Blow up of Solutions for a Class of Nonlinear Wave Equations

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Abstract
In this work, we study the blow up of solutions to the initial boundary value problem for a class of nonlinear wave equations with a damping term.

1 Introduction
In this work, we study the blow up of solutions of initial boundary value problem for a class of nonlinear wave equations with a damping term:

\begin{equation}
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \text{div} \sigma(\nabla u) + \Delta u_t - \Delta^2 u \quad \text{in} \quad \Omega \times (0, +\infty), \\
\left. u \right|_{\partial \Omega} &= 0, \quad \left. \frac{\partial u}{\partial \nu} \right|_{\partial \Omega} = 0 \quad \text{on} \quad (0, +\infty), \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{aligned}
\end{equation}

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a sufficiently smooth boundary \( \partial \Omega \), \( \nu \) is the outward normal to the boundary and \( \sigma(s) \) are given nonlinear functions.

The study of nonlinear evolution equations with linear damping or dissipative term has been considered by many authors; see [1]–[7]. In our study, we establish a blow up result for solutions with negative energy. The proof of our technique is similar to the one in [7].

2 Blow up of solution
For this purpose, we define

\begin{equation}
\begin{aligned}
E(t) &= \frac{1}{2} \| u(t) \|_2^2 + \frac{1}{2} \| \Delta u(t) \|_2^2 + \int_0^t \int_{\partial \Omega} \sigma(s) \cdot ds dx, \quad t \geq 0, 
\end{aligned}
\end{equation}

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where
\[ \sigma(s) = \nabla \omega(s), \quad \omega(s) \in C^1(\mathbb{R}^n), \quad s \in \mathbb{R}^n, \quad \sigma(s) \cdot s \leq k \int_{[0,s]} \sigma(\tau) \cdot d\tau \leq -k\beta|s|^{m+1}, \]

\[ \cdot \] denotes the dot product in \( \mathbb{R}^n \), the integrals in (4) and (5) are line integrals along arbitrary curves connecting 0 and \( \nabla u \) (respectively 0 and \( s \)) in \( \mathbb{R}^n \), \( k > 2 \) and \( \beta > 0 \) are constants, also \( 1 < m \leq 3 \).

**Theorem 1** Let \( u \) be the solution of problem (1) – (3). Assume that the following conditions are valid:

\[ u_0 \in H^2_0(\Omega), \quad u_1 \in L^2(\Omega), \]

\[ E(0) = \frac{1}{2} \| u_1 \|^2 + \frac{1}{2} \| \Delta u_0 \|^2 + \int_{\Omega} \int_{[0, \nabla u]} \sigma(s) \cdot ds \, dx < 0. \]

Then the solution \( u \) blows up in finite time

\[ T \leq \left\{ \begin{array}{ll}
\frac{3-m}{2} + \frac{3-m}{2} \int_{t_1}^{t} \frac{1}{C_8(a-1)b^{a-1}(t)} \, dt, & m < 3, \\
t_1 \cdot \exp \frac{1}{C_8(a-1)b^{a-1}(t_1)}, & m = 3,
\end{array} \right. \]

where \( t_1 \) and \( y \) will be defined respectively by (17) and (18), \( C_8 \) and \( \alpha > 1 \) are constants to be defined later.

**Proof.** By multiplying equation (1) by \( u_t \) and integrating the new equation over \( \Omega \), we obtain

\[ E'(t) + \| \nabla u(t) \|^2_2 = 0, \]

\[ E(t) \leq E(0) < 0, \quad t \geq 0. \]

