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Endpoint Values of Wavelets on an Interval

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Abstract

The aim of this work is to find an approximative method for computing the scaling functions constructed at the endpoints of an interval, using the inner product in $L^2([0,1])$ of the scaling function and its derivative.

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Introduction

As it is known, on the interval $[0,1]$ the values of the scaling functions, the wavelets at the endpoints are not zero [2]. Unfortunately, there exists no method to calculate them due to the homogeneous algebraic system obtained by the relation (4) [5].

The idea of this work is to make a little detour, by making use of the scalar product in $L^2([0,1])$ of the scaling function constructed on an interval and its derivative, to solve the algebraic system obtained in order to have a better approximation of these functions at the endpoints 0 and 1.

Multiresolution analysis

A multiresolution analysis on $L^2([0,1])$ is given by an increasing sequence $\{V_j\}_{j\geq j_0}$, $j, j_0 \in \mathbb{Z}$, of closed subspaces of $L^2([0,1])$ satisfying the following properties:

$$\bigcup_{j=j_0}^{\infty} V_j \text{ is dense in } L^2([0,1]), \tag{1}$$

$$\forall f \in L^2([0,1]),\ \forall j, j_0 \in \mathbb{Z},\ j \geq j_0, \quad \text{we have } \ f(x) \in V_j \Rightarrow f(2x) \in V_{j+1}, \tag{2}$$

$$\{\Phi_{jk},\ j \geq j_0,\ k = 0,1,\ldots,2^j - 1\} = \{\phi^L_{jk},\ k = 0,1,\ldots,N-1\} \tag{3}$$
$$\cup\{\phi_{jk},\ k = N,\ldots,2^j - N - 1\} \cup \{\phi^R_{jk},\ k = -N,\ldots,-1\}$$

is a system representing an orthonormal basis of V_j, where

$$\phi^L_{jk}(x) = 2^{\frac{j}{2}}\phi^L_k(2^j x), \quad \phi_{jk}(x) = 2^{\frac{j}{2}}\phi(2^j x - k), \quad \phi^R_{jk}(x) = 2^{\frac{j}{2}}\phi^R_k(2^j x)$$

are, respectively, the scaling functions of the endpoint 0, the internal ones and of the endpoint 1, given by,

for $x \geq 0$, $k = 0,1,\ldots,N-1$, we have $\Phi_k(x) = \phi^L_k(x)$,

$$\phi^L_k(x) = \sqrt{2}\sum_{l=0}^{N-1} H^L_{k,l}\phi^L_l(2x) + \sqrt{2}\sum_{m=N}^{N+2k} h^L_{k,m}\phi(2x-m); \tag{4}$$

for $x \geq 0$, $k = N,\ldots,2^j - N - 1$, we have $\Phi_k(x) = \phi_k(x)$,

$$\phi_k(x) = \phi(x-k) = \sqrt{2}\sum_{q=-N+1}^{N} h_q\phi(2x - 2k - q); \tag{5}$$

for $x \leq 0$, $k = -N,\ldots,-1$, we have $\Phi(x) = \phi^R_k(x)$,

$$\phi^R_k(x) = \sqrt{2}\sum_{l=-N}^{-1} H^R_{k,l}\phi^R_l(2x) + \sqrt{2}\sum_{m=2k-N+1}^{-N-1} h^R_{k,m}\phi(2x-m). \tag{6}$$

In an analogous way, we define W_j, the complementary subspace of V_j in V_{j+1}, as the subspace generated by the orthonormal basis

$$\begin{aligned}\{\Psi_{jk},\ j &\geq j_0,\ k = 0,1,\ldots,2^j - 1\} = \{\psi^L_{jk},\ k = 0,1,\ldots,N-1\}\\ \cup\{\psi_{jk},\ k &= N,\ldots,2^j - N - 1\} \cup \{\psi^R_{jk},\ k = -N,\ldots,-1\},\end{aligned}$$

where

$$\psi^L_{jk}(x) = 2^{\frac{j}{2}}\psi^L_k(2^j x), \quad \psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \quad \psi^R_{jk}(x) = 2^{\frac{j}{2}}\psi^R_k(2^j x)$$

are, respectively, the wavelets of the endpoint 0, the internal ones and of the endpoint 1, satisfying the algebraic systems,

for $x \geq 0$, $k = 0,1,\ldots,N-1$, we have $\Psi_k(x) = \psi^L_k(x)$,

$$\psi^L_k(x) = \sqrt{2}\sum_{l=0}^{N-1} G^L_{k,l}\phi^L_l(2x) + \sqrt{2}\sum_{m=N}^{N+2k} g^L_{k,m}\phi(2x-m); \tag{4'}$$

for $x \geq 0$, $k = N, \ldots, 2^j - N - 1$, we have $\Psi_k(x) = \psi_k(x)$,

$$\psi_k(x) = \psi(x-k) = \sqrt{2} \sum_{q=-N+1}^{N} g_q \phi(2x - 2k - q); \tag{5'}$$

pour $x \leq 0$, $k = -N, \ldots, -1$, we have $\Psi(x) = \psi_k^R(x)$,

$$\psi_k^R(x) = \sqrt{2} \sum_{l=-N}^{-1} G_{k,l}^R \phi_l^R(2x) + \sqrt{2} \sum_{m=2k-N+1}^{-N-1} g_{k,m}^R \phi(2x - m). \tag{6'}$$

