5–10 July 2004, Antalya, Turkey — Dynamical Systems and Applications, Proceedings, pp. 577–581

Boundedness of Weighted Composition Operators between Two Different L^p Spaces

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Abstract

In this paper we will consider the weighted composition operators uC_φ between two different $L^p(X, \Sigma, \mu)$ spaces and the boundedness of the weighted composition operators has been investigated in the $1 \leq p \leq q < \infty$ case.

AMS Subject Classification: Primary 47B20; Secondary 47B38.
Key words: Weighted composition operator, Conditional expectation, Multiplication operator.

1 Preliminaries and notation

Let (X, Σ, μ) be a sigma finite measure space. By $L(X)$, we denote the linear space of all Σ-measurable functions on X. When we consider any subsigma algebra $\mathcal{A}$ of Σ, we assume they are completed; *i.e.*, $\mu(A) = 0$ implies $B \in \mathcal{A}$ for any $B \subset A$. For any sigma finite algebra $\mathcal{A} \subseteq \Sigma$ and $1 \leq p \leq \infty$ we abbreviate the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ to $L^p(\mathcal{A})$, and denote its norm by $\| \cdot \|_p$. We define the support of a measurable function f as $\sigma(f) = \{x \in X;\ f(x) \neq 0\}$. We understand $L^p(\mathcal{A})$ as a subspace of $L^p(\Sigma)$ and as a Banach space. Here functions which are equal μ-almost everywhere are identical. An atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subset A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. It is easy to see that every $\mathcal{A}$- measurable function $f \in L(X)$ is constant μ- almost everywhere on A. So for each $f \in L(X)$ and each atom A we have

$$\int_A f \, d\mu = f(A)\mu(A).$$

A measure with no atoms is called *non-atomic*. We can easily check the following well known facts (see [14]):

(a) Every sigma finite measure space (X, Σ, μ) can be decomposed into two disjoint sets B and Z, such that μ is non-atomic over B and Z is a countable union of atoms of finite measure. So we can write X as follows:

$$X = B \cup (\cup_{n\in\mathbb{N}} A_n),$$

where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of disjoint atoms and B is a non-atomic set.

(b) Suppose that an $\mathcal{A}$- measurable set K is non-atomic and that $\mu(K) > 0$. Then, for any number a with $0 < a < \mu(K)$, there is an $\mathcal{A}$- measurable subset K_a of K such that $\mu(K_a) = a$.

(c) Suppose $1 \leq p < q < \infty$. If an $\mathcal{A}$- measurable set K is non-atomic and such that $\mu(K) > 0$, there exists a function $f_0 \in L^p(\mathcal{A})$ such that $\int_K |f_0|^q \, d\mu = \infty$.

Associated with each sigma algebra $\mathcal{A} \subseteq \Sigma$, there exists an operator $E(\cdot|\mathcal{A}) = E^{\mathcal{A}}(\cdot)$, which is called *conditional expectation* operator, on the set of all non-negative measurable functions f or for each $f \in L^p$ for any p, $1 \leq p \leq \infty$, and is uniquely determined by the conditions

(i) $E^{\mathcal{A}}(f)$ is $\mathcal{A}$- measurable, and

(ii) if A is any $\mathcal{A}$- measurable set for which $\int_A f \, d\mu$ exists, we have $\int_A f \, d\mu = \int_A E^{\mathcal{A}}(f) \, d\mu$.

This operator is at the central idea of our work, and we list here some of its useful properties:

E1. $E^{\mathcal{A}}(f \cdot g) = f \cdot E^{\mathcal{A}}(g)$, for all $f \in L^p(\mathcal{A})$.

E2. $E^{\mathcal{A}}(1) = 1$.

E3. $|E^{\mathcal{A}}(fg)|^2 \leq E^{\mathcal{A}}(|f|^2)E^{\mathcal{A}}(|g|^2)$.

