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Application of the Haar Wavelets for Solution of Linear Integral Equations

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Abstract

An efficient algorithm based on the Haar wavelet approach for numerical solution of linear integral equations is proposed. The method is applicable for different kinds of integral equations (Fredholm and Volterra equations, integro-differential equations, weakly singular integral equations). Five test problems, for which the exact solution is known, are considered. The calculations indicate that the accuracy of the obtained solutions is quite high even when the number of calculation points is small.

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Key words: Integral equations; Haar wavelets; Galerkin method; Collocation method.

1 Introduction

For numerical solution of linear integral equations traditional quadrature formula methods and spline approximations are used. In the case of these methods systems of linear equations must be solved. For big matrices this requires a huge number of arithmetic operations and a large storage capacity. A lot of computing time is saved if we succeed in replacing the fully populated transform matrix with a sparse matrix. One possibility for this gives the wavelet method; the wavelet bases lead to a sparse matrix representation since

(i) the basis functions are usually orthogonal;

(ii) most of the functions have a small interval of support.

Presumably the first paper in which the wavelet method was applied for solving integral equations belongs to Beylkin *et al.* [1] from 1991. After that important

contributions to this subject are due to many authors (*cf.*, *e.g.*, [2]–[18]). That the interest to the wavelet approach has lasted up to the immediate past indicate the papers [14]–[18] which were published in 2003.

In [1]–[18] mostly Fredholm and Volterra integral equations are discussed. In [3, 6, 8, 17] also weakly singular equations and in [16] the Hammerstein equation are analyzed. Solution of integro-differential equations is considered in [9, 17]. In the majority of these papers the continuous operator is discretized into matrix form by Galerkin's procedure; in [13, 18] the collocation method is used.

Various wavelet bases are applied. In addition to the conventional Daubechies wavelets the adaptive Battle-Lemarie wavelets [7], Hermite-type trigonometric wavelets [12], Haar wavelets [13], linear B-splines [14], Walsh functions [17], Cohen [10] and Albert [16] wavelets have been used.

In our opinion insufficient attention is paid to the Haar functions, which are mathematically the most simple wavelets. Probably the reason for this is the fact that Haar wavelets are discontinuous. Nevertheless Chen and Hsiao [19, 20] demonstrated that these wavelets can be successfully applied for solving differential equations, since here very sparse transform matrices have been obtained. This approach was developed further in [21].

The main aim of the present paper is to propose for numerical solution of integral equations a simple method based on the Haar wavelets. The recommended technique is applicable with minor changes to Fredholm, Volterra and integro-differential equations; also singular integral equations can be treated in the same way. The method is tested with the aid of the five numerical examples, for which the exact solution is known. Error estimates show that the accuracy of computations is very high even when the mode number is small.

2 Haar wavelet method

Let us confine to the time interval $t \in [0, 1]$. The Haar wavelet family is

$$h_i(t) = \begin{cases} 1 & \text{for } t \in [\tau_1, \tau_2)\,, \\ -1 & \text{for } t \in [\tau_2, \tau_3)\,, \\ 0 & \text{elsewhere}\,. \end{cases} \tag{2.1}$$

Here the notations

$$\tau_1 = \frac{k}{m}\,, \quad \tau_2 = \frac{k+0.5}{m}\,, \quad \tau_3 = \frac{k+1}{m} \tag{2.2}$$

are introduced. The integer $m = 2^j$, $j = 0, 1, \ldots, J$, indicates the level of the wavelet; $k = 0, 1, \ldots, m-1$ is the translation parameter. The integer J determines

the maximal level of resolution. The index i is calculated from the formula $i = m + k + 1$; the minimal value for which (2.1) holds is $i = 2$ (then $m = 1$, $k = 0$); the maximal value is $i = 2M$ where $M = 2^J$. The index $i = 1$ corresponds to the scaling function of the Haar wavelet $h_1(t) \equiv 1$.