From the relation $V_{j+1} = V_j \oplus W_j$, and from (1), we obtain

$$V_J \ \oplus \ \bigoplus_{j=J}^{\infty} W_j = L^2([0,1]), \quad J \geq j_0, \quad 2^{j_0} \geq 2N,$$

which gives the expansion of any function f of $L^2([0,1])$,

$$f(x) = P_J f(x) + \sum_{j=J}^{\infty} Q_j f(x), \tag{7}$$

where P_J is the orthogonal projection of the function f onto V_J and Q_j its orthogonal projection onto W_j.

Moments of the scaling functions on an interval

Denote by $m_k^{L,i}$ the moment of order i of the function ϕ_k^L defined by

$$m_k^{L,i} = \int_0^{\infty} x^i \phi_k^L(x)\, dx.$$

From the scaling relation (4), we obtain for $i = 0$ the following algebraic system

$$m_k^{L,0} = \sqrt{2} \sum_{l=0}^{N-1} H_{k,l}^L \frac{m_k^{L,0}}{2} + \sqrt{2} \sum_{m=N}^{N+2k} h_{k,m}^L \frac{M_0}{2},$$

where $M_0 = 1$. In fact, we have $m \geq N$, hence

$$\int_0^{\infty} \phi(x-m)\, dx = \int_{-m}^{\infty} \phi(x)\, dx = \int_{-\infty}^{+\infty} \phi(x)\, dx = 1,$$

which implies that for different values of i, we have the following algebraic system [8],

$$2^i\sqrt{2}m_k^{L,i} = \sum_{l=0}^{N-1} H_{k,l}^L m_k^{L,i} + \sum_{m=N}^{N+2k} h_{k,m}^L \left(\sum_{j=0}^{i} \binom{i}{j} m^j M_{i-j}\right),$$

where M_p is the moment of order p of the scaling function on $\mathbb{R}$ and $\binom{i}{j} = \frac{i!}{j!(i-j)!}$.

In the same way, we find the moments $m_k^{R,i}$ of the scaling functions ϕ_k^R [8],

$$2^i\sqrt{2}m_k^{R,i} = \sum_{l=-N}^{-1} H_{k,l}^R m_k^{R,i} + \sum_{m=2k-N+1}^{-N-1} h_{k,m}^R \left(\sum_{j=0}^{i} \binom{i}{j} m^j M_{i-j}\right).$$

Applications

The moments of the scaling functions ϕ^L, ϕ^R et ϕ for Daubechies's wavelets D_4 with two vanishing moments [8, 9], with $i = 0, 1, 2$, are

i	$m_0^{L,i}$	$m_1^{L,i}$	$m_{-2}^{R,i}$	$m_{-1}^{R,i}$	M_i
0	0.3620521	1.001445	1.089843	1.295480	1.000000
1	-0.1509356	1.032428	-1.995769	-0.7217156	0.6339746
2	-0.3873851	1.166270	3.499148	0.5874012	0.4019238

The moments of the scaling functions on an interval
for $N = 2$ (D_4 Daubechies)

Scalar product of the scaling functions on an interval

Denote by θ_{jkl} the matrix of the scalar product $\langle \Phi'_{jk}, \Phi_{jl}\rangle$ given by

$$\begin{aligned}\theta_{jkl} &= \int_0^1 \Phi'_{jk}(x)\Phi_{jl}(x)\,dx = 2^j \int_0^{2^j} \Phi'_k(x)\Phi_l(x)\,dx \\ &= 2^j \int_0^{\infty} \Phi'_k(x)\Phi_l(x)\,dx = 2^j\theta_{kl},\end{aligned}$$

let Θ be the matrix defined by $\Theta = \theta_{kl}$, where

$$\theta_{kl} = \langle \Phi'_k, \Phi_l\rangle = \int_0^{\infty} \Phi'_k(x)\Phi_l(x)\,dx$$

The construction of the scaling functions on an interval shows us that the matrix $\Theta = \theta_{kl}$ depends on nine submatrices given as follows,

$$\Theta = \theta_{kl} = \begin{pmatrix} \theta^{LL} & \theta^{LI} & \theta^{LR} \\ \theta^{IL} & \theta^{II} & \theta^{IR} \\ \theta^{RL} & \theta^{RI} & \theta^{RR} \end{pmatrix},$$

