Solving Integral Equations by Petrov-Galerkin Method and Using Hermite-type $3 - 1$ Elements

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Abstract

In this paper, we use the Petrov-Galerkin method for solving Fredholm integral equations of the second kind on $[0, 1]$ that the trial space is piecewise Hermite-type cubic polynomials and the test space is piecewise linear polynomials, and for showing the efficiency of method, we use numerical examples.

Key words: The Petrov-Galerkin method, Piecewise polynomials, Regular pair, Trial space, Test space.

1 Introduction

In this paper, we solve Fredholm integral equations of the second kind given in the form

$$u(t) - (Ku)(t) = f(t), \qquad t \in [0, 1], \tag{1}$$

where

$$(Ku)(t) = \int_0^1 k(t, s)u(s)\, ds,$$

the function $f \in L^2[0, 1]$, the kernel $k \in L^2([0, 1] \times [0, 1])$ are given and $u \in L^2[0, 1]$ is the unknown function to be determined.

Numerical methods including least square, collocation and Galerkin methods for equation (1) are used and their analysis may be found in [1, 2, 3]. The Petrov-Galerkin method is established in [4] for equation (1). In [4] it has been shown that the Petrov-Galerkin method includes the Galerkin, collocation and least square methods. One of the advantages of the Petrov-Galerkin method is that it allows us to achieve the same order of convergence as the Galerkin method with much less computational cost by choosing the test spaces to be spaces of piecewise polynomials of lower degree. In [5] we used continuous and discontinuous Lagrange-type $k - 0$ elements for solving equation (1) on $[0, 1]$.

This paper is organized as follows: In Section 2, we review the Petrov-Galerkin method for equation (1). In Section 3 we use Hermite-type $3-1$ elements for solving equation (1) on $[0, 1]$.

2   The Petrov-Galerkin method

In this section we follow the paper [4] with a brief review of the Petrov-Galerkin method.

Let X be a Banach space and X^* be its dual space of continuous linear functionals. For each positive integer n, we assume that $X_n \subset X$, $Y_n \subset X^*$ and X_n, Y_n are finite dimensional vector spaces with

$$\dim X_n = \dim Y_n, \quad n = 1, 2, \ldots \tag{2}$$

Also X_n, Y_n satisfy condition (H) : For each $x \in X$ and $y \in X^*$, there exist $x_n \in X_n$ and $y_n \in Y_n$ such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$.

Define, for $x \in X$, an element $P_n x \in X_n$ called the *generalized best approximation* from X_n to x with respect to Y_n, by the equation

$$(x - P_n x, y_n) = 0 \qquad \text{for all } \ y_n \in Y_n. \tag{3}$$

It is proved in [4] that, for each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if

$$Y_n \cap X_n^\perp = \{0\}. \tag{4}$$

Under this condition, P_n is a projection, *i.e.*, $P_n^2 = P_n$.

Assume that, for each n, there is a linear operator $\Pi_n : X_n \to Y_n$ with $\Pi_n X_n = Y_n$ satisfying the following two conditions:

(H-1) for all $x_n \in X_n$, $\|x_n\| \leq C_1 (x_n, \Pi_n x_n)^{1/2}$,

(H-2) for all $x_n \in X_n$, $\|\Pi_n x_n\| \leq C_2 \|x_n\|$.

If a pair of sequences of spaces $\{X_n\}$ and $\{Y_n\}$ satisfy (H-1) and (H-2), we call $\{X_n, Y_n\}$ a *regular pair*. It is proved in [4] that, if a regular pair $\{X_n, Y_n\}$ satisfies $\dim X_n = \dim Y_n$ and condition (H), then the corresponding generalized projection P_n satisfies:

(1) for all $x \in X$, $\|P_n x - x\| \to 0$ as $n \to \infty$,

(2) there is a constant $C > 0$ such that $\|P_n\| < C$, $n = 1, 2, \ldots,$

(3) for some constant $C > 0$ independent of n, $\|P_n x - x\| \le C\|Q_n x - x\|$,

where $Q_n x$ is the best approximation from X_n to x.