Simple calculations show that

$$\int_0^1 h_i(t)h_l(t)dt = \begin{cases} 1/m & \text{for } i = l\,, \\ 0 & \text{for } i \neq l\,, \end{cases} \tag{2.3}$$

consequently, the functions $h_i(t)$ are orthogonal.

Next we discretize the functions $h_i(t)$ by dividing the interval $t \in [0, 1]$ into $2M$ parts of equal length $\Delta t = 1/(2M)$ and introduce the collocation points

$$t_l = (l - 0.5)/(2M)\,, \quad l = 1, 2, \ldots, 2M\,. \tag{2.4}$$

Following Chen and Hsiao [19, 20] the coefficients matrix $H_{il} = h_i(t_l)$ is introduced (this is a $2M \times 2M$ matrix). A function $u(t)$ which is defined in the interval $t \in [0, 1]$ can be expanded into the Haar wavelet series:

$$u(t) = \sum_{i=1}^{2M} a_i h_i(t)\,, \tag{2.5}$$

where a_i are the wavelet coefficients. The discrete form of this equation is

$$u(t_l) = \sum_{i=1}^{2M} a_i h_i(t_l) = \sum_{i=1}^{2M} a_i H_{il}\,,$$

or in a matrix presentation $u = aH$ where u and a are $2M$ dimensional row vectors.

3 Fredholm integral equations

A linear Fredholm integral equation has the form

$$u^*(x^*) - \int_\alpha^\beta K^*(x^*, t^*)u^*(t^*)\,dt^* = f^*(x^*), \quad x^* \in [\alpha, \beta]\,,$$

where the kernel K^* and the right-hand side function f^* are prescribed. Since the Haar wavelets are defined only for $t \in [0, 1]$, the transformation

$$t^* = (\beta - \alpha)t + \alpha, \qquad x^* = (\beta - \alpha)x + \alpha \tag{3.1}$$

must be accomplished. By doing this we find

$$u(x) - \int_0^1 K(x,t)u(t)\,dt = f(x), \quad x \in [0,1]\,, \tag{3.2}$$

where $u(t) = u^*(t^*)$, $u(x) = u^*(x^*)$, $f(x) = f^*(x^*)$, $K(x,t) = (\beta - \alpha)K^*(x^*, t^*)$. The formula (3.1) is applicable only if α and β are finite (the cases $\alpha \to -\infty$ or $\beta \to \infty$ are not considered in this paper).

Replacing (2.5) into (3.2) we find

$$\sum_{i=1}^{2M} a_i h_i(x) - \sum_{i=1}^{2M} a_i G_i(x) = f(x)\,, \tag{3.3}$$

where

$$G_i(x) = \int_0^1 K(x,t)h_i(t)\,dt\,. \tag{3.4}$$

Next we shall evaluate the wavelet coefficients a_i in the following two ways:

(i) Collocation method. Satisfying (3.3) only at the collocation points (2.4) we get a system of linear equations

$$\sum_{i=1}^{2M} a_i[h_i(x_l) - G_i(x_l)] = f(x_l)\,, \quad l = 1, 2, \ldots, 2M\,. \tag{3.5}$$

The matrix form of this system is

$$a(H - G) = F \tag{3.6}$$

where $G_{il} = G_i(x_l)$, $F_l = f(x_l)$.

(ii) Galerkin method. For realizing this approach each term of (3.3) is multiplied by $h_l(x)$ and the result is integrated over $x \in [0,1]$. Due to the orthogonality condition (2.3) we obtain

$$\frac{a_l}{m_1} - \sum_{i=1}^{2M} a_i \Gamma_{il} = \int_0^1 f(x)h_l(x)\,dx\,. \tag{3.7}$$

Here $l = m_1 + k_1 + 1$, $m_1 = 2^{j_1}$, $j_1 = 0, 1, \ldots, J$, $k_1 = 0, 1, \ldots, m_1 - 1$ and

$$\Gamma_{il} = \int_0^1 G_i(x)h_l(x)\,dx\,. \tag{3.8}$$

Example 3.1 Let us solve Eq. (3.2) for $K(x,t) = x + t$, $f(x) = x^2$. Taking into account (2.1) and evaluating the integrals (3.4) we find

$$G_i(x) = \begin{cases} x + 0.5 & \text{for } i = 1\,, \\ -\frac{1}{4m^2} & \text{for } i > 1\,. \end{cases}$$

If we apply the collocation method, then the vector a can be calculated from the system (3.6).