Let

\[ F(t) = \| u(t) \|^2_2 + \int_0^t \| \nabla u(\tau) \|^2_2 \, d\tau, \]

then

\[ F'(t) = 2(u, u_t) + \| \nabla u(t) \|^2_2, \]

\[ F''(t) = 2 \left( \| u(t) \|^2_2 - \| \Delta u(t) \|^2_2 - \int_{\Omega} \sigma(\nabla u) \cdot \nabla u \, dx \right) \]

\[ \geq 2 \left( \| u(t) \|^2_2 - \| \Delta u(t) \|^2_2 - k \int_{[0, \nabla u]} \sigma(s) \cdot ds \, dx \right) \]

\[ \geq 2 \left( 2 \| u(t) \|^2_2 - (k - 2) \int_{[0, \nabla u]} \sigma(s) \cdot ds \, dx - 2E(0) \right) \]

\[ \geq 2 \left( 2 \| u(t) \|^2_2 + (k - 2)\beta \| \nabla u(t) \|^{m+1}_{m+1} - 2E(0) \right) , \quad t > 0, \]

(10)
where the assumption (5) and the fact that
\[
k\int_{\Omega} \int_{[0,\nabla u]} \sigma(s) \cdot ds \, dx \leq 2E(0) - \|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 + (k-2) \int_{\Omega} \int_{[0,\nabla u]} \sigma(s) \cdot ds \, dx
\]
have been used. Taking the inequality (10) and integrating this, we obtain
\[
F'(t) \geq 2(k-2)\beta \int_0^t \|\nabla u(\tau)\|_{m+1}^{m+1} \, d\tau - 4E(0)t + F'(0), \quad t > 0.
\] (11)
After this calculation, we could add the inequalities (10) with (11), then we get
\[
F''(t) + F'(t) \geq 4\|u_t(t)\|_2^2 + 2(k-2)\beta \left( \|\nabla u(t)\|_{m+1}^{m+1} + \int_0^t \|\nabla u(\tau)\|_{m+1}^{m+1} \, d\tau \right)
- 4E(0)(1 + t) + F'(0) = g(t), \quad t > 0.
\] (12)
Take \( p = \frac{m+3}{2} \), obviously \( 2 < p < m + 1 \) and \( p' = \frac{m+3}{m+1} \) (\(< 2\)). By using the Young inequality and the Sobolev-Poincaré inequality,
\[
|(u, u_t)| \leq \frac{1}{p} \|u(t)\|_p^p + \frac{1}{p'} \|u_t(t)\|_{p'}^{p'}
\leq C_1 \left( \|\nabla u(t)\|_{m+1}^m + \|u_t(t)\|_2^2 \right),
\]
\[
|(u, u_t)|^{1/\mu} \leq C_2 \left( \|\nabla u(t)\|_{m+1} + \|u_t(t)\|_2 \right), \quad t > 0,
\] (13)
where in this inequality and in the sequel \( C_i \) \( (i = 1, 2, \ldots) \) denote positive constants independent of \( t \), \( \mu = \frac{m+3}{2(m+1)} \) (< 1). By the Sobolev-Poincaré inequality and the Hölder inequality
\[
\|\nabla u(t)\|_{m+1}^m \geq C_3 \left( \|u(t)\|_2^{m+1} \right), \quad t > 0,
\] (14)
\[
\int_0^t \|\nabla u(\tau)\|_{m+1}^{m+1} \, d\tau \geq C_4 t^{1-\mu} \left( \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \right)^{\frac{m+1}{2}}.
\] (15)
By using the inequalities (13)–(15), we obtain
\[
g(t) \geq C_5 \left( 3\|\nabla u(t)\|_{m+1}^m + \|u_t(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_{m+1}^{m+1} \, d\tau \right) - 4E(0)t + F'(0)
\geq C_6 \left( \left( (u, u_t) \right)^{1/\mu} + \left( \|u(t)\|_2^{m+1} \right)^{\frac{m+1}{2}} + \left( \|\nabla u(t)\|_2^2 \right)^{\frac{m+1}{2}} + t^{\frac{1-m}{2}} \left( \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau \right)^{\frac{m+1}{2}} \right)
- 4E(0)t + F'(0)
\]
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\[ \geq C t^{\frac{1-m}{2}} \left( \| (u, u_t) \|^\alpha + \left( \| u(t) \|_2^2 \right)^\alpha + \left( \| \nabla u(t) \|_2^2 \right)^\alpha + \left( \int_0^t \| \nabla u(\tau) \|_2^2 d\tau \right)^\alpha \right) \]

\[ - 4E(0)t + F'(0) - C t^{\frac{1-m}{2}}, \quad t \geq 1, \tag{16} \]

where in this inequality and in the sequel \( \alpha = \frac{1}{\mu} > 1 \).