The Petrov-Galerkin method for equation (1) is a numerical method for finding $u_n \in X_n$ such that

$$(u_n - K u_n, y_n) = (f, y_n) \quad \text{for all} \ \ y_n \in Y_n. \tag{5}$$

If $\{X_n, Y_n\}$ is a regular pair with a linear operator $\Pi_n : X_n \to Y_n$, then equation (5) may be rewritten as

$$(u_n - K u_n, \Pi_n x_n) = (f, \Pi_n x_n) \quad \text{for all} \ \ x_n \in X_n. \tag{6}$$

Furthermore, equation (5) is equivalent to

$$u_n - P_n K u_n = P_n f. \tag{7}$$

Equation (7) can also be derived from the fact that $P_n x = 0$ for an $x \in X$ if and only if $(x, y_n) = 0$ for all $y_n \in Y_n$.

Now, assume $u_n \in X_n$ and $\{b_i\}_{i=1}^n$ is a basis for X_n and $\{b_j^*\}_{j=1}^n$ is a basis for Y_n. Therefore the Petrov-Galerkin method on $[0,1]$ for equation (1) is

$$(u_n - K u_n, b_j^*) = (f, b_j^*), \quad j = 1, \dots, n. \tag{8}$$

Let $u_n(t) = \sum_{i=1}^n a_i b_i(t)$. Then equation (1) leads to determining $\{a_1, a_2, \dots, a_n\}$ as the solution of the linear system

$$\sum_{i=1}^n a_i \left\{ \int_0^1 b_i(t) b_j^*(t)\, dt - \int_0^1 \int_0^1 K(s,t) b_i(s) b_j^*(t)\, ds\, dt \right\}$$

$$= \int_0^1 f(t) b_j^*(t)\, dt, \qquad j = 1, \dots, n. \tag{9}$$

The Petrov-Galerkin methods using regular pairs $\{X_n, Y_n\}$ of piecewise polynomial spaces are called Petrov-Galerkin elements. If we use piecewise polynomials of degree k and k' for the spaces X_n and Y_n respectively, we call the corresponding Petrov-Galerkin elements $k - k'$ elements. In Section 3 we solve the equation (1) using Hermite-type $3 - 1$ elements.

3 Hermite-type 3-1 elements

We subdivide the interval $[0, 1]$ into n subintervals by a sequence of points $0 = t_0 < t_1 < \cdots < t_n = 1$. Denote $I_i = [t_{i-1}, t_i]$ and $h_i = t_i - t_{i-1}$ and let X_n be the space of piecewise Hermite-type cubic polynomials, that is:

$$
\begin{aligned}
X_n \;=\; & \{x_n \in C^1[0,1] : \; x_n\big|_{I_i} \text{ is a cubic polynomial determined by} \\
& x_n^l(t_{i-1}),\; x_n^l(t_i), \quad l = 0, 1, \quad i = 1, \ldots, n\} \\
=\; & \operatorname{span}\{b_1(t), b_2(t), \ldots, b_{2n+2}(t)\}.
\end{aligned}
$$

Using Hermite interpolation, for each $x_n \in X_n$ it holds that

$$
x_n(t) = \sum_{j=1}^{n+1}\{x_n(t_{j-1})b_{2j-1}(t) + x_n'(t_{j-1})b_{2j}(t)\},
$$

where

$$
b_j(t) \;=\; \begin{cases} \phi_j(\tau)(h_1)^{j-1}, & \tau = \frac{t-t_0}{h_1},\; t \in I_1, \\[2mm] & \qquad\qquad\qquad j = 1, 2, \\[2mm] 0, & t \notin I_1, \end{cases}
$$