In the case of the Galerkin method we shall go out from (3.7).

Evaluating the integrals (3.8) we obtain

$$\Gamma_{11} = 1\,, \quad \Gamma_{1l} = -\tfrac{1}{4m_1^2} \quad \text{for} \quad l > 1\,,$$
$$\Gamma_{i1} = -\tfrac{1}{4m^2} \quad \text{for} \quad i > 1\,, \qquad \Gamma_{il} = 0 \quad \text{for} \quad i, l > 1\,.$$

It follows from (3.7) that

$$\sum_{i=2}^{2M} \frac{a_i}{m^2} = \frac{4}{3} \quad \text{for} \quad l = 1\,, \tag{3.9}$$

$$a_l = -\frac{a_1}{4m_1} - \frac{1}{4m_1^2}(2k_1 + 1) \quad \text{for} \quad l > 1\,. \tag{3.10}$$

Replacing a_l from (3.10) into (3.9) we get

$$a_1 \sum_{i=2}^{2M} \frac{1}{m^3} + \sum_{i=2}^{2M} \frac{2k+1}{m^4} = -\frac{16}{3}\,. \tag{3.11}$$

Since

$$\begin{aligned} \sum_{i=2}^{2M} \frac{1}{m^3} &= 1 + \left(\frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64}\right) + \cdots + 2^J \frac{1}{2^{3J}} \\ &= 1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{2J}} = \frac{4}{3}\left(1 - \frac{1}{(2M)^2}\right) \end{aligned}$$

and

$$\sum_{i=2}^{2M} \frac{2k+1}{m^4} = 1 + \sum_{k=0}^{1} \frac{2k+1}{2^4} + \sum_{k=0}^{2} \frac{2k+1}{2^8} + \cdots + \sum_{k=0}^{M-1} \frac{2k+1}{2^{4J}} = \frac{4}{3}\left(1 - \frac{1}{(2M)^2}\right),$$

it follows from (3.11) that

$$a_1 = -\frac{5 + (2M)^{-2}}{1 - (2M)^{-2}}\,.$$

The coefficients a_l for $l > 1$ can be calculated according to (3.10); the function $u(x)$ is evaluated from (2.5).

Computations were carried out for different values of J. These results were compared with the exact solution

$$u_{ex} = x^2 - 5x - 17/6\,.$$

The accuracy of the results was estimated by the error function

$$e_J = \max_{1\le l\le 2M}(|u(t_l) - u_{ex}(t_l)|)\,, \tag{3.12}$$

where t_l is defined by (2.4).

The errors e_J both for collocation and Galerkin method are shown in Table 1.

Table 1. Errors of solving (3.2) for $K = x + t$, $f(x) = x^2$.

		e_J	
J	$2M$	collocation	Galerkin
2	8	$7.2E-2$	$1.1E-1$
3	16	$1.7E-2$	$2.6E-2$
4	32	$4.3E-3$	$6.5E-3$
5	64	$1.3E-3$	$1.6E-3$

This example demonstrates the simplicity of the Haar wavelet solution: for evaluating the wavelet coefficients in the case of the collocation method a system of linear equations must be solved; as to the Galerkin method then analytic expressions for these coefficients were obtained. For getting the necessary accuracy the number of calculation points is quite small.

