from [1], where the escape rate $\gamma$ is given by $\gamma=-P_1(\phi)$. Thus, the escape rate is $\gamma=\log\left(\lambda_1^{-1}\right)$, where $\lambda_1^{-1}=t_1$ is the least real positive solution of $P_{Q_1}(t)=0$.

If $\beta=0$, then the determinant $D(t,0)$ corresponds to the kneading determinant described in [9], where $t_0^{-1}=\lambda_0$ is the growth number of $F$, *i.e.*, the spectral radius of the transition matrix $A$. Consequently, $\log(\lambda_0)$ is the topological entropy of the map $F$. ■

Relative to this family of IFS's, the next statement give us a rule of construction of the characteristic polynomials between the periodic symbolic sequences of periods $p$ and $p+1$, with $p\geq 2$.

**Theorem 5** *Given a periodic sequence of period $p$ whose polynomial $P_{Q_\beta}(t)$ has degree $n=p+1$, then the polynomials corresponding to the periodic sequences of period $p+1$ and to the sequence that lies in the hole, level $p+1$, have the following rule of construction*

$$P_{Q_\beta}(t) \xrightarrow{L} P_{Q_\beta}(t)+\tau_p\,t^p-\left(a^\beta+b^\beta+c^\beta\right)\tau_p\,t^{p+1}+\left(a^\beta c^\beta\tau_p+b^\beta\tau_{p+1}\right)t^{p+2},$$

$$P_{Q_\beta}(t) \xrightarrow{M} P_{Q_\beta}(t)-b^\beta\tau_p\,t^{p+1}+\left(b^\beta c^\beta\tau_p+b^\beta\tau_{p+1}\right)t^{p+2},$$

$$P_{Q_\beta}(t) \xrightarrow{H} P_{Q_\beta}(t)-b^\beta\tau_p\,t^{p+1},$$

$$P_{Q_\beta}(t) \xrightarrow{R} P_{Q_\beta}(t)-\left(b^\beta+c^\beta\right)\tau_p\,t^{p+1}+\left(c^{2\beta}\tau_p+b^\beta\tau_{p+1}\right)t^{p+2}.$$

The proof is a consequence of Theorems 3 and 4, Eq. (4.1) and by analysis of the lexicographical order. The above theorem allows us to construct a tree of characteristic polynomials order with the variation of the parameter $a$.

**Remark 1** According to Theorem 1, we have $\dim_H(E)=\beta$, where $\beta$ is the unique solution of $P_{Q_\beta}(1)=0$. As a consequence of Theorems 1 and 5, we obtain the behavior of the Hausdorff dimension to the attractors of this family of IFS's, which is dependent on the contraction ratios, compare with [6] and [15]. We note that, the precise behavior is just obtained for periodic, eventually periodic orbits or for the orbits that lie in the hole. On the other hand, we also have the behavior of the topological entropy and the escape rate. The graph of the topological entropy depending on the parameters has a structure of a Cantor function.

**Remark 2** The theory presented in this paper with respect to periodic, eventually periodic orbits or to the orbits that lie in the holes is also valid for aperiodic orbits. In this case, the invariant coordinates associated with the turning points and with the discontinuity points are formal power series. The computation of the topological entropy, the Hausdorff dimension and the escape rate is done by approximation using: periodic, eventually periodic orbits or the orbits that lie in the holes.

Note that the parameter $a$ associated to each periodic sequence $((L)\, S_1^{(1)} \ldots S_{p-1}^{(1)})^\infty$ of period $p$ is given by $F_{a,b,c}^p(y_1) = y_1$. To the sequence $(L)\, S_1^{(1)} \ldots S_{p-1}^{(1)}\, H$, which lie in the hole, the interval of variation of the parameter $a$ is given by the condition

$$y_2 < F_{a,b,c}^p(y_1) < y_3.$$

Relative to the eventually periodic orbits, we obtain the topological entropy, the Hausdorff dimension and the escape rate using finite matrices. The next table shows examples of eventually periodic sequences to this family and the correspondent parameter $a^{-1}$, with $b^{-1} = 9$ and $c^{-1} = 3$, the topological entropy, the Hausdorff dimension and the escape rate.

| $a^{-1}$ | seq. event. perdc. | $h_{top}(F_{a,b,c})$ | $\dim_H(E)$ | $\gamma(E,F)$ |
|---|---|---|---|---|
| 1.125 | $(L)\, M^\infty$ | $\log(2)$ | 1 | 0 |
| 2.25 | $(L)\, RL^\infty$ | $\log(2.61803...)$ | 0.681446... | 0.405465... |
| 3.0 | $(L)\, RRL^\infty$ | $\log(2.87939...)$ | 0.736918... | 0.352695... |
| 3.375 | $(L)\, R^\infty$ | $\log(3)$ | 0.773916... | 0.300105... |