$k = 0, 1, \dots, N-1$,
$l = 0, 1, \dots, N-1$,

$$\theta_{kl}^{LL} = \int_0^\infty \phi_k'^L(x)\phi_l^L(x)\,dx,$$

$k = 0, 1, \dots, N-1$,
$l = N, \dots, 2^j - N - 1$,

$$\theta_{kl}^{LI} = \int_0^\infty \phi_k'^L(x)\phi(x-l)\,dx,$$

$k = N, \dots, 2^j - N - 1$,
$l = 0, 1, \dots, N-1$,

$$\theta_{kl}^{IL} = \int_0^\infty \phi'(x-k)\phi_l^L(x)\,dx,$$

$k = N, \dots, 2^j - N - 1$,
$l = N, \dots, 2^j - N - 1$,

$$\theta_{kl}^{II} = \int_0^\infty \phi'(x-k)\phi(x-l)\,dx,$$

$k = N, \dots, 2^j - N - 1$,
$l = -N, \dots, -1$,

$$\theta_{kl}^{IR} = \int_{-\infty}^0 \phi'(x-k+2^j)\phi_l^R(x)\,dx,$$

$k = -N, \dots, -1$,
$l = N, \dots, 2^j - N - 1$,

$$\theta_{kl}^{RI} = \int_{-\infty}^0 \phi_k'^R(x)\phi(x-l+2^j)\,dx,$$

$k = -N, \dots, -1$,
$l = -N, \dots, -1$,

$$\theta_{kl}^{RR} = \int_{-\infty}^0 \phi_k'^R(x)\phi_l^R(x)\,dx.$$

The above relation between the scaling functions shows that the elements of the submatrices $\theta^{LL}, \theta^{LI}, \dots, \theta^{RR}$ depend on the elements of the internal block θ^{II} of the matrix.

Lemma 1 *The matrix $\Theta = \theta_{kl}$ is a band matrix with a half band width $2N-1$.*

In fact, due to the compact and disjoint supports of the functions of the endpoints ϕ_k^L, ϕ_k^R, we have

$$\theta^{LR} = \int_0^{\infty} \phi_k'^L(x)\phi_k^R(x)\,dx = 0,$$

as well as

$$\theta^{RL} = \int_{-\infty}^{0} \phi_k'^R(x)\phi_l^L(x)\,dx = 0.$$

Lemma 2 *The elements of the internal matrix θ^{II} are antisymmetric,*

$$\theta_{kl}^{II} = -\theta_{lk}^{II},$$

moreover, they satisfy the algebraic system

$$\theta_{kl}^{II} = \theta_{2k,2l}^{II} + \frac{1}{2}\sum_{r=1}^{N} d_{2r-1}(\theta_{2k,2l+2r-1}^{II} + \theta_{2k+2r-1,2l}^{II}). \tag{8}$$

In fact, applying to the expression

$$\theta_{kl}^{II} = \int_0^{\infty} \phi'(x-k)\phi(x-l)\,dx \tag{9}$$

an integration by parts, we obtain

$$\theta_{kl}^{II} = \phi(x-k)\phi(x-l)\mid_0^{\infty} - \int_0^{\infty} \phi'(x-l)\phi(x-k)\,dx = -\theta_{lk}^{II},$$

moreover, the scaling relation (5) applied to equation (9) leads us directly to system (8); found also in [1].

Lemma 3 *The elements of the matrix θ^{LI} satisfy the algebraic system*

$$\theta_{kl}^{LI} = 2\sum_{p=0}^{N-1}\sum_{q=-N+1}^{N} H_{kp}^{L} h_q \theta_{p,2l+q}^{LI} + 2\sum_{m=N}^{N+2k}\sum_{q=-N+1}^{N} h_{km}^{L} h_q \theta_{m,2l+q}^{II}. \tag{10}$$

Moreover, we have

$$\theta_{kl}^{LI} = -\theta_{lk}^{IL}.$$

In fact, let

$$\theta^{LI}_{kl} = \int_0^\infty \phi'^L_k(x)\phi(x-l)\,dx. \tag{11}$$

The scaling relations (4) and (5) applied to equation (11) give the system (10); an integration by parts of this equation leads us to the relation $\theta^{LI}_{kl} = -\theta^{IL}_{lk}$.

Let us note that the calculation of the elements de la matrix θ^{LI} is done in a simple and straightforward way, beginning with the elements θ^{LI}_{kl} such that $2l+q > 3N-3$, for all $k = 0, 1, \ldots, N-1$, and in an analogous way we find the matrices θ^{RI} and θ^{IR}.