4 Eigenvalues and eigenfunctions

Consider the homogeneous Fredholm equation

$$u(x) - \lambda \int_0^1 K(x,t)u(t)\,dt = 0\,, \tag{4.1}$$

where λ is a parameter. Equation (3.6) obtains now the form $a(H - \lambda G) = 0$. This system has a nontrivial solution only for some values of λ (eigenvalues). The corresponding functions $u(x)$ are eigenfunctions.

The method of solution presented in Sec. 3 is still applicable also in this case. The details are explained by means of the following example.

Example 4.1 Here the equation

$$u^*(x^*) - \lambda \int_0^{\pi} \cos(x^* + t^*) u^*(t^*)\, dt^* = 0$$

is solved. By the change of variables $x^* = \pi x$, $t^* = \pi t$, $u^*(x^*) = u(x)$ this equation can be put into the form

$$u(x) - \pi\lambda \int_0^1 \cos \pi(x + t) u(t)\, dt = 0\,. \tag{4.2}$$

Carrying out the integration in (3.4) we obtain

$$G_1(x) = -\frac{2}{\pi} \sin \pi x \tag{4.3}$$

and

$$G_i(x) = \frac{1}{\pi}\big\{2\sin[\pi(x + \tau_2)] - \sin[\pi(x + \tau_1)] - \sin[\pi(x + \tau_3)]\big\}\,.$$

Making use of (2.2) and applying trigonometric transformations, this equation can be rewritten in the form

$$G_i(x) = \frac{4}{\pi} \sin^2 \frac{\pi}{4m} \sin\left[\pi\left(x + \frac{2k+1}{2m}\right)\right], \quad i = 2, 3, \ldots, 2M\,. \tag{4.4}$$

If we want to apply the collocation method then again the matrix $G_{il} = G_i(x_l)$ is formed. For solving the matrix equation $a(H - \pi\lambda G) = 0$ the MATLAB program EIG was used.

Now let us solve (4.2) with the Galerkin method. This procedure leads to the equation

$$\sum_{i=1}^{2M} a_i \int_0^1 h_i(x) h_l(x)\, dx = \pi\lambda \sum_{i=1}^{2M} a_i \Gamma_{il}\,,$$

which is view of (2.3) gets the form

$$\frac{a_l}{m_1} = \pi\lambda \sum_{i=1}^{2M} a_i \Gamma_{il}\,, \quad l = 1, \ldots, 2M\,. \tag{4.5}$$

Taking into consideration (3.8) we get

$$\begin{aligned}
&\Gamma_{11} = -\frac{4}{\pi^2}\,,\\
&\Gamma_{1l} = \frac{8}{\pi^2}\sin^2\frac{\pi}{4m_1}\cos\left(\pi\frac{2k_1+1}{2m_1}\right), \quad l = k_1 + m_1 + 1 > 1\,,\\
&\Gamma_{i1} = \frac{8}{\pi^2}\sin^2\frac{\pi}{4m}\cos\pi\left(\frac{2k+1}{2m}\right), \quad i = k + m + 1 > 1\,,\\
&\Gamma_{il} = -\frac{16}{\pi^2}\sin^2\frac{\pi}{4m}\sin^2\frac{\pi}{4m_1}\cos\left[\pi\left(\frac{2k+1}{2m}+\frac{2k_1+1}{2m_1}\right)\right],\ i,l > 1\,.
\end{aligned} \tag{4.6}$$

For solving the system again the EIG program was used (this program calculates also the eigenfunctions).

The exact solution of the problem is

$$\lambda_1 = \frac{2}{\pi}\,, \quad \lambda_2 = -\frac{2}{\pi}\,, \quad u_1(x) = \cos\pi x\,, \quad u_2(x) = \sin\pi x\,,$$

where u_1, u_2 are normed eigenfunctions.