The next matrices are the weighted transition matrices associated with each eventually periodic sequence

$$Q_\beta((L)\, M^\infty) = \begin{bmatrix} a^\beta & b^\beta & 0 & c^\beta \\ a^\beta & b^\beta & 0 & c^\beta \\ 0 & 0 & b^\beta & c^\beta \\ 0 & 0 & b^\beta & c^\beta \end{bmatrix}, \qquad Q_\beta((L)\, RL^\infty) = \begin{bmatrix} a^\beta & b^\beta & 0 & c^\beta \\ a^\beta & b^\beta & 0 & c^\beta \\ a^\beta & b^\beta & 0 & c^\beta \\ 0 & b^\beta & 0 & c^\beta \end{bmatrix},$$

$$Q_\beta((L)\, RRL^\infty) = \begin{bmatrix} a^\beta & b^\beta & 0 & c^\beta & 0 \\ a^\beta & b^\beta & 0 & c^\beta & 0 \\ a^\beta & b^\beta & 0 & c^\beta & 0 \\ a^\beta & b^\beta & 0 & 0 & c^\beta \\ 0 & b^\beta & 0 & 0 & c^\beta \end{bmatrix}, \quad Q_\beta((L)\, R^\infty) = \begin{bmatrix} a^\beta & b^\beta & 0 & c^\beta \\ a^\beta & b^\beta & 0 & c^\beta \\ a^\beta & b^\beta & 0 & c^\beta \\ a^\beta & b^\beta & 0 & c^\beta \end{bmatrix}.$$

The next figures show the behavior of the growth rate of $F_{a,b,c}$, which is given by $\exp\left(h_{top}\left(F_{a,b,c}\right)\right)$, the Hausdorff dimension and the escape rate to this family, with

$$\frac{9}{8} \leq a^{-1} \leq \frac{27}{8},\ b^{-1} = 9 \text{ and } c^{-1} = 3.$$

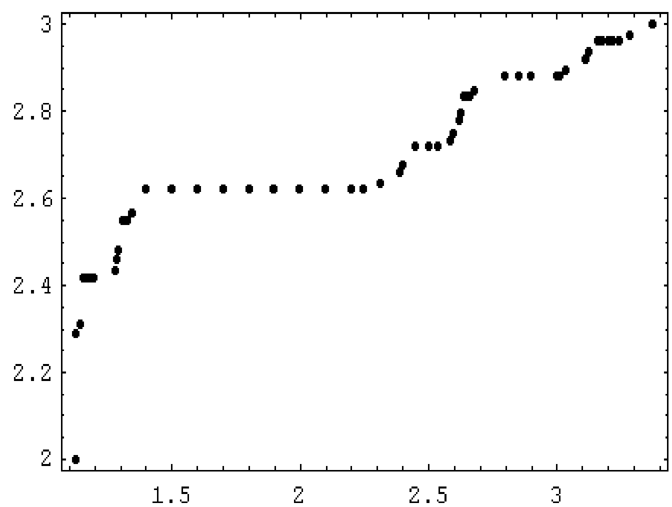

Figure 1: Growth rate of the map $F_{a,b,c}$ depending on the parameter $a$

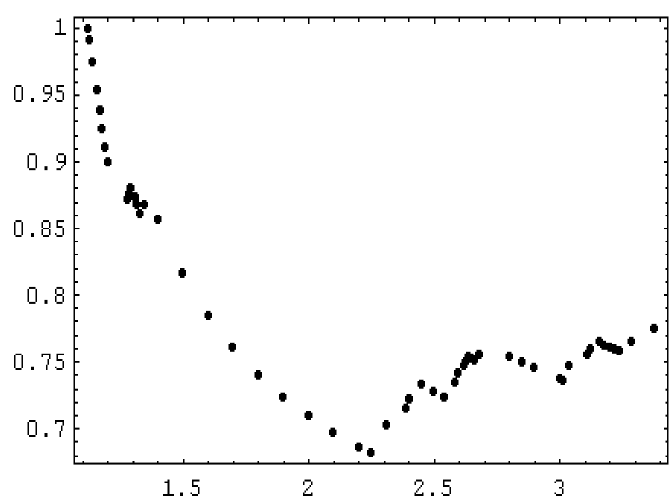

Figure 2: Hausdorff dimension of $E$ depending on the parameter $a$.

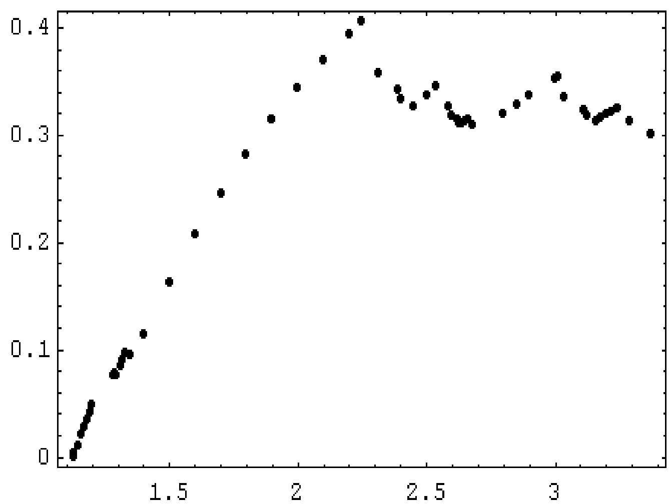


Figure 3: Escape rate of $(E, F)$ depending on the parameter $a$.

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