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Solving Integral Equations of the Second Kind by Using Wavelet Basis in the PG Method

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Abstract

In this paper, we use the Petrov–Galerkin (PG) method for solving Fredholm integral equations of the second kind whose trial space and test space are Alpert's multiwavelets. This method yields a linear system having numerically sparse coefficient matrices and their condition numbers are bounded. At last, for showing the efficiency of the method, we use numerical examples.

Keywords: Integral equations, The wavelet Petrov–Galerkin method, Regular pairs, Trial space, Test space.

1 Introduction

In this paper, we solve Fredholm integral equations of the second kind given in the form

$$u(t) - (Ku)(t) = f(t), \qquad t \in [0,1], \tag{1}$$

where

$$(Ku)(t) = \int_0^1 k(t,s)u(s)\,ds.$$

The function $f \in L^2[0,1]$, the kernel $k \in L^2([0,1]\times[0,1])$ are given and $u \in L^2[0,1]$ is the unknown function to be determined.

The Petrov–Galerkin method for equation (1) has been studied in [4]. We have seen from [4] that one of the advantages of the Petrov–Galerkin method is that it allows us to achieve the same order of convergence as the Galerkin method with much less computational cost by choosing the test spaces to be spaces of piecewise polynomials of lower degree than the trial space. In [5], we used continuous and discontinuous Lagrange type $k-0$ elements with $1 \le k \le 5$ for equation (1).

In [1], Alpert constructed a class of wavelet bases in $L^2[0,1]$ and applied it to approximate the solution of equation (1). The numerical method employed in [1] was the Galerkin method. In [2], the wavelet Petrov–Galerkin schemes based on discontinuous orthogonal multiwavelets were described. The results of this method yield integrals that are not solved easily and for this problem in [3] Discrete Wavelet Petrov–Galerkin (DWPG) method was described. In this paper we use Alpert's multiwavelets by using Petrov–Galerkin method for solving equation (1).

We organize this paper as follows. In Section 2, we review the Petrov–Galerkin method for equation (1). In Section 3, we describe the wavelet basis we use for the piecewise polynomials spaces considered here and, at last, in Section 4 we use the wavelet Petrov–Galerkin method with this multiwavelet basis for solving equation (1).

2 The Petrov–Galerkin method

In this section we follow the paper [4] with a brief review of the Petrov–Galerkin method.

Let X be a Banach space and X^* be its dual space of continuous linear functionals. For each positive integer n, we assume that $X_n \subset X$, $Y_n \subset X^*$ and X_n, Y_n are finite dimensional vector spaces with

$$\dim X_n = \dim Y_n, \quad n = 1, 2, \ldots \tag{2}$$

Also X_n, Y_n satisfy condition (H): For each $x \in X$ and $y \in X^*$, there exist $x_n \in X_n$ and $y_n \in Y_n$ such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$.

The Petrov-Galerkin method for equation (1) is a numerical method for finding $u_n \in X_n$ such that

$$(u_n - Ku_n, y_n) = (f, y_n) \quad \text{for all } \ y_n \in Y_n. \tag{3}$$

Definition. For $x \in X$, an element $P_n x \in X_n$ is called the generalized best approximation from X_n to x with respect to Y_n if it satisfies the equation

$$(x - P_n x, y_n) = 0 \quad \text{for all } \ y_n \in Y_n. \tag{4}$$

It is proved in [4] that for each $x \in X$ the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if

$$Y_n \cap X_n^{\perp} = \{0\}. \tag{5}$$

Under this condition, P_n is a projection, *i.e.*, $P_n^2 = P_n$ and equation (3) is equivalent to

$$u_n - P_n K u_n = P_n f. \tag{6}$$

Assume that, for each n, there is a linear operator $\Pi_n : X_n \to Y_n$ with $\Pi_n X_n = Y_n$ satisfying the following two conditions

(H-1) for all $x_n \in X_n$, $\|x_n\| \leq C_1(x_n, \Pi_n x_n)^{1/2}$,

(H-2) for all $x_n \in X_n$, $\|\Pi_n x_n\| \leq C_2\|x_n\|$.

If a pair of space sequences $\{X_n\}$ and $\{Y_n\}$ satisfies (H-1) and (H-2), we call $\{X_n, Y_n\}$ a *regular pair*. Then X_n and Y_n are respectively trial space and test space.

3 Alpert's multiwavelets

In this section, we follow the paper [1] with a brief review of the Alpert's wavelets.

For k a positive integer, and for $m = 0, 1, \ldots$ we define a space S_m^k of piecewise polynomials functions,

$$\begin{aligned} S_m^k = \{f : &\textit{the restriction of } f \textit{ to the interval } \ (2^{-m}n, 2^{-m}(n+1)) \\ &\textit{is a polynomial of degree less than } k, \textit{for } n = 0, 1, \ldots, 2^m - 1 \\ &\textit{and } f \textit{ vanishes elsewhere}\}. \end{aligned}$$

It is apparent than $\dim(S_m^k) = 2^m k$ and

$$S_0^k \subset S_1^k \subset \cdots \subset S_m^k \subset \cdots$$

The orthogonal complement of S_m^k in S_{m+1}^k is denoted by R_m^k so that $\dim(R_m^k) = 2^m k$ and

$$S_m^k \bigoplus R_m^k = S_{m+1}^k \ , \quad R_m^k \perp S_m^k \ .$$

Let $h_1, h_2, \ldots, h_k$ be an orthonormal basis for R_0^k, therefore, since R_0^k is orthogonal to S_0^k, the first k moments of $h_1, h_2, \ldots, h_k$ vanish, that is,

$$\int_0^1 h_j(x) x^i dx = 0, \quad i = 0, 1, \ldots, k-1.$$

The functions $h_1, h_2, \ldots, h_k$ for $k = 1, 2, 4$ are as follows:

— — — — — — — — — — — — — — — $k = 1$ — — — — — — — — — — — — — — —

$$h_1(x) = \begin{cases} -1, & 0 < x < .5, \\ \\ 1, & .5 < x < 1, \end{cases}$$

— — — — — — — — — — — — — — — $k = 2$ — — — — — — — — — — — — — — —

$$h_1(x) = \begin{cases} \sqrt{3}(4x-1), & 0 < x < .5, \\ \\ \sqrt{3}(3-4x), & .5 < x < 1, \end{cases} \qquad h_2(x) = \begin{cases} 6x-1, & 0 < x < .5, \\ \\ 6x-5, & .5 < x < 1, \end{cases}$$

— — — — — — — — — — — — — — — $k = 4$ — — — — — — — — — — — — — — —

$$h_1(x) = \begin{cases} \sqrt{\frac{15}{17}}(3 - 56x + 216x^2 - 224x^3), & 0 < x < .5, \\ \sqrt{\frac{15}{17}}(-61 + 296x - 456x^2 + 224x^3), & .5 < x < 1, \end{cases}$$

$$h_2(x) = \begin{cases} \frac{1}{\sqrt{21}}(-11 + 270x - 1320x^2 + 1680x^3), & 0 < x < .5, \\ \frac{1}{\sqrt{21}}(-619 + 2670x - 3720x^2 + 1680x^3), & .5 < x < 1, \end{cases}$$

$$h_3(x) = \begin{cases} \sqrt{\frac{35}{68}}(2 - 60x + 348x^2 - 512x^3), & 0 < x < .5, \\ \sqrt{\frac{35}{68}}(-222 + 900x - 1188x^2 + 512x^3), & .5 < x < 1, \end{cases}$$

$$h_4(x) = \begin{cases} \sqrt{\frac{5}{84}}(-2 + 72x - 492x^2 + 840x^3), & 0 < x < .5, \\ \sqrt{\frac{5}{84}}(-418 + 1608x - 2028x^2 + 840x^3), & .5 < x < 1. \end{cases}$$

— —

Therefore, we have

$$R_0^k = \text{Span}\,\{h_1, \ldots, h_k\} \tag{7}$$

and

$$R_m^k = \text{Span}\,\{h_{j,m}^n;\ j = 1, \ldots, k,\ n = 0, 1, \ldots, 2^m - 1\}, \tag{8}$$

where

$$h_{j,m}^n(x) = 2^{\frac{m}{2}} h_j(2^m x - n), \quad j = 1, \ldots, k, \ \ m, n \in \mathbb{Z}. \tag{9}$$

Let $\{u_1, \ldots, u_k\}$ be the orthonormal Legendre polynomials adjusted to the interval [0,1], then for a fixed value of m

$$B_k = \{b_j\}_{j=1}^{2^m k} =$$
$$\{u_j,\ j = 1, \ldots, k\} \cup \{h_{j,p}^n :\ p = 0, 1, \ldots, m-1,\ n = 0, 1, \ldots, 2^p - 1,\ j = 1, \ldots, k\}$$

is an orthonormal system for S_m^k.

4 The wavelet Petrov–Galerkin method

In this method, we choose $X_n = S_m^k$ as trial space and $Y_n = S_{m'}^{k'}$ as test space where k, k', m, m' are positive integers such that $k' < k$ and $n = 2^m k = 2^{m'} k'$. This condition is equivalent to $\frac{k}{k'} = 2^q$ and $m' = m + q$ for some non-negative integer q. If $q = 0$, we have the Galerkin method.

Now, assume $u_n \in X_n$ and $\{b_i\}_{i=1}^n$ is a basis for X_n and $\{b_j^*\}_{j=1}^n$ is a basis for Y_n. Therefore the Petrov–Galerkin method on $[0, 1]$ for equation (1) is

$$(u_n - Ku_n, b_j^*) = (f, b_j^*), \qquad j = 1, \ldots, n. \tag{10}$$

Let $u_n(t) = \sum_{i=1}^n a_i b_i(t)$ and the equation (1) leads to determining $\{a_1, a_2, \cdots, a_n\}$ as the solution of the linear system

$$\sum_{i=1}^n a_i \left\{ \int_0^1 b_i(t) b_j^*(t)\, dt - \int_0^1 \int_0^1 K(s,t) b_i(s) b_j^*(t)\, ds\, dt \right\}$$
$$= \int_0^1 f(t) b_j^*(t)\, dt, \qquad j = 1, \ldots, n. \tag{11}$$

In the sequel, we test this method by an example.

Example.

$$u(t) - \int_0^1 (-\frac{1}{3} e^{2t-5s/3}) u(s)\, ds = e^{2t+1/3}, \qquad 0 \le t \le 1,$$

with exact solution $u(t) = e^{2t}$. In the following tables we computed $\|u_n(t) - u(t)\|_2$ for different k, k', m, m' such that $k' < k$ and $\frac{k}{k'} = 2^q$ and $m' = m + q$.

m	m'	$k = 2, k' = 1$	$k = 4, k' = 2$
1	2	0.0164051	0.000339447
2	3	0.00422859	0.0000223377
3	4	0.00172423	$3.40864 * 10^{-6}$
4	5	0.00175073	$3.48285 * 10^{-6}$

m	m'	$k = 4, k' = 1$
1	3	0.0000235729
2	4	$3.07707 * 10^{-6}$
3	5	$3.03923 * 10^{-6}$
4	6	$3.09844 * 10^{-6}$

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