Some numerical results obtained by the wavelet method are shown in Table 2. The eigenvalues $\lambda_1 = -\lambda_2$ (and their error in percents) are calculated both by the collocation and Galerkin methods. In the last column the error function (3.12) of the normed eigenfunctions is given (it turned out that the error was the same both for the collocation and Galerkin approach). Again high accuracy even with a small number of calculation points is stated.

Table 2. Eigenvalues of Eq. (4.2) and error of the normed eigenfunctions.

		$\lambda_1 = -\lambda_2$		
J	$2M$	collocation	Galerkin	e_J
2	8	0.6407 (0.64%)	0.6449 (1.3%)	$1.9E-2$
3	16	0.6376 (0.15%)	0.6387 (0.32%)	$4.8E-3$
4	32	0.6369 (0.04%)	0.6371 (0.07%)	$1.2E-3$
5	64	0.6367 (0.01%)	0.6367 (0.01%)	$3.0E-4$

5 Volterra integral equation

The Volterra integral equation is

$$u(x) - \int_0^x K(x,t)u(t)\,dt = f(x)\,, \quad 0 \le x \le 1\,. \tag{5.1}$$

Its discrete form is

$$u(x_l) - \int_0^{x_l} K(x_l,t)u(t)\,dt = f(x_l)\,, \tag{5.2}$$

where $x_l = (l-0.5)/(2M)$, $l = 1,2,\ldots,2M$, are the collocation points.

We proceed as in Sec. 3, Eq. (3.5) holds. The matrix $G_{il} = G_i(x_l)$ is now defined as

$$G_{il} = \int_0^{t_l} K(x_l,t)h_i(t)\,dt\,. \tag{5.3}$$

By computing these integrals the following cases should be distinguished:

$$\begin{aligned}
&\textbf{(i)} \quad G_{il} = 0 \quad \text{if} \quad x_l < \tau_1\,,\\
&\textbf{(ii)} \quad G_{il} = \int_{\tau_1}^{x_l} K(x_l,t)\,dt \quad \text{if} \quad \tau_1 \le x_l \le \tau_2\,,\\
&\textbf{(iii)} \quad G_{il} = \int_{\tau_1}^{\tau_2} K(x_l,t)\,dt - \int_{\tau_2}^{x_l} K(x_l,t)\,dt \quad \text{if} \quad \tau_2 \le x_l \le \tau_3\,,\\
&\textbf{(iv)} \quad G_{il} = \int_{\tau_1}^{\tau_2} K(x_l,t)\,dt - \int_{\tau_2}^{\tau_3} K(x_l,t)\,dt \quad \text{if} \quad \tau_3 \le x_l \le 1\,,
\end{aligned} \tag{5.4}$$

where the quantities τ_1, τ_2, τ_3 are defined with the formulas (2.2).

In this and in the following sections the Galerkin solution is not considered.

Example 5.1 Let us take $K(x,t) = 3 - x - t$, $f(x) = x^3$. Evaluating the integrals (5.4) we obtain

(i) $G_{il} = 0$ for $x_l < \tau_1$,

(ii) $G_{il} = \dfrac{1}{2}(x_l - \tau_1)(6 - 3x_l - \tau_1)$ for $\tau_1 \le x_l \le \tau_2$,

(iii) $G_{il} = (3 - x_l)(2\tau_2 - \tau_1 - x_l) + \dfrac{1}{2}(x_l^2 - 2\tau_2^2 + \tau_1^2)$ for $\tau_2 \le x_l \le \tau_3$,

(iv) $G_{il} = \dfrac{1}{4m^2}$ for $\tau_3 \le x_l$.