E4. If $f > 0$ then $E^{\mathcal{A}}(f) > 0$.

Properties E1. and E2. imply that $E^{\mathcal{A}}(\cdot)$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. Suppose that φ is a mapping from X into X which is measurable, (*i.e.*, $\varphi^{-1}(\Sigma) \subseteq \Sigma$) such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ (we write $\mu \circ \varphi^{-1} \ll \mu$, as usual). Let h be the Radon-Nikodym derivative $h = \frac{d\mu\circ\varphi^{-1}}{d\mu}$. If we put $\mathcal{A} = \varphi^{-1}(\Sigma)$, it is easy to show that for each non-negative Σ-measurable function f or for each $f \in L^p(\Sigma)$ ($p \geq 1$), there exists a Σ-measurable function g such that $E^{\varphi^{-1}(\Sigma)}(f) = g \circ \varphi$. We can assume that the support of g lies in the support of h, and there exists only one g with this property. We then write $g = E^{\varphi^{-1}(\Sigma)}(f) \circ \varphi^{-1}$, though we make no assumptions regarding the invertibility of φ (see [3]). For a deeper study of the properties of E see the paper [8].

2 Some results on weighted composition operators between two L^p-spaces

In this section we characterize the functions u and transformations φ that induce weighted composition operators between L^p-spaces by using some properties of the conditional expectation operator, the pair (u, φ) and the measure space (X, Σ, μ). The results in the case $1 \leq p \leq q < \infty$ are apparently new.

Theorem 1 *Suppose* $1 \leq p < q < \infty$, $u \in L(X)$ *and* $\mathcal{K}_{p,q} = \mathcal{K}_{p,q}(\mathcal{A}, \Sigma) = \{u \in L(X) : uL^p(\mathcal{A}) \subseteq L^q(\Sigma)\}$. *Then* $u \in \mathcal{K}_{p,q}$ *if and only if* u *satisfies the following two conditions:*

(i) $E^{\mathcal{A}}(|u|^q) = 0$ *on* B.

(ii) $\sup_{n\in\mathbb{N}} \frac{\left(E^{\mathcal{A}}(|u(A_n)|^q)\right)^{\frac{s}{q}}}{\mu(A_n)} < \infty$, *where* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$.

Proof. Suppose that both (i) and (ii) hold. Put $b = \sup_{n\in\mathbb{N}} \frac{\left(E^{\mathcal{A}}(|u(A_n)|^q)\right)^{\frac{s}{q}}}{\mu(A_n)}$. Then, for each $f \in L^p(\mathcal{A})$, we have

$$\|u \cdot f\|_q^q = \int_X E^{\mathcal{A}}(|u|^q)|f|^q \, d\mu = \sum_{n\in\mathbb{N}} \int_{A_n} E^{\mathcal{A}}(|u|^q)|f|^q \, d\mu$$

$$= \sum_{n\in\mathbb{N}} \left(\frac{\left(E^{\mathcal{A}}(|u(A_n)|^q)\right)^{\frac{s}{q}}}{\mu(A_n)} \right)^{\frac{q}{s}} (|f(A_n)|^p \mu(A_n))^{\frac{q}{p}} \leq b^{\frac{q}{s}} \sum_{n\in\mathbb{N}} (|f(A_n)|^p \mu(A_n))$$

$$= b^{\frac{q}{s}} \sum_{n\in\mathbb{N}} \int_{A_n} |f|^p \, d\mu \leq b^{\frac{q}{s}} \int_X |f|^p \, d\mu \leq b^{\frac{q}{s}} \|f\|_p^p .$$