Lemma 4 *The elements of the matrix θ^{LL} satisfy the algebraic system*

$$\begin{aligned}\theta^{LL}_{kl} &= 2\sum_{p=0}^{N-1}\sum_{q=0}^{N-1} H^L_{kp}H^L_{lq}\theta^{LL}_{pq} + 2\sum_{p=0}^{N-1}\sum_{s=N}^{N+2l} H^L_{kp}h^L_{ls}\theta^{LI}_{ps} \\ &+ 2\sum_{m=N}^{N+2k}\sum_{q=0}^{N-1} h^L_{km}H^L_{lq}\theta^{IL}_{mq} + 2\sum_{m=N}^{N+2k}\sum_{s=N}^{N+2l} h^L_{km}h^L_{ls}\theta^{II}_{ms}\,,\end{aligned} \tag{12}$$

moreover, we have the relation

$$\theta^{LL}_{kl} + \theta^{LL}_{lk} = -\phi^L_k(0)\phi^L_l(0).$$

In fact, let

$$\theta^{LL}_{kl} = \int_0^\infty \phi'^L_k(x)\phi^L_l(x)\,dx. \tag{13}$$

The scaling relation (4) applied to equation (13) gives the algebraic system (12) formed by $N \times N$ equations. The second term in (13) depends on elements of the matrices θ^{LI}, θ^{IL} and θ^{II}; found also in [5].

If we apply an integration by parts to the equation (13), we obtain

$$\theta^{LL}_{kl} = \phi^L_k(x)\phi^L_l(x)\mid_0^\infty - \int_0^\infty \phi'^L_l(x)\phi^L_k(x)\,dx,$$

thus

$$\theta^{LL}_{kl} = -\phi^L_k(0)\phi^L_l(0) - \theta^{LL}_{lk}.$$

Corollary 1 *The values of the scaling functions at the endpoint of an interval are given by*

$$\phi^L_k(0) = \pm\sqrt{-2\theta^{LL}_{kk}}. \tag{14}$$

It suffices to take $k = l$ in the above expression.
In the same way we find

$$\phi_k^R(0) = \pm\sqrt{2\theta_{kk}^{RR}}. \tag{14'}$$

We should notice that the calculation of the function ϕ_k^L at the point 0 using the scaling relation (4) is practically impossible, due to the homogeneity of the system of equations; on the contrary, the relation (14) gives the approximate values of the scaling function at the endpoint, of course, after having solved the algebraic system (12). The same reasoning applies to the function ϕ_k^R at the point 1.

Let

$$\theta^{LL} = \begin{pmatrix} \theta_{00}^{LL} & \theta_{01}^{LL} & \cdots & \theta_{0,N-1}^{LL} \\ \theta_{10}^{LL} & \theta_{11}^{LL} & \cdots & \theta_{1,N-1}^{LL} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{N-1,0}^{LL} & \theta_{N-1,1}^{LL} & \cdots & \theta_{N-1,N-1}^{LL} \end{pmatrix},$$

also, one has

$$\theta^{RR} = \begin{pmatrix} \theta_{2^j-N,2^j-N}^{RR} & \theta_{2^j-N,2^j-N+1}^{RR} & \cdots & \theta_{2^j-N,2^j-1}^{RR} \\ \theta_{2^j-N+1,2^j-N}^{RR} & \theta_{2^j-N+1,2^j-N+1}^{RR} & \cdots & \theta_{2^j-N+1,2^j-1}^{RR} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{2^j-1,2^j-N}^{RR} & \theta_{2^j-1,2^j-N+1}^{RR} & \cdots & \theta_{2^j-1,2^j-1}^{RR} \end{pmatrix}.$$

Applications

$$\theta_2^{LL} = \begin{pmatrix} -1.96344 & -1.52529 \\ 0.93546 & -0.04429 \end{pmatrix}, \quad \theta_2^{RR} = \begin{pmatrix} 0.08996 & -0.79412 \\ 0.31493 & 0.63712 \end{pmatrix}.$$

It is easy to see that the scaling functions ϕ_0^L and ϕ_1^L are positives in a neighbourhood of 0, while the functions ϕ_0^R and ϕ_1^R are of the opposite sign, hence from the relations (14) and (14′) we have

$$\begin{aligned} \phi_0^L(0) &= 1.981636, \\ \phi_1^L(0) &= 0.2976239, \\ \phi_{-2}^R(0) &= -0.4241698, \\ \phi_{-1}^R(0) &= 1.128822. \end{aligned}$$

For the different values of N (the number of vanishing moments of the wavelets on an interval), one proceeds in the same way to obtain the approximate values of the scaling functions at the endpoints. The values of the wavelets at the endpoints are deduced in a very simple way.

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