To find the exact solution of the problem eq. (5.1) is differentiated twice with respect to x. Doing this we get the differential equation

$$u'' - (3 - 2x)u' + 3u = 6x$$

with the initial conditions $u(0) = u'(0) = 0$. This equation was integrated by the fourth order Runge-Kutta method and the result was taken for $u_{ex}(x)$. Computations gave for the error function (3.12) the values:

$$e_2 = 1.2E-2\,, \quad e_3 = 3.1E-3\,, \quad e_4 = 7.9E-4\,, \quad e_5 = 2.0E-4\,.$$

6 Integro-differential equation

For the sake of concreteness let us consider equation of the following type

$$u'(x) + p(x)u(x) = \int_0^1 K(x,t)[\alpha u(t) + \beta u'(t)]\,dt + f(x)\,, \tag{6.1}$$

where α, β are constants and $p(x)$, $f(x)$ prescribed functions. To this equation belongs the initial condition $u(0) = \gamma$.

According to the method suggested by Chen and Hsiao [19, 20] we do not develop into the Haar series the function $u(x)$, but its derivative $u'(x)$:

$$u'(t) = \sum_{i=1}^{2M} a_i h_i(t)\,, \quad u(t) = \sum_{i=1}^{2M} a_i S_i(t) + u(0)\,, \tag{6.2}$$

where $S_i(t) = \displaystyle\int_0^t h_i(t)\,dt$.

A substitution of (6.2) into (6.1) gives

$$\sum_{i=1}^{2M} a_i\big[h_i(x)+p(x)S_i(x)-\alpha R_i(x)-\beta G_i(x)\big] = -p(x)u(0)+\alpha u(0)Q(x)+f(x)\,. \tag{6.3}$$

Here $G_i(x)$ is defined by (3.4), besides the following notations are introduced

$$Q(x) = \int_0^1 K(x,t)\,dt\,, \qquad R_i(x) = \int_0^1 K(x,t)S_i(t)\,dt\,.$$

Next we discretize (6.3) by assuming $x = x_l = (l-0.5)/(2M)$, $l = 1, 2, \ldots, 2M$. It is convenient to pass to the matrix notation assuming that

$$H_{il} = h_i(x_l)\,, \quad G_{il} = G_i(x_l)\,, \quad Q_l = Q(x_l)\,, \quad R_{il} = R_i(x_l)\,, \quad V_{il} = p(x_l)S_i(x_l)\,.$$

In [19] the matrix S was written in the form $S = PH$, where P is the integration matrix. In the same paper an algorithm, which allows to compute P for an arbitrary J, is given.

If p, f and Q are understood as $2M$-vectors, then the matrix form of (6.3) is

$$a(H + V - \alpha R - \beta G) = -\gamma p + \alpha\gamma Q + f\,. \tag{6.4}$$

Example 6.1 Consider the case $p(x) = 2x$, $K(x,y) = x + t$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $f(x) = 2x^4 + 2x^3 + 3x^2 - 17/12$. Equation (6.4) gets the form

$$a(H + V - G) = f\,. \tag{6.5}$$

Here $V_{il} = 2x_l(PH)_{il}$ and according to (3.4):

$$G_{il} = \begin{cases} x_l + 0.5 & \text{for } i = 1\,, \\ -1/(4m^2) & \text{for } i > 1\,. \end{cases}$$

After (6.5) is solved the functions $u(x)$ and $u'(x)$ can be calculated from the formulas (6.2).

The results of the computation were compared with the exact solution, which is $u_{ex} = x^2(1+x)$. The error estimate (3.12) was: $e_2 = 1.6E-2$, $e_3 = 4.0E-3$, $e_4 = 1.0E-3$, $e_5 = 2.6E-4$, $e_6 = 6.5E-5$.

7 Weakly singular integral equation

In this section the integral equation

$$u(x) = \int_0^x \frac{K(x,t)}{(x-t)^\alpha} u(t)\,dt + f(x)\,, \quad 0 < \alpha < 1\,,\ 0 \le t \le x \le 1, \tag{7.1}$$

is considered. This is a weakly singular equation. Since it is also a Volterra equation, then we can here apply the results of Sec. 5.