Hence $u \in \mathcal{K}_{p,q}$. Now suppose that $u \in \mathcal{K}_{p,q}$. So the operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ given by $M_u f = u \cdot f$ is a bounded linear operator. Assume that $\mu(\{x \in B : E^{\mathcal{A}}(|u(x)|^q) \neq 0\}) > 0$. Then there exists a positive number δ such that $\mu(\{x \in B : E^{\mathcal{A}}(|u(x)|^q) \geq \delta\}) > 0$. Put $K = \{x \in B : E^{\mathcal{A}}(|u(x)|^q) \geq \delta\}$. Since K is non-atomic, by (c) we can find $f_0 \in L^p(\mathcal{A})$ such that $\int_K |f_0|^q \, d\mu = \infty$. Then we have

$$\infty > \|M_u f_0\|_q^q \geq \int_K E^{\mathcal{A}}(|u|^q)|f_0|^q \, d\mu \geq \delta \int_K |f_0|^q \, d\mu = \infty ,$$

which is a contradiction. In other words, $E^{\mathcal{A}}(|u|^q) = 0$ on B. Now we prove that (ii) also holds. For any $n \in \mathbb{N}$ put $f_n = \frac{1}{\mu(A_n)^{\frac{1}{p}}} \chi_{A_n}$. It is clear that $f_n \in L^p(\mathcal{A})$ and

$\|f_n\|_p = 1$. Hence we have

$$\frac{\left(E^{\mathcal{A}}(|u(A_n)|^q)\right)^{\frac{1}{q}}}{\mu(A_n)^{\frac{1}{s}}} = \left\{\frac{1}{\mu(A_n)^{\frac{q}{p}}} E^{\mathcal{A}}(|u(A_n)|^q)\mu(A_n)\right\}^{\frac{1}{q}}$$

$$= \left\{\frac{1}{\mu(A_n)^{\frac{q}{p}}} \int_{A_n} E^{\mathcal{A}}(|u|^q)\, d\mu\right\}^{\frac{1}{q}} = \left\{\int_X E^{\mathcal{A}}(|uf_n|^q)\, d\mu\right\}^{\frac{1}{q}} = \|M_u f_n\|_q \leq \|M_u\| \ .$$

Since this holds for any $n \in \mathbb{N}$, it follows that $b \leq \|M_u\|^s < \infty$.

Corollary 2 *Suppose $1 \leq p < q < \infty$ and $u \in L(X)$. Then M_u from $L^p(\Sigma)$ into $L^q(\Sigma)$ is a bounded linear operator if and only if u satisfies the following two conditions:*

(i) $u = 0$ *on* B.

(ii) $\sup\limits_{n\in\mathbb{N}} \frac{|u(A_n)|^s}{\mu(A_n)} < \infty$, *where* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$.

Theorem 3 *Suppose $1 \leq p < q < \infty$, $u \in L(X)$ and $\varphi : X \to X$ is a non-singular measurable transformation. Then the pair (u, φ) induces a weighted composition operator uC_φ from $L^p(\Sigma)$ into $L^q(\Sigma)$ if and only if the following conditions hold:*

(i) $J = 0$ *on* B, *where* $J = hE^{\varphi^{-1}(\Sigma)}(|u|^q) \circ \varphi^{-1}$.

(ii) $\sup\limits_{n\in\mathbb{N}} \frac{|J(A_n)|^{\frac{s}{q}}}{\mu(A_n)} < \infty$, *where* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$.

Proof. Since $\|uC_\varphi f\|_q = \|M_{\sqrt[q]{J}} f\|_q$ for all $f \in L^p(\Sigma)$, so by Corollary 2 the theorem holds.

Corollary 4 *Under the same assumptions as in Theorem 3, φ induces a composition operator $C_\varphi : L^p(\Sigma) \to L^q(\Sigma)$ if and only if the following conditions hold:*

(i) $h = 0$ *on* B.

(ii) $\sup\limits_{n\in\mathbb{N}} \frac{|h(A_n)|^{\frac{s}{q}}}{\mu(A_n)} < \infty$, *where* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$.

Acknowledgment. The author would like to thank to the Prof. Jabbarzadeh for his comments and suggestions improving the contents of the paper.

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