Since \(-4E(0)t + F'(0) - C t^{\frac{1-m}{2}} \to \infty \) as \( t \to \infty \), there must be a \( t_1 \geq 1 \) such that

\[- 4E(0)t + F'(0) - C t^{\frac{1-m}{2}} \geq 0 \quad \text{as} \quad t \geq t_1. \tag{17} \]

Let

\[ y(t) = F'(t) + F(t), \tag{18} \]

then from the inequality (11) and the equality (8) we obtain \( y(t) > 0 \) as \( t \geq t_1 \). By using the inequality

\[ (a_1 + \cdots + a_\ell)^n \leq 2^{(n-1)(\ell-1)}(a_1^n + \cdots + a_\ell^n), \]

where \( a_i \geq 0 \) \( (i = 1, \ldots, \ell) \) and \( n > 1 \) are real numbers, by virtue of (17) and using the inequality (16), we get

\[ g(t) \geq C s t^{\frac{1-m}{2}} y^\alpha(t), \quad t \geq t_1. \tag{19} \]

So combining (12) with (19) gives

\[ y(t) \geq C s t^{\frac{1-m}{2}} y^\alpha(t), \quad t \geq t_1. \tag{20} \]

Therefore, there exists a positive constant

\[ T = \begin{cases} \frac{\alpha}{2} - \frac{3-m}{2} t_1^{\frac{3-m}{2}} \left( \frac{2}{\alpha} \right)^{\frac{3-m}{2}} \left( t_1 \right)^{\frac{3-m}{2}} \alpha, & m < 3, \\ t_1 \cdot \exp \left( \frac{1}{\alpha} \right) & m = 3, \end{cases} \tag{21} \]

such that

\[ y(t) \to \infty \quad \text{as} \quad t \to T^- \tag{22} \]

By using (8), (9) and (22), we obtain

\[ 2\| u(t) \|_2^2 + \| u_t(t) \|_2^2 + \| \nabla u(t) \|_2^2 + \int_0^t \| \nabla u(\tau) \|_2^2 d\tau \geq F'(t) + F(t) \to \infty \quad \text{as} \quad t \to T^- \tag{23} \]

So (23) implies

\[ \| u(t) \|_2^2 + \| u_t(t) \|_2^2 + \| \nabla u(t) \|_2^2 + \int_0^t \| \nabla u(\tau) \|_2^2 d\tau \to \infty \quad \text{as} \quad t \to T^- \]

This completes the proof.
Example 1 Take $\sigma(s) = a|s|^{m-1}s$, where $a < 0$, $1 < m < 3$ are real numbers. Obviously $\sigma(s) = \nabla \omega(s)$, where $\omega(s) = \frac{a}{m+1}|s|^{m+1} \in C^1(\mathbb{R}^n)$, $s \in \mathbb{R}^n$. A simple verification shows that

$$\sigma(s) \cdot s = k \int_{[0,s]} \sigma(\tau) \cdot d\tau = -k|s|^{m+1},$$

where $k = m + 1 > 2$, $\beta = -\frac{a}{m+1} > 0$. If $u_0 \in H_0^2(\Omega)$, $u_1 \in L_2(\Omega)$ are such that $E(0) < 0$, then by Theorem 1 the solution of the corresponding problem (1) – (3) blows up in finite time.

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References