$$
b_{2i+j}(t) \;=\; \begin{cases} \phi_{j+2}(\tau)(h_i)^{j-1}, & \tau = \frac{t-t_{i-1}}{h_i},\; t \in I_i, \\[2mm] \phi_j(\tau)(h_{i+1})^{j-1}, & \tau = \frac{t-t_i}{h_{i+1}},\; t \in I_{i+1}, \quad \begin{cases} i = 1, \ldots, n-1, \\ j = 1, 2, \end{cases} \\[2mm] 0, & t \notin I_i \bigcup I_{i+1}, \end{cases}
$$

$$
b_{2n+j}(t) \;=\; \begin{cases} \phi_{j+2}(\tau)(h_n)^{j-1}, & \tau = \frac{t-t_{n-1}}{h_n},\; t \in I_n, \\[2mm] & \qquad\qquad\qquad j = 1, 2, \\[2mm] 0, & t \notin I_n, \end{cases}
$$

and

$$
\begin{aligned}
\phi_1(\tau) &= (1-\tau)^2(2\tau+1), \\
\phi_2(\tau) &= \tau(1-\tau)^2, \\
\phi_3(\tau) &= \tau^2(3-2\tau), \\
\phi_4(\tau) &= (\tau-1)\tau^2.
\end{aligned}
$$

Now, let Y_n be the space of piecewise linear polynomials, that is,

$$
Y_n = \operatorname{span}\{b_1^*(t), b_2^*(t), \ldots, b_{2n+2}^*(t)\},
$$

where

$$
b^*_{2i+1}(t) \;=\; \begin{cases} 1, & t \in [t_i - \frac{h_i}{2}, t_i + \frac{h_{i+1}}{2}], \\[2mm] 0, & \text{otherwise,} \end{cases} \qquad i = 0, 1, \ldots, n,
$$

$$
b^*_{2i+2}(t) \;=\; \begin{cases} t - t_i, & t \in [t_i - \frac{h_i}{2}, t_i + \frac{h_{i+1}}{2}], \\[2mm] 0, & \text{otherwise,} \end{cases} \qquad i = 0, 1, \ldots, n,
$$

$$
h_0 \;=\; h_{n+1} = 0.
$$

Then $\dim X_n = \dim Y_n = 2n + 2$ and in [4] it is proved that $\{X_n, Y_n\}$ form a regular pair.

Let B be a $(2n+2) \times (2n+2)$ matrix with entries

$$
b_{ij} = (b_i(t), b_j^*(t)), \qquad i, j = 1, \ldots, 2n+2.
$$

A direct computation gives

$$
B = \begin{pmatrix}
L_0 & N_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
M_1 & L_1 & N_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & M_2 & L_2 & N_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & M_{n-2} & L_{n-2} & N_{n-1} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & M_{n-1} & L_{n-1} & N_n \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & M_n & L_n
\end{pmatrix},
$$

where

$$
L_0 \;=\; \begin{pmatrix} \frac{13}{32} h_1 & \frac{29}{320} h_1^2 \\[2mm] \frac{11}{192} h_1^2 & \frac{1}{60} h_1^3 \end{pmatrix}, \qquad
L_n \;=\; \begin{pmatrix} \frac{13}{32} h_n & \frac{-29}{320} h_n^2 \\[2mm] \frac{-11}{192} h_n^2 & \frac{1}{60} h_n^3 \end{pmatrix},
$$

$$
L_i \;=\; \begin{pmatrix} \frac{13}{32} h_i + \frac{13}{32} h_{i+1} & \frac{-29}{320} h_i^2 + \frac{29}{320} h_{i+1}^2 \\[2mm] \frac{-11}{192} h_i^2 + \frac{11}{192} h_{i+1}^2 & \frac{1}{60} h_i^3 + \frac{1}{60} h_{i+1}^3, \end{pmatrix}, \quad i = 1, 2, \ldots, n-1,
$$

$$
M_i \;=\; \begin{pmatrix} \frac{3}{32} h_i & \frac{11}{320} h_i^2 \\[2mm] \frac{-5}{192} h_i^2 & \frac{-3}{320} h_i^3 \end{pmatrix}, \quad N_i = \begin{pmatrix} \frac{3}{32} h_i & \frac{-11}{320} h_i^2 \\[2mm] \frac{5}{192} h_i^2 & \frac{-3}{320} h_i^3 \end{pmatrix}, \quad i = 1, \ldots, n.
$$