Example 7.1 Let us consider the Abel integral equation

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\,dt = 1\,, \qquad 0 \le t \le x \le 1\,. \tag{7.2}$$

Replacing (2.5) into (7.2) we get

$$\sum_{i=1}^{2M} a_i G_i(x_l) = 1\,, \qquad l = 1, \ldots, 2M\,, \tag{7.3}$$

where

$$G_i(x) = \int_0^x \frac{h_i(t)}{\sqrt{x-t}} dt\,.$$

The matrix formulation of this equation is

$$aG = E\,, \tag{7.4}$$

where $G_{il} = G_i(x_l)$ and E is a $2M$-dimensional unit vector.

Evaluating the integrals $G_{il} = G_i(x_l)$ we obtain

(i) $G_{il} = 0$ for $x_l \le \tau_1$,

(ii) $G_{il} = 2\sqrt{x_l - \tau_1}$ for $\tau_1 \le x_l \le \tau_2$,

(iii) $G_{il} = -4\sqrt{x_l - \tau_2} + 2\sqrt{x_l - \tau_1}$ for $\tau_2 \le x_l \le \tau_3$,

(iv) $G_{il} = -4\sqrt{x_l - \tau_2} + 2\sqrt{x_l - \tau_1} + 2\sqrt{x_l - \tau_3}$ for $\tau_3 \le x_l \le 1$.

The exact solution of (7.2) is

$$u_{ex} = \frac{1}{\pi\sqrt{x}}\,.$$

The computations which were carried out for the Haar wavelet approach showed that near the singularity $x = 0$ a great number of collocation points must be taken. This situation is demonstrated in Fig. 1: only for $J = 7$ (or $2M = 256$) the accuracy of the wavelet solutions can be regarded more or less satisfactory. If we move away from $x = 0$ the convergence is again good. So for the interval $0.1 \le x \le 1$ the error estimates (3.12) are $e_4 = 1.9E-2$, $e_5 = 1.3E-3$, $e_6 = 3.4E-4$, $e_7 = 8.8E-5$.

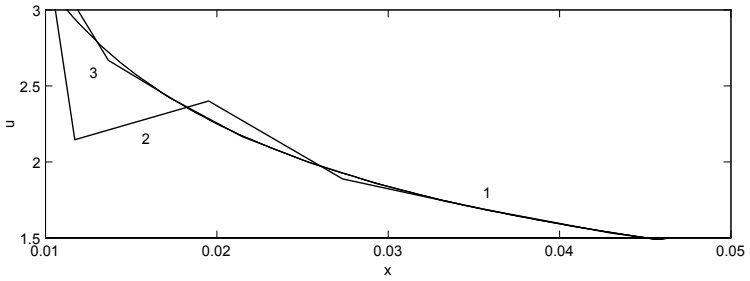

Figure 1: Solution of the Example 7.1 in vicinity of the singular point $x = 0$; 1 – exact solution; 2,3 – wavelet solutions for $J = 6$ and $J = 7$.

8 Conclusion

In this work the Haar wavelet method for solution of linear integral equations is proposed. A method of solution which is applicable for different kind of integral equations, is worked out. Fredholm and Volterra equations,also integro-differential and singular integral equations are considered. Two variants of solution, which are based on the collocation and Galerkin technics are proposed. The elaborated method is very simple and – as it follows from the test problems – high precision of results can be obtained with a small number of calculation points.

The calculations show that by doubling the number of the calculation points the error function decreases ~ 4 times. This result follows also from analytical

considerations. Approximation with the Haar wavelets is equivalent with the approximation for piecewise constant functions [13]. Therefore the results on the convergence rate obtained for the piecewise constant approximation can be transferred to the Haar wavelet approach. It is shown in [22]–[24] that in the case of Fredholm and Volterra equations the convergence rate is $O(M^{-2})$, which coincides with our numerical results.

It should be noted that in the case of Haar wavelets we have to solve systems of linear equations with a smaller condition number as for other methods based on piecewise constant approximation. Therefore the solution via Haar wavelets is more effective than conventional solutions with the same stepsize [13].

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