For solving equation (1), we seek a function $u_n \in X_n$, and it can be written as

$$u_n(t) = \sum_{j=1}^{2n+2} c_j b_j(t). \tag{10}$$

From (5), the Petrov-Galerkin method for equation (1) is

$$\left(u_n(t) - \int_0^1 k(t,s) u_n(s)\, ds, b_i^*(t) \right) = \left(f(t), b_i^*(t) \right), \tag{11}$$

$$i = 1, \ldots, 2n+2.$$

If we substitute (10) in (11), we have

$$\sum_{j=1}^{2n+2} c_j \left\{ (b_j(t), b_i^*(t)) - \left(\int_0^1 k(t,s) b_j(t)\, ds, b_i^*(t) \right) \right\} = \left(f(t), b_i^*(t) \right), \tag{12}$$

$$i = 1, \ldots, 2n+2.$$

Now, we approximate $k(t,s)$ and $f(t)$ in X_n:

$$k(t,s) = \sum_{p=1}^{n+1} \{ k(t_{p-1}, s) b_{2p-1}(t) + k_t(t_{p-1}, s) b_{2p}(t) \},$$

$$f(t) = \sum_{q=1}^{n+1} \{ f(t_{q-1}) b_{2q-1}(t) + f'(t_{q-1}) b_{2q}(t) \},$$

and then substitute in (12). Therefore, this leads to determining $\{c_1, c_2, \cdots, c_{2n+2}\}$ as the solution of the linear system

$$\sum_{j=1}^{2n+2} c_j \{ b_{ji} - \sum_{p=1}^{n+1} d_{pj} b_{2p-1,i} + e_{pj} b_{2p,i} \} = \sum_{q=1}^{n+1} f(t_{q-1}) b_{2q-1,i} + f'(t_{q-1}) b_{2q,i},$$

$$i = 1, 2, \ldots, 2n+2,$$

where

$$\begin{cases} d_{pj} = \int_0^1 k(t_{p-1}, s) b_j(s)\, ds, \\[2mm] e_{pj} = \int_0^1 k_t(t_{p-1}, s) b_j(s)\, ds, \end{cases} \quad \begin{aligned} j &= 1, \ldots, 2n+2, \\ p &= 1, \ldots, n+1. \end{aligned}$$

Example 1

$$u(t) - \int_0^1 (t+s) u(s)\, ds = (t/2) - (1/3), \qquad 0 \le t \le 1,$$

with exact solution $u(t) = t$.

Example 2

$$u(t) - \int_0^1 (t^2 e^{s(t-1)})u(s)\,ds = (1-t)e^t + t, \qquad 0 \le t \le 1,$$

with exact solution $u(t) = e^t$.

Example 3

$$u(t) - \int_0^1 (-\frac{1}{3}e^{2t-5s/3})u(s)\,ds = e^{2t+1/3}, \qquad 0 \le t \le 1,$$

with exact solution $u(t) = e^{2t}$.

In the following table we computed $\|u_n(t_i) - u(t_i)\|_2$ for $n = 1, 2, 4, 10$ with equally spaced points:

n	**Example 1**	**Example 2**	**Example 3**
1	$4.66082 * 10^{-15}$	0.00902821	0.116
2	$2.28836 * 10^{-14}$	0.00061158	0.00715332
4	$1.17916 * 10^{-13}$	0.0000443491	0.00051260
10	$4.89515 * 10^{-13}$	$1.552805 * 10^{-6}$	0.0000